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1. A Co-Author of a best selling book on Basic Mathematics for Polytechnic Students.
2. A Consultant for over a decade for Higher Education in USA and several other countries.

Over 25 years of experience in Academic Publishing...

With over more than 25 years of total experience in previous company, to bring out more than 1200 titles in Engineering, Polytechnic, Pharmacy, Computer Science and Information Technology, Sachin Shah is a name synonymous with quality and innovative content.

A publishing career that started with handwritten cyclostyled notes back in 1992, Sachin Shah has to his credit setting up and expansion of one of the leading companies in higher education publishing.

KEY Role in PREVIOUS WELL KNOWN & highly REPUTED PUBLISHING COMPANY ...

(1) I have being the major share holder of previous company. Throughout the inception, I have been actively involved in :

- a. **Selection of Authors from various Universities,**
- b. **Technical expertise required, Refining the language & other things necessary for easy understanding of concepts.**
- c. **Complete editorial support to many Authors.**
- d. **Actively involved in entire set up right from Getting Authors, production, establishment of strong distribution network.**

Hence due to my expertise in various aspects of Publication, led the previous publication to attain maximum popularity in a very short span of time.

(2) I had a wide experience of working with the Best Authors. Unique Ability to explore the full potential of Authors has always being my strong point throughout.

(3) Over a span of 25 years & visiting across many engineering colleges, meeting numerous faculties during my journey in previous Publication, has given many "best sellers" to Students of Savitribai Phule Pune University (SPPU), Polytechnic students of Maharashtra State (MSBTE), Mumbai University (MU) and Gujarat Technological University (GTU).

Engineering Mathematics-I

First Year Engineering

GROUP A & GROUP B: Semester I (BTBS101)

(Dr. Babasaheb Ambedkar Technological University)

Strictly as per the New Syllabus of Dr. Babasaheb Ambedkar Technological University w.e.f. academic year 2018-2019

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Engineering Mathematics-I

Santosh R. Mitkari, Mahesh V. Ghotkar, Balasaheb B. Gadekar

(Group A & B, Semester I, First Year Engineering, Dr. Babasaheb Ambedkar Technological University)

(BFE1) (FB1) (BATU1A)

▶ **Note from Author**
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Preface

"Our Passion is for Lucidity. We don't mean simple mindness.
If people can't understand it, why write it."

Dear students,

We are extremely happy to come out with this edition of **"Engineering Mathematics - I"** for you, the First year engineering students of Dr. BATU. We are very glad for your overwhelming response of our Engineering Mathematics–I/II books from Previous syllabus. The purpose of this book is to introduce Engineering Mathematics- I in a simple and easy way. This book has been designed for the use of Firstyear engineering students to remove the fear of the subject. This text book has been written strictly as per the New Syllabus prescribed by University of Dr. Babasaheb Ambedkar Technological University. During our twenty years teaching experience, we had fully understood the need of the students and hence we had taken a great care to present the subject in the most clear, interesting and complete form, from the students point of view.

A large number of examples have been solved in simple and honest reasoning explained step by step which anybody can understand easily with concentration. Generally, a student has difficulty to understand the mathematical concepts only because, the author skips steps which he assumes the students to be familiar with. We avoided such gaps and all the necessary and useful part has been arranged in a proper sequence. Reading this book will give you a great satisfaction to understand the concepts of the subject. The example in each article has been carefully graded. Attention is called to examples occurring where a slight variation in statement changes the nature of problem. We hope that this book will serve as great introductions fo engineering students. We discussed all kinds of problems. We had shown that how to solve problems in a simple way through this book.

We sincerely thank to Prof. S.L. Bhilare, Assistant Executive Director (JSPM's Tathawade campus), Dr. R. K. Jain, Principal RSCOE, Pune, Dr. H. V. Vankudre, Principal BVCOEW, Pune , Prof. A.S. Devasthali Vice-Principal RSCOE, Pune. and Prof. A.M. Pawar, HOD, for continuous support in all respect. We are thankful to Mr. Sachin Shah for the encouragement and support that they have extended to us. We are als thankful to staff members of Tech-Neo Publications for their efforts to make this book as good it is.

The suggestions for improvement of the text will be thankfully welcome.

- Autho

SYLLABUS

Teaching Scheme :
Lecture: 3 hrs/week
Tutorial: 1 hr/week

Examination Scheme:
Internal Assessment : 20 Marks
Mid Term Test : 20 Marks
End Semester Exam : 60 Marks
(Duration : 03 hrs)

Unit 1 : Linear Algebra-Matrices

Inverse of a matrix by Gauss-Jordan method; Rank of a matrix; Normal form of a matrix ; Consistency of non- homogeneous and homogeneous system of linear equations; Eigen values and eigen vectors ; Properties of eigen values and eigen vectors (without proofs); Cayley- Hamilton's theorem (without proof) and its applications. **[6Hours]**

(Refer chapters 1, 2 and 3)

Unit 2 : Partial Differentiation

Partial derivatives of first and higher orders; Homogeneous functions – Euler's Theorem for functions containing two and three variables (with proofs); Total derivatives; Change of variables. **[06 Hours]**

(Refer chapter 4)

Unit 3 : Applications of Partial differentiation

Jacobians - properties; Taylor's and Maclaurin's theorems (without proofs) for functions of two variables; Maxima and minima of functions of two variables; Lagrange's method of undetermined multipliers. **[06Hours]**

(Refer chapter 5)

Unit 4 : Reduction Formulae and Curve Tracing

Reduction formulae for $\int_0^{\pi/2} \sin^n x \, dx$, $\int_0^{\pi/2} \cos^n x \, dx$, $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$; Tracing of the curves given in Cartesian, parametric and polar forms. **[06Hours]**

(Refer chapters 6 and 7)

Unit 5 : Multiple Integrals

Double integration in Cartesian and polar co-ordinates; Evaluation of double integrals by changing the order of integration and changing to polar form; Triple integral; Applications of multiple integrals to find area as double integral , volume as triple integral and surface area. **[08 Hours]**

(Refer chapters 8 and 9)

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UNIT I

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UNIT 1

Linear Algebra-Matrices

➢➢ Syllabus :

Inverse of a matrix by Gauss-Jordan Method; Rank of a matrix; Normal form of a matrix.

Consistency of non- homogeneous and homogeneous system of linear equations.

Eigen values and eigen vectors; Properties of eigen values and eigen vectors (without proofs); Cayley-Hamilton's theorem (without proof) and its applications.

- **Chapter 1 : Matrices**
- **Chapter 2 : System of Linear Equations**
- **Chapter 3 : Eigen Values and Eigen Vectors**

CHAPTER 1

UNIT I

Matrices

Syllabus

Inverse of a matrix by Gauss-Jordan Method; Rank of a matrix; Normal form of a matrix.

1.1 Introduction

In the engineering field matrix theory is used in various areas. It has a special relationship with systems of linear equations which occurs in many engineering processes. Matrix theory occurs in many branches of applied mathematics such as algebraic and differential equations, mechanics, theory of electric circuits, nuclear physics, aerodynamics and astronomy.

Matrix theory play a very vital role in communication theory, network analysis, theory of structures, quantum mechanics, biology, sociology, economics, psychology, statistics etc.

1.2 Definition

An ordered set of mn numbers (elements) arranged in a rectangular array of m rows and n columns and enclosed by a pair of brackets [ ] or parentheses ( ) is called a **matrix of order m × n** (read as m by n).

$$A = \begin{bmatrix} a_{11} & a_{12} & -- & -- & a_{1n} \\ a_{21} & a_{22} & -- & -- & a_{2n} \\ -- & -- & -- & -- & -- \\ -- & -- & -- & -- & -- \\ a_{m1} & a_{m2} & -- & -- & a_{mn} \end{bmatrix}$$

In short we can write it as,

$A = [a_{ij}]$

Where, $i = 1, 2, 3, \ldots\ldots, m \leftarrow$ Row index

$j = 1, 2, 3, \ldots\ldots, n \leftarrow$ Column index

Example

$$A = \begin{array}{cc} \begin{bmatrix} 2 & 5 & 7 \\ 3 & 2 & -3 \end{bmatrix} & \begin{array}{l} \text{Row I} \\ \text{Row II} \end{array} \\ \begin{array}{ccc} \text{Column I} & \text{Column II} & \text{Column III} \end{array} & \end{array}$$

Order of matrix is number of rows × number of columns.

$\therefore$ Order of A $= 2 \times 3$

$$B = \begin{array}{cc} \begin{bmatrix} 5 & -7 & 9 \\ 3 & 2 & -1 \\ 0 & 7 & 3 \end{bmatrix} & \begin{array}{l} \text{Row I} \\ \text{Row II} \\ \text{Row III} \end{array} \\ \begin{array}{ccc} \text{Column I} & \text{Column II} & \text{Column III} \end{array} & \end{array}$$

$\therefore$ Order of matrix is 3×3.

Matrices are denoted by capital letters A, B, C etc.

1.3 Types of Matrices

Matrices are in different forms. Following are the different types of matrices.

1.3.1 Rectangular Matrix

A matrix is said to be rectangular matrix if number of rows of matrix is not equal to number of columns of matrix.

Examples

(i)

$$A = \begin{array}{c} \begin{bmatrix} 2 & 5 & -3 \\ 3 & -7 & 2 \end{bmatrix} \begin{array}{l} \rightarrow \text{Row I} \\ \rightarrow \text{Row II} \end{array} \\ \begin{array}{ccc} \text{Column I} & \text{Column II} & \text{Column III} \end{array} \end{array}$$

Here order of matrix A is 2×3.

(ii) $B = \begin{bmatrix} 2 & 9 \\ 1 & 2 \\ 3 & -5 \end{bmatrix}$

Here order of matrix B is 3×2.

1.3.2 Row matrix or Row Vector

A matrix is said to be row matrix or row vector in which only one row and any number of columns.

Order of row matrix is $1 \times n$

Examples

(i) $A = [1, -3]$, order of matrix A is 1×2

(ii) $B = [5 \ \ -9 \ \ 11]$, order of matrix B is 1×3

(iii) $C = [1 \ \ -2 \ \ 7 \ \ -5 \ \ 1]$, order of matrix C is 1×5

1.3.3 Column Matrix or Column Vector

A matrix is said to be column matrix or column vector in which only one column and any number of rows.

Order of column matrix is $m \times 1$

Examples

(i) $A = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, order matrix A is 2×1

(ii) $B = \begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix}$, order matrix B is 3×1

(iii) $C = \begin{bmatrix} 2 \\ -6 \\ 0 \\ -4 \\ 7 \end{bmatrix}$, order matrix C is 5×1

1.3.4 Null Matrix (Zero Matrix)

A matrix is said to be null matrix or zero matrix if all elements of a matrix are zero.

It is denoted as O or Z.

Examples

(i) $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, order of matrix is 2×2

(ii) $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, order of matrix is 3×3

(iii) $Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, order of matrix is 4×3

1.3.5 Transpose of a Matrix

A matrix obtained from a given matrix A by interchanging rows and columns is called transpose of A. It is denoted as A^T or A'

$$B = \begin{bmatrix} 2 & 3 & 7 \\ 9 & -2 & 5 \\ 0 & 6 & -4 \end{bmatrix} \quad \text{Then} \quad B^T = \begin{bmatrix} 2 & 9 & 0 \\ 3 & -2 & 6 \\ 7 & 5 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 7 & -3 & 2 \\ -1 & 0 & 0 \\ 2 & -1 & -9 \end{bmatrix} \quad \text{Then} \quad A^T = \begin{bmatrix} 1 & 7 & -1 & 2 \\ 5 & -3 & 0 & -1 \\ 4 & 2 & 0 & -9 \end{bmatrix}$$

1.3.6 Square Matrix

A matrix is said to be a square matrix if number of rows and number of columns are equal.

Order of square matrix is $n \times n$ or simply n.

Examples

(i) $A = \begin{bmatrix} 2 & 5 \\ 7 & 9 \end{bmatrix}$

A is a square matrix of order 2×2 or simply 2.

In a square matrix of order 2.

$\begin{bmatrix} 2 & 9 \\ 1 & 3 \end{bmatrix}$ ← Diagonal of a matrix

(ii) $B = \begin{bmatrix} -2 & 5 & -7 \\ 2 & 3 & 9 \\ 11 & 0 & -2 \end{bmatrix}$ ← Diagonal of a matrix

order of matrix B is 3×3

Matrix B is a square matrix of order 3

1.3.7 Diagonal Matrix

A square matrix is said to be diagonal matrix if all non-diagonal elements are zero.

It is denoted by D.

Examples

(i) $D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$ Is a diagonal matrix of order 2 x 2

(ii) $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ Is a diagonal matrix of order 3 x 3

1.3.8 Trace of Matrix

The trace of a square matrix is the sum of its diagonal elements

For example if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then trace of a matrix $A = a_{11} + a_{22} + a_{33}$

1.3.9 Scalar Matrix

A diagonal matrix is said to be scalar matrix if all diagonal elements are equal.

It is denoted by K.

Examples

$K = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ Is a scalar matrix of order 2 x 2

$K = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ Is a scalar matrix of order 3 x 3

1.3.10 Unit (Identity) Matrix

A square matrix is said to be unit matrix if all diagonal (leading) elements are unity and all non diagonal elements are zero.

If unit matrix of order n, it is denoted by I_n

Examples

(i) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$;

It is an unit (identity) matrix of order 2

(ii) $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

It is an unit (identity) matrix of order 3

1.3.11 Upper Triangular Matrix

A square matrix is said to be upper triangular matrix if all elements below leading (principal) diagonal are zero.

It is denoted by U

Examples

(i) $U = \begin{bmatrix} 4 & 0 & 9 \\ 0 & -7 & 3 \\ 0 & 0 & 5 \end{bmatrix}$ ← Leading diagonal

It is upper triangular matrix of order 3 x 3

(ii) $U = \begin{bmatrix} 4 & 5 & 8 & 7 \\ 0 & 5 & 3 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ← Leading diagonal

It is upper triangular matrix of order 4 x 4

1.3.12 Lower Triangular Matrix

A square matrix is said to be lower triangular matrix if all elements above leading (principal) diagonal are zero.

It is denoted by L

Examples

(i) $L = \begin{bmatrix} 4 & 0 & 0 \\ -3 & -7 & 0 \\ -6 & 0 & 5 \end{bmatrix}$ ← Leading diagonal

It is lower triangular matrix of order 3 x 3

(ii)

$$L = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 5 & -3 & 0 & 0 \\ 2 & 1 & -8 & 0 \\ 0 & 9 & 7 & -1 \end{bmatrix}$$ Leading diagonal

It is lower triangular matrix of order 4 x 4

1.3.13 Triangular Matrix

A square matrix is said to be triangular matrix if it is either upper triangular matrix or lower triangular matrix.

1.3.14 Symmetric Matrix

A square matrix is said to be symmetric matrix if $A = A^T$.

A^T – Transpose of a matrix (Refer Section 1.3.5)

Examples : (i) $B = \begin{bmatrix} 2 & 9 \\ 9 & 7 \end{bmatrix}$ then

$$B^T = \begin{bmatrix} 2 & 9 \\ 9 & 7 \end{bmatrix}$$

Observe that $B = B^T$,

$\therefore$ B is symmetric matrix of order 2×2

(ii) $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

and

$A^T = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ we can observe that $A = A^T$

$\therefore$ A is symmetric matrix

1.3.15 Skew-Symmetric Matrix

A square matrix is said to be skew symmetric or antisymmetric matrix if $a_{ij} = -a_{ij}$.

Examples

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}; \qquad B = \begin{bmatrix} 0 & 7 & 2 \\ -7 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

Observe that, in a **skew symmetric matrix leading diagonal elements are zero.**

Also, $A = -A^T$

1.4 Algebra of Matrices

We shall study the algebraic operations such as equality, addition, subtraction, multiplication of matrices.

1.4.1 Equality of Two Matrices

Two matrices A and B of same order are said to be equal if corresponding elements of matrix A and matrix B are same.

Examples

(i) $A = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 7 & -1 \end{bmatrix}; \; B = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 7 & -1 \end{bmatrix}$

Since corresponding elements are same

$\therefore$ A = B

(ii) $A = \begin{bmatrix} 1 & 2 \\ 3 & c \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} a & 2 \\ b & 4 \\ 2 & -3 \end{bmatrix}$

A = B if and only if corresponding elements of A and B are equals.

$$\left.\begin{aligned} \therefore\ 1 &= a,\ 2=2 & \therefore\ a &= 1 \\ 3 &= b,\ c=4 & b &= 3 \\ 2 &= 2, -3=-3 & c &= 4 \end{aligned}\right\}$$

1.4.2 Properties of Equality of Matrices

If A, B and C are three matrices of same order then these satisfies following properties :

(i) **Reflective property**

A = A

(ii) **Symmetric property**

If A = B then B = A

(iii) **Transitive property**

If A = B and B = C then A = C

1.5 Addition of Matrices

If A and B are two matrices of the same order, then their addition A + B is defined as the matrix, obtained by adding of the corresponding elements of A and B.

If order of A and order of B are same then only, addition A + B is defined, otherwise it is not defined.

Examples

(I) If $A = \begin{bmatrix} 4 & 2 & -5 \\ 7 & -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

Observe that order of matrix A and matrix B is same.

$\therefore$ A + B exist

$$\therefore A+B = \begin{bmatrix} 4 & 2 & -5 \\ 7 & -3 & 2 \end{bmatrix} + \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+5 & 2+(-3) & -5+2 \\ 7+1 & -3+0 & 2+3 \end{bmatrix} \quad \left\{\begin{matrix}\text{Adding corresponding} \\ \text{elements of A and B}\end{matrix}\right\}$$

$$A+B = \begin{bmatrix} 9 & -1 & -3 \\ 8 & -3 & 5 \end{bmatrix}$$

(II) If $A = \begin{bmatrix} 2 & -7 & 2 \\ 1 & -3 & 5 \\ -3 & 9 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -7 & -8 \\ 9 & 6 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$\therefore A+B = \begin{bmatrix} 2 & -7 & 2 \\ 1 & -3 & 5 \\ -3 & 9 & 4 \end{bmatrix} + \begin{bmatrix} 5 & -7 & -8 \\ 9 & 6 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+5 & -7+(-7) & 2+(-8) \\ 1+9 & -3+6 & 5+0 \\ -3+1 & 9+2 & 4+(-1) \end{bmatrix}$$

(adding corresponding elements of A and B)

$$A+B = \begin{bmatrix} \mathbf{7} & \mathbf{-14} & \mathbf{-6} \\ \mathbf{10} & \mathbf{3} & \mathbf{5} \\ \mathbf{-2} & \mathbf{11} & \mathbf{3} \end{bmatrix}$$

1.5.1 Properties of Matrix Addition

If A, B and C are three matrices of same order then

(i) **Commutative law : A + B = B + A**

$\therefore$ Matrix addition is commutative

(ii) **Associative law : A + (B + C) = (A + B) + C**

$\therefore$ Matrix addition is associative

(iii) **Existence of identity :** A + O = O + A = A

[O is null matrix]

O is additive identity.

(iv) **Existence of inverse**

A + (– A) = (– A) + A = O (O is null matrix)

$\therefore$ – **A is additive inverse of matrix A.**

(v) **Cancellation laws**

$A + B = A + C \Rightarrow B = C$ (left cancellation)

$B + A = C + A \Rightarrow B = C$ (Right cancellation)

1.6 Subtraction of Matrices

If A and B are two matrices of the same order then their subtraction A – B is defined as the matrix obtained by subtracting the elements of second matrix from the corresponding elements of the first matrix.

Examples

(I) If $A = \begin{bmatrix} 5 & 7 \\ 3 & 2 \\ 1 & -3 \end{bmatrix}$, and $B = \begin{bmatrix} 9 & 2 \\ 5 & -8 \\ -6 & 4 \end{bmatrix}$

Order of matrix A and matrix B are same.

$\therefore$ A – B exist

$$A-B = \begin{bmatrix} 5 & 7 \\ 3 & 2 \\ 1 & -3 \end{bmatrix} - \begin{bmatrix} 9 & 2 \\ 5 & -8 \\ -6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5-9 & 7-2 \\ 3-5 & 2-(-8) \\ 1-(-6) & -3-4 \end{bmatrix} \quad \begin{bmatrix}\text{subtracting elements of II}^{\text{nd}} \\ \text{matrix from corresponding} \\ \text{elements of I}^{\text{st}}\text{ matrix}\end{bmatrix}$$

$$A-B = \begin{bmatrix} -4 & 5 \\ -2 & 10 \\ 7 & -7 \end{bmatrix} \qquad \text{(Result)}$$

(II) If $A = \begin{bmatrix} 5 & 7 & 9 \\ 1 & -2 & 5 \\ 3 & -4 & -6 \end{bmatrix}$ and

$$B = \begin{bmatrix} 11 & 7 & -9 \\ -3 & -8 & 13 \\ 3 & 0 & -12 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 5 & 7 & 9 \\ 1 & -2 & 5 \\ 3 & -4 & -6 \end{bmatrix} - \begin{bmatrix} 11 & 7 & -9 \\ -3 & -8 & 13 \\ 3 & 0 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 5-11 & 7-7 & 9-(-9) \\ 1-(-3) & -2-(-8) & 5-13 \\ 3-3 & -4-0 & -6-(-12) \end{bmatrix}$$

$$\begin{bmatrix}\text{Subtracting elements of II}^{nd}\\ \text{matrix from corresponding}\\ \text{elements of I}^{st}\text{ matrix}\end{bmatrix}\left\{\begin{matrix}\therefore -6-(-12)=-6+12=6\\ \text{and } 9-(-9)=9+9=18\\ -2-(-8)=-2+8=6\end{matrix}\right\}$$

$$A - B = \begin{bmatrix} -6 & 0 & 18 \\ 4 & 6 & -8 \\ 0 & -4 & 6 \end{bmatrix} \checkmark \qquad \text{...Ans.}$$

1.6.1 Properties of Matrix Subtraction

If A, B and C are three matrices then,

(i) Subtraction of matrices is not commutative

$A - B \neq B - A$

(ii) Subtraction of matrices is not associative.

$(A - B) - C \neq A - (B - C)$

1.7 Scalar Multiplication

If A is a matrix and k is any scalar then scalar multiplication kA is a matrix obtained by multiplying every element of matrix A by k.

This operation also known as 'scaling of a matrix'.

Examples :

(i) $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \\ 9 & -3 \end{bmatrix}$

$$\therefore kA = k\begin{bmatrix} 2 & 3 \\ 5 & 7 \\ 9 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} k\times 2 & k\times 3 \\ k\times 5 & k\times 7 \\ k\times 9 & k\times(-3) \end{bmatrix} \left(\begin{matrix}\text{multiply each element}\\ \text{of A by k}\end{matrix}\right)$$

$$kA = \begin{bmatrix} 2k & 3k \\ 5k & 7k \\ 9k & -3k \end{bmatrix} \qquad \text{(Result)}$$

$$\text{Also (ii) } 2A = 2\begin{bmatrix} 2 & 3 \\ 5 & 7 \\ 9 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 2 & 2\times 3 \\ 2\times 5 & 2\times 7 \\ 2\times 9 & 2\times(-3) \end{bmatrix} \left(\begin{matrix}\text{multiply each element}\\ \text{of A by 2}\end{matrix}\right)$$

$$2A = \begin{bmatrix} 4 & 6 \\ 10 & 14 \\ 18 & -6 \end{bmatrix} \checkmark \qquad \text{...Ans.}$$

1.7.1 Properties of Scalar Multiplication

(i) If k = 0 then kA = 0 (null matrix)

(ii) To find matrix $\frac{A}{k}$ consider, $\frac{1}{k}\cdot A$; $(k \neq 0)$

(iii) If every element of matrix is a multiple of same common constant then we can take common out that constant

Examples

$$\begin{bmatrix} 2 & 8 \\ 6 & -4 \\ 12 & 0 \end{bmatrix} = \begin{bmatrix} 2\times 1 & 2\times 4 \\ 2\times 3 & 2\times(-2) \\ 2\times 6 & 2\times 0 \end{bmatrix} = 2\begin{bmatrix} 1 & 4 \\ 3 & -2 \\ 6 & 0 \end{bmatrix}$$

i.e. converse of scalar multiplication also true.

(iv) **Commutative law**

$k \cdot A = A \cdot k$ where k is scalar and A is matrix

(v) **Associative law**

If m, n are scalar and A is matrix then,

$(mA)\,n = m(A \cdot n)$

$= m\,(n \cdot A) = (m\,n)\,A$

(vi) **Distributive law**

$k\,(A + B) = kA + kB$

$(m + n)A = mA + nA$

where k, m, n are scalar and A is a matrix.

Example 1.7.1

If $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$

(i) **Find 2A + 3B – 4I, Where I is the unit matrix**

(ii) **Find 3A – 2B**

Solution : Given matrices,

Matrix	Order	
$A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$	2×2	...(1)
$B = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$	2×2	...(2)

Also unit matrix I is,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{with same order as } 2\times 2 \qquad \text{...(3)}$$

(i) Since orders of matrices A, B and I are same.

$\therefore$ 2A + 3B – 4I exist.

$$\therefore 2A + 3B - 4I = 2\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} + 3\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

[from Equations (1), (2), (3)]

$$= \begin{bmatrix} 2\times 2 & 2\times 3 \\ 2\times 4 & 2\times 7 \end{bmatrix} + \begin{bmatrix} 3\times 1 & 3\times 3 \\ 3\times 4 & 3\times 6 \end{bmatrix} - \begin{bmatrix} 4\times 1 & 4\times 0 \\ 4\times 0 & 4\times 1 \end{bmatrix}$$

(by scalar multiplication)

$$= \begin{bmatrix} 4 & 6 \\ 8 & 14 \end{bmatrix} + \begin{bmatrix} 3 & 9 \\ 12 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3 & 6+9 \\ 8+12 & 14+18 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix}\text{Addition of first}\\ \text{two matrices}\end{bmatrix}$$

$$= \begin{bmatrix} 7 & 15 \\ 20 & 32 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7-4 & 15-0 \\ 20-0 & 32-4 \end{bmatrix} \qquad \text{[subtraction of two matrices]}$$

$$\mathbf{2A + 3B - 4I} = \begin{bmatrix} \mathbf{3} & \mathbf{15} \\ \mathbf{20} & \mathbf{28} \end{bmatrix} \checkmark \quad \text{This is required matrix.}$$

(ii) $3A - 2B = 3\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} - 2\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$ [From Equations (1) and (2)]

$$= \begin{bmatrix} 3\times 2 & 3\times 3 \\ 3\times 4 & 3\times 7 \end{bmatrix} - \begin{bmatrix} 2\times 1 & 2\times 3 \\ 2\times 4 & 2\times 6 \end{bmatrix} \left(\begin{matrix}\text{by scalar}\\ \text{multiplication}\end{matrix}\right)$$

$$= \begin{bmatrix} 6 & 9 \\ 12 & 21 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 8 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 6-2 & 9-6 \\ 12-8 & 21-12 \end{bmatrix} \qquad \text{(subtraction of two matrices)}$$

$$\mathbf{3A - 2B} = \begin{bmatrix} \mathbf{4} & \mathbf{3} \\ \mathbf{4} & \mathbf{9} \end{bmatrix} \checkmark \qquad \text{This is required matrix.}$$

1.8 Multiplication of Matrices

Multiplication of two matrices A and B is possible only if the number of columns in the first matrix is equal to number of rows in the second matrix.

Examples

(i) If order of matrices : A is 3×2 , B is 2×3 and C is 3×3.

same

$\therefore$ *For AB order is* 3×2 ; 2×3 $= 3\times 3$

Order of product

$\therefore$ **AB exist** and order of product is 3×3

For AC order is

not same

3×2 ; 3×3

$\therefore$ **AC does** not exist.

same

$\therefore$ *For CA order is* 3×3 ; 3×2 $= 3\times 2$

Order of product

$\therefore$ **CA exist** and order of CA is 3×2

same

$\therefore$ *For BC order is* 2×3 ; 3×3 $= 2\times 3$

Order of product

$\therefore$ **BC exist** and order of BC is 2×3

In general

If A is a matrix of order $m\times n$ and B is a matrix of order $n\times r$ then order of product AB is,

same

$\therefore$ *Order of AB* $\quad m\times n$; $n\times r$ $= m\times r$

Order of product

(I) Matrix Multiplication Row by Column Method

The rule of multiplication of two conformable matrices is called **row-by-column method.**

$$\text{Consider, } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

Orders of A and B are 3×3 and 3×2 respectively. AB is defined and is of order

same

3×3 ; 3×2 $= 3\times 2$

Order of product

Where,

c_{11} = sum of products of elements of 1st row of A and 1st column of B

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11}\,b_{11} + a_{12}\,b_{21} + a_{13}\,b_{31}$$

c_{12} = sum of products of elements of 1st row of A and 2nd column of B

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{11}\,b_{12} + a_{12}\,b_{22} + a_{13}\,b_{32}$$

c_{21} = sum of products of elements of 2nd row of A and 1st column of B

$$= \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{21}\,b_{11} + a_{22}\,b_{21} + a_{23}\,b_{31}$$

c_{22} = sum of products of elements of 2nd row of A and 2nd column of B

$$= \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{21}\,b_{12} + a_{22}\,b_{22} + a_{23}\,b_{32}$$

c_{31} = sum of products of elements of 3rd row of A and 1st column of B

$$= \begin{bmatrix} a_{31} & a_{32} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{31}\,b_{11} + a_{32}\,b_{21} + a_{33}\,b_{31}$$

c_{32} = sum of products of elements of 3rd row of A and 2nd column of B

$$= \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{31}b_{12} + a_{32}\,b_{22} + a_{33}\,b_{32}$$

Thus,

$$AB = \begin{bmatrix} a_{11}\,b_{11} + a_{12}\,b_{21} + a_{13}\,b_{31} & a_{11}\,b_{12} + a_{12}\,b_{22} + a_{13}\,b_{32} \\ a_{21}\,b_{11} + a_{22}\,b_{21} + a_{23}\,b_{31} & a_{21}\,b_{12} + a_{22}\,b_{22} + a_{23}\,b_{32} \\ a_{31}\,b_{11} + a_{32}\,b_{21} + a_{33}\,b_{31} & a_{31}\,b_{12} + a_{32}\,b_{22} + a_{33}\,b_{32} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

(II) Another useful notation to remember matrix multiplication

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} \to R_1 \\ \to R_2 \\ \to R_3 \end{matrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix};$$

Where, R_1 = 1st Row, R_2 = 2nd Row, R_3 = 3rd Row

$$\text{and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = [C_1 \quad C_2]$$

Where, C_1 = 1st column; C_2 = 2nd column

$$AB = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} [C_1 \quad C_2] = \begin{bmatrix} R_1\,C_1 & R_1\,C_2 \\ R_2\,C_1 & R_2\,C_2 \\ R_3\,C_1 & R_3\,C_2 \end{bmatrix}$$

1.8.1 Properties of Matrix Multiplication

If A, B, C, are three matrices.

(i) Commutative Law

$A \cdot B = B \cdot A$ is not always true

Multiplication of matrices is not commutative

In general, $A \cdot B \neq B \cdot A$

For two diagonal matrices A and B

$AB = BA$

(ii) Associative Law

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

(iii) Distributive Law

$$A \cdot (B + C) = A \cdot B + A \cdot C$$
$$(B + C) \cdot A = B \cdot A + C \cdot A$$

Also,

$$A \cdot (B - C) = A \cdot B - A \cdot C$$
$$(B - C) \cdot A = B \cdot A - C \cdot A$$

Provided all required operations are exist.

Note that

If $AB = AC$ then $B = C$ is not true **always.**

Note

(I) If A and B are square matrices then AB and BA both exist.

(II) If order of A is $m \times n$ and order of B is $n \times m$ then AB and BA both exist.

(III) For square matrices A, B, I (I- unit matrix) then

(i) $(A + B)^2 = A^2 + AB + BA + B^2$

(ii) $(A - B)^2 = A^2 - AB - BA + B^2$

(iii) $(A + B)(A - B) = A^2 - AB + BA - B^2$

1.8.2 Power of Matrix

If A is any square matrix then

$A^2 = A \cdot A$

$A^3 = A \cdot A \cdot A = A^2 \cdot A = A \cdot A^2$

Similarly we can find other powers of square matrix A.

Order of any power of matrix A is equal to order of matrix A.

1.9 Examples on Multiplication

Example 1.9.1

If $A = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 6 \\ -4 & -12 \end{bmatrix}$

Show that AB is null matrix

Solution :

Step I : Given matrices are,

Matrix	Order	
$A = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}$	2×2	...(1)
$B = \begin{bmatrix} 2 & 6 \\ -4 & -12 \end{bmatrix}$	2×2	...(2)

same

$\therefore$ *Order AB is* $2 \times 2\ ;\ 2 \times 2 = 2 \times 2$

Order of AB

$\therefore$ AB exist.

Step II

$$AB = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \end{matrix} \begin{bmatrix} 2 & 6 \\ -4 & -12 \end{bmatrix}$$

$C_1 \quad C_2$ [From Equations (1) and (2)]

$$= \begin{bmatrix} R_1\,C_1 & R_1\,C_2 \\ R_2\,C_1 & R_2\,C_2 \end{bmatrix} \quad \ldots \text{ Standard form} \quad \ldots(3)$$

Now,

$$R_1\,C_1 = \underbrace{(4 \times 2)}_{8} + \underbrace{(2 \times (-4))}_{-8} = 8 + (-8) = \mathbf{0}$$

$$R_1\,C_2 = \underbrace{(4 \times 6)}_{24} + \underbrace{(2 \times (-12))}_{-24} = 24 + (-24) = \mathbf{0}$$

$$R_2C_1 = \underbrace{(8 \times 2)}_{16} + \underbrace{(4 \times (-4))}_{-16} = 16 + (-16) = \mathbf{0}$$

$$R_2\,C_2 = \underbrace{(8 \times 6)}_{48} + \underbrace{(4 \times (-12))}_{-48} = 48 + (-48) = \mathbf{0}$$

Substituting all these values in Equation (3)

$$\therefore AB = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark \quad \ldots\textbf{Ans.}$$

This shows that AB is null matrix.

1.10 Determinant of the Matrix

Determinant of a square matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

1.10.1 Non-singular matrix

A square matrix A is said to be non singular matrix if $|A| \neq 0$. Otherwise it is said to be singular matrix.

1.10.2 Examples on Determinant of Matrix

Example 1.10.1

If $A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$ Show that the matrix AB is a non singular.

Solution : Given :

Matrix	Order
$A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$	2×3
$B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$	3×2

same

$\therefore$ Order of AB $= (2 \times 3) \times (3 \times 2)$

Order of AB $= 2 \times 2$

$$AB = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-2\times 0)+(0\times 2)+(1\times 1) & (-2\times 1)+(0\times 3)+(1\times 1) \\ (1\times 0)+(2\times 2)+(3\times 1) & (1\times 1)+(2\times 3)+(3\times 1) \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+1 & -2+0+1 \\ 0+4+3 & 1+5+3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -1 \\ 7 & 10 \end{bmatrix}$$

This shows that matrix AB is non-singular.

1.11 Minor of an Element

In a determinant a minor of an element is a determinant by omitting row and column in which that elements present.

Example : $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Minor of an element $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Minor of an element $a_{12} = M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Minor of an element $a_{13} = M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Minor of an element $a_{21} = M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

Minor of an element $a_{22} = M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

Minor of an element $a_{23} = M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

Minor of an element $a_{31} = M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

Minor of an element $a_{32} = M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

Minor of an element $a_{33} = M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

1.12 Cofactor of an Element

Cofactor of an element is minor of that element with the sign prefixed by the rule $(-1)^{i+j}$, where i and j are the number of rows and columns in which that elements presents.

Example : $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Cofactor of $a_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Cofactor of $a_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Cofactor of $a_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Cofactor of $a_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

Cofactor of $a_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

Cofactor of $a_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

Cofactor of $a_{31} = (-1)^{3+1} M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

Cofactor of $a_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

Cofactor of $a_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

1.13 Adjoint of a Matrix

If $A = [a_{ij}]$ is a square matrix, then the matrix obtained by replacing the elements a_{ij} by the cofactors of a_{ji} of [A], is known as adjoint of a matrix and is denote as "adj.A".

Adj A = $[A_{ji}]$

Where A_{ji} = cofactor of a_{ji} in $|A^T|$

To find adjoint of A, find transpose of A i.e. A^T and replace elements of A^T by the cofactors of the corresponding elements of $|A^T|$.

> **Note :** $A^{-1} = \frac{1}{|A|} \times$ Adj. A

1.14 Elementary Transformations

1.14.1 Row Elementary Transformation

1. The interchange of i^{th} row and j^{th} row (R_{ij} or $R_i \leftrightarrow R_j$)
2. The multiplication of each elements of i^{th} row by a non-zero elements k (kR_i).
3. The addition of a constant multiple of the elements of any row to the corresponding elements of any other row ($R_i + kR_j$).

1.14.2 Column Elementary Transformation

1. The interchange of i^{th} column and j^{th} column (C_{ij} or $C_i \leftrightarrow C_j$).
2. The multiplication of each elements of i^{th} column by a non zero element k, (kC_i).
3. The addition of constant multiple of the elements of any column to the corresponding elements of any other column ($C_i + kC_j$).

1.14.3 Equivalent Matrix

If a matrix B is obtained from the matrix A by using elementary transformation then the matrix B is said to be equivalent matrix of A.

We can write A ~ B.

Example 1.14.1

Find the inverse of matrix by adjoint method.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

OR Find the adjoint of matrix A.

Solution : Given matrix is,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \text{Compare with} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We know,

$$A^{-1} = \frac{1}{|A|} \text{adj} \cdot A \qquad \dots(1)$$

$$|A| = (1)\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - (2)\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} + (3)\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$$

$$= 1\,[\underbrace{(4)(6)}_{24} - \underbrace{(5)(5)}_{10}] - 2\,[\underbrace{(2)(6)}_{12} - \underbrace{(3)(5)}_{15}]$$

$$+ 3\,[\underbrace{(2)(5)}_{10} - \underbrace{(3)(4)}_{12}]$$

$$= 1\,\underbrace{[24-25]}_{1} - 2\,\underbrace{[12-15]}_{-3} + 3\,\underbrace{[10-12]}_{-2}$$

$$= 1(-1) - 2(-3) + 3(-2) = -1 + 6 - 6 = -1$$

i.e. $|A| = -1$ ...(2)

i.e. $|A| \neq 0$ $\therefore A^{-1}$ exist

Minors of elements	Cofactors of elements
$a_{11}(=1) = \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix}$ $= (4)(6) - (5)(5)$ $= 24 - 25 = -1 = M_{11}$	$a_{11} = (-1)^{1+1} M_{11}$ $= (1)(-1)$ $= -1 = C_{11}$
$a_{12}(=2) = \begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix}$ $= (2)(6) - (3)(5)$ $= 12 - 15 = -3 = M_{12}$	$a_{12} = (-1)^{1+2} M_{12}$ $= (-1)(-3)$ $= 3 = C_{12}$
$a_{13}(=3) = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$ $= (2)(5) - (3)(4)$ $= 10 - 12 = -2 = M_{13}$	$a_{13} = (-1)^{1+3} M_{13}$ $= (1)(-2)$ $= -2 = C_{13}$
$a_{21}(=2) = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$ $= (2)(6) - (5)(3)$ $= 12 - 15 = -3 = M_{21}$	$a_{21} = (-1)^{2+1} M_{21}$ $= (-1)(-3)$ $= 3 = C_{21}$
$a_{22}(=4) = \begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix}$ $= (1)(6) - (3)(3)$ $= 6 - 9 = -3 = M_{22}$	$a_{22} = (-1)^{2+2} M_{22}$ $= (1)(-3)$ $= -3 = C_{22}$
$a_{23}(=5) = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$ $= (1)(5) - (3)(2)$ $= 5 - 6 = -1 = M_{23}$	$a_{23} = (-1)^{2+3} M_{23}$ $= (-1)(-1)$ $= 1 = C_{23}$
$a_{31}(=3) = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$ $= (2)(5) - (4)(3)$ $= 10 - 12 = -2 = M_{31}$	$a_{31} = (-1)^{3+1} M_{31}$ $= (1)(-2)$ $= -2 = C_{31}$
$a_{32}(=5) = \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$ $= (1)(5) - (2)(3)$ $= 5 - 6 = -1 = M_{32}$	$a_{32} = (-1)^{3+2} M_{32}$ $= (-1)(-1)$ $= 1 = C_{32}$
$a_{33}(=6) = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$ $= (1)(4) - (2)(2)$ $= 4 - 4 = 0 = M_{33}$	$a_{33} = (-1)^{3+3} M_{33}$ $= (1)(0)$ $= 0 = C_{33}$

By these values,

$$\text{Matrix of cofactors} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

Adjoint of matrix A = Transpose of matrix of cofactors

$$= \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix} \qquad \text{...(3)}$$

From Equations (1), (2), (3),

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \mathbf{1} & \mathbf{-3} & \mathbf{2} \\ \mathbf{-3} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{0} \end{bmatrix} \checkmark \qquad \text{...Ans.}$$

Example 1.14.2

Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Using adjoint matrix

Solution :

Given matrix is,

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} \text{ compare with } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We know,

$$A^{-1} = \frac{1}{|A|} \text{adj } A \qquad \text{...(1)}$$

$$|A| = (1)\begin{vmatrix} 3 & -1 \\ -1 & 0 \end{vmatrix} - (-3)\begin{vmatrix} -3 & -1 \\ 2 & 0 \end{vmatrix} + 2\begin{vmatrix} -3 & 3 \\ 2 & -1 \end{vmatrix}$$

$$= 1\,[\underbrace{(3)(0)}_{0} - \underbrace{(-1)(-1)}_{1}] + 3\,[\underbrace{(-3)(0)}_{0} - \underbrace{(2)(-1)}_{-2}]$$

$$+ 2\,[\underbrace{(-3)(-1)}_{3} - \underbrace{(2)(3)}_{6}]$$

$$= 1\,\underbrace{[0-1]}_{-1} + 3\,\underbrace{[0+2]}_{2} + 2\,\underbrace{[3-6]}_{-3} = 1(-1) + 3(2) + 2(-3)$$

$$|A| = -1 + 6 - 6 = -1 \qquad \text{...(2)}$$

$|A| \neq 0 \quad \therefore A^{-1}$ exist.

Minors of elements	Cofactors of elements
$a_{11}(=1) = \begin{vmatrix} 3 & -1 \\ -1 & 0 \end{vmatrix}$ $= (3)(0) - (-1)(-1)$ $= 0 - 1 = -1 = M_{11}$	$a_{11} = (-1)^{1+1} M_{11}$ $= (1)(-1)$ $= -1 = C_{11}$
$a_{12}(=-3) = \begin{vmatrix} -3 & -1 \\ 2 & 0 \end{vmatrix}$ $= (-3)(0) - (2)(-1)$ $= 0 + 2 = 2 = M_{12}$	$a_{12} = (-1)^{1+2} M_{12}$ $= (-1)(2)$ $= -2 = C_{12}$
$a_{13}(=2) = \begin{vmatrix} -3 & 3 \\ 2 & -1 \end{vmatrix}$ $= (-3)(-1) - (2)(3)$ $= 3 - 6 = -3 = M_{13}$	$a_{13} = (-1)^{1+3} M_{13}$ $= (1)(-3)$ $= -3 = C_{13}$
$a_{21}(=-3) = \begin{vmatrix} -3 & 2 \\ -1 & 0 \end{vmatrix}$ $= (-3)(0) - (-1)(2)$ $= 0 + 2 = 2 = M_{21}$	$a_{21} = (-1)^{2+1} M_{21}$ $= (-1)(2)$ $= -2 = C_{21}$
$a_{22}(=3) = \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix}$ $= (1)(0) - (2)(2)$ $= 0 - 4 = -4 = M_{22}$	$a_{22} = (-1)^{2+2} M_{22}$ $= (1)(-4)$ $= -4 = C_{22}$
$a_{23}(=-1) = \begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix}$ $= (1)(-1) - (2)(-3)$ $= -1 + 6 = 5 = M_{23}$	$a_{23} = (-1)^{2+3} M_{23}$ $= (-1)(5)$ $= -5 = C_{23}$
$a_{31}(=2) = \begin{vmatrix} -3 & 2 \\ 3 & -1 \end{vmatrix}$ $= (-3)(-1) - (3)(2)$ $= 3 - 6 = -3 = M_{31}$	$a_{31} = (-1)^{3+1} M_{31}$ $= (1)(-3)$ $= -3 = C_{31}$
$a_{32}(=-1) = \begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix}$ $= (1)(-1) - (-3)(2)$ $= -1 + 6 = 5 = M_{32}$	$a_{32} = (-1)^{3+2} M_{32}$ $= (-1)(5)$ $= -5 = C_{32}$
$a_{33}(=0) = \begin{vmatrix} 1 & -3 \\ -3 & 3 \end{vmatrix}$ $= (1)(3) - (-3)(-3)$ $= 3 - 9 = -6 = M_{33}$	$a_{33} = (-1)^{3+3} M_{33}$ $= (1)(-6)$ $= -6 = C_{33}$

By these values,

$$\text{Matrix of cofactors} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ -2 & -4 & -5 \\ -3 & -5 & -6 \end{bmatrix} \text{...(3)}$$

Adjoint of matrix A = transpose of matrix of cofactors

$$= \begin{bmatrix} -1 & -2 & -3 \\ -2 & -4 & -5 \\ -3 & -5 & -6 \end{bmatrix}$$

From Equations (1), (2) and (3)

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} -1 & -2 & -3 \\ -2 & -4 & -5 \\ -3 & -5 & -6 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} & \mathbf{5} \\ \mathbf{3} & \mathbf{5} & \mathbf{6} \end{bmatrix} \checkmark \qquad \text{...Ans.}$$

1.15 Inverse of a Matrix

If A is a non-singular matrix then there exist a matrix A^{-1} such that $AA^{-1} = I = A^{-1}A$.

The matrix A^{-1} is known as inverse of matrix A. Inverse of a matrix is unique.

1.15.1 Inverse of a Matrix by using Elementary Transformation

If A is a nonsingular square matrix then A^{-1} exist.

$$AA^{-1} = I$$

Using row transformation on L.H.S. matrix A can reduce into unit matrix, same transformation operate on R.H.S. matrix I, it gives,

$$IA^{-1} = B \Rightarrow A^{-1} = B$$

1.15.2 Working Rule to Find A^{-1} by Elementary Row Transformation

Step I : Find the order of given square matrix A. Suppose order of matrix A is n

Consider, $AA^{-1} = I$

where I is a unit matrix of order n.

Step II : Reduce L.H.S. A by using elementary row transformation into unit matrix.

Same transformation operate on R.H.S. matrix I.

As matrix A reduced into unit matrix I

Corresponding R.H.S. reduced matrix B gives A^{-1} as,

$$IA^{-1} = B \Rightarrow A^{-1} = B$$

Note : To reduce A into unit matrix use following steps.

(I) Obtain 1 at a_{11} by using any row transformation.

(II) Obtain all zeros below a_{11} using I[st] row and ro[w] transformations.

(III) Obtain 1 at a_{22} by using row transformation but d[o] not use I[st] row.

(IV) Obtain all zeros below and above a_{22}, using II[nd] ro[w] and row transformations.

(V) Obtain 1 at a_{33} by using row transformation but d[o] not use first two rows.

(VI) Obtain all zeros below and above a_{33} using III[rd] ro[w] and row transformation.

(VIII) Use similar steps for diagonal elements until we g[et] unit matrix.

1.15.3 Gauss – Jordan Method

Those elementary row transformations which reduc[e] a given square matrix A to the unit matrix, when applied [to] unit matrix I give the inverse of A.

1.15.4 Working rule of Gauss-Jordan Method to find A^{-1}

Step I : Find the order of given square matrix A. [If] order of matrix A is n then consider [A ⋮ I[n]]

Where I_n is a unit matrix of order n.

Step II : Reduce A into unit matrix by using elementar[y] row transformations.

As A reduced into unit matrix I corresponding [I is] converted into A^{-1}.

Ex. 1.15.1

Find the inverse of matrix

$\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **by elementary transformations**

Solution :

Given matrix is,

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}; \qquad |A| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

By solving determinant,

$$= \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= 2 \times \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1 \times \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 2 \times [0-1] - 1 \times [1-1] + 3 \times [1-0]$$

$= 2(-1) - 1 \times (0) + 3(1) = -2 - 0 + 3 = 1 \neq 0$

$\therefore |A| \neq 0 \;\therefore\; A^{-1}$ exist.

Consider,

$AA^{-1} = I$ where I is limit matrix of order 3

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reduce L.H.S. matrix A into identity matrix I by using row elementary transformations.

Same transformations operate on R.H.S. matrix.

Operate R_{12} (To obtain 1 at a_{11})

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $R_2 - 2R_1$ and $R_3 - R_1$ (To obtain zeros below a_{11})

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Operate $R_3 - R_2$ (To obtain below and above a_{22})

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

Operate $-R_3$ (To obtain 1 at a_{33})

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$I A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \quad (\text{Since } I A^{-1} = A^{-1})$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \checkmark$$

...Ans.

Example 1.15.2

Using the Gauss - Jordan method find the inverse of the matrix.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Solution : The order of given square matrix A is 3,

$\therefore$ Consider

$$\left[A \,\vdots\, I_3 \right] = \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

Where I_3 is unit matrix of order 3

... (**Target :** Convert (reduce) matrix A into unit matrix by using row transformations)

Operate $R_2 - R_1$ and $R_3 + 2R_1$

(To obtain all zeros below a_{11})

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 1 & 2 & 0 & 1 \end{array}\right]$$

Operate $\frac{1}{2}R_2$ and $\frac{1}{2}R_3$ (To obtain 1 at a_{22})

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 1 & 0 & \frac{1}{2} \end{array}\right]$$

Operate $R_1 - R_2$ and $R_3 + R_2$

(To obtain all zeros below and above a_{22})

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 6 & \frac{3}{2} & \frac{-1}{2} & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

Operate $-\frac{1}{2}R_3$ (To obtain 1 at a_{33})

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 6 & \frac{3}{2} & \frac{-1}{2} & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

Operate $R_1 - 6R_3$ and $R_2 + 3R_3$

(To obtain all zeros above and below a_{33})

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{5}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{5}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

A converted into unit matrix I_3 — Corresponding reduced matrix gives A^{-1}

Here A converted (reduced) into I_3 (unit matrix)

$$\therefore A^{-1} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \checkmark$$

...Ans.

Example 1.15.3 **Dr. BATU - Nov. 16**

Use Gauss-Jordan method to find A^{-1}, where

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Solution :

The order of given square matrix A is 3 :

Consider,

$$\left[A \,\vdots\, I_3 \right] = \left[\begin{array}{ccc:ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

(where I_3 is unit matrix of order 3)

Target : Convert (Reduce) matrix A into unit matrix by using row transformation.

Operate $\frac{1}{2}R_1$ (To obtain 1 at a_{11})

$$\left[\begin{array}{ccc:ccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 1 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Operate $R_2 - 5R_1$ (To obtain all zeros below a_{11})

$$\left[\begin{array}{ccc:ccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{5}{2} & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \quad \left(\begin{array}{c}\text{observe that} \\ \text{At } a_{22} \\ \text{already } 1\end{array}\right)$$

Operate $R_3 - R_2$ (To obtain all zeros below a_{22})

$$\left[\begin{array}{ccc:ccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{5}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{5}{2} & -1 & 1 \end{array}\right]$$

Operate $2R_3$ (To obtain 1 at a_{33})

$$\left[\begin{array}{ccc:ccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{5}{2} & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right]$$

Operate $R_1 + \frac{1}{2}R_3$ and $R_2 - \frac{5}{2}R_3$

(To obtain all zeros above)

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right]$$

A converted into unit matrix I_3 — Corresponding reduced matrix gives A^{-1}

Here A reduced (converted) into I_3 (unit matrix)

$$\therefore A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \checkmark$$

...Ans.

Example 1.15.4 : **Dr. BATU - May 15**

Use Gauss-Jordan method of find A^{-1}, where

$$A = \begin{bmatrix} 8 & 4 & -3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution : The order of given square matrix A is 3.

Consider,

$$[A \,\vdots\, I_3] = \left[\begin{array}{ccc|ccc} 8 & 4 & -3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

Where I_3 is unit matrix of order 3

Target : Reduce (convert) matrix A into unit matrix by using row transformation.

Operate $\frac{1}{8}R_1$ (To obtain 1 at a_{11})

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{3}{8} & \frac{1}{8} & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

Operate $R_2 - 2R_1$ and $R_3 - R_1$ (To obtain all zeros below a_{11})

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{3}{8} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{7}{4} & -\frac{1}{4} & 1 & 0 \\ 0 & \frac{3}{2} & \frac{11}{8} & -\frac{1}{8} & 0 & 1 \end{array}\right]$$

Operate $R_2 \leftrightarrow R_3$ (or R_{23}) (**Target :** obtain 1 at a_{22})

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{-3}{8} & \frac{1}{8} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{11}{8} & \frac{-1}{8} & 0 & 1 \\ 0 & 0 & \frac{7}{4} & \frac{-1}{4} & 1 & 0 \end{array}\right]$$

Operate $\frac{2}{3}R_2$ (To obtain 1 at a_{22})

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{3}{8} & \frac{1}{8} & 0 & 0 \\ 0 & 1 & \frac{11}{12} & -\frac{1}{12} & 0 & \frac{2}{3} \\ 0 & 0 & \frac{7}{4} & -\frac{1}{4} & 1 & 0 \end{array}\right]$$

Operate $R_1 - \frac{1}{2}R_2$ (To obtain all zeros below and above a_{22})

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{5}{6} & \frac{1}{6} & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{11}{12} & -\frac{1}{12} & 0 & \frac{2}{3} \\ 0 & 0 & \frac{7}{4} & -\frac{1}{4} & 1 & 0 \end{array}\right]$$

Operate $\frac{4}{7}R_3$ (To obtain 1 at a_{33})

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{5}{6} & \frac{1}{6} & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{11}{12} & -\frac{1}{12} & 0 & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{7} & \frac{4}{7} & 0 \end{array}\right]$$

Operate $R_1 + \frac{5}{6}R_3$ and $R_2 - \frac{11}{12}R_3$

(To obtain all zeros above a_{33})

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{21} & \frac{10}{21} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{21} & -\frac{11}{21} & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{7} & \frac{4}{7} & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{5}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

A converted into unit matrix I_3 — Corresponding reduced matrix gives A^{-1}

Here A reduced (converted) into I_3 (unit matrix)

$$\therefore A^{-1} = \begin{bmatrix} \frac{1}{21} & \frac{10}{21} & \frac{-1}{3} \\ \frac{1}{21} & \frac{-11}{21} & \frac{2}{3} \\ \frac{-1}{7} & \frac{4}{7} & 0 \end{bmatrix} \checkmark \quad \text{...Ans.}$$

Exercise 1

1. **Find inverse by row elementary transformation.**

(i) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Ans. : $A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

(ii) Find the inverse of matrix by

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

Ans. : $A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

(iii) Find the inverse of the matrix $A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Ans. : $A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

(iv) Find inverse of matrix $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

Ans. : $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

2. **Using Gauss Jordan Method, find the inverse of the matrix.**

(i) Find A^{-1} if $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

Ans. : $A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$

(ii) Find the inverse of matrix

$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & -1 & 1 \end{bmatrix}$

Ans. : $A^{-1} = \frac{1}{4}\begin{bmatrix} 7 & -2 & 1 \\ 2 & 0 & -2 \\ -5 & 2 & 1 \end{bmatrix}$

(iii) Find the inverse of matrix

$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix}$

Ans. : $A^{-1} = \frac{-1}{5}\begin{bmatrix} 0 & -1 & -1 \\ 5 & 5 & -5 \\ 5 & 2 & -3 \end{bmatrix}$

(iv) Find the inverse of matrix

$A = \begin{bmatrix} 4 & -5 & 1 \\ 3 & 1 & -2 \\ 1 & 4 & 1 \end{bmatrix}$

Ans. : $A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$

(v) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 2 & 3 \\ 1 & 4 & 1 \end{bmatrix}$

Ans. : $A^{-1} = \frac{1}{-26}\begin{bmatrix} -10 & 14 & -2 \\ 4 & -3 & -7 \\ -6 & -2 & 4 \end{bmatrix}$

1.16 Rank of Matrix

The matrix $A_{m \times n}$ is said to be of rank r if.

(i) At least one minor of order r is non-vanishi
and

(ii) Every minors of order (r + 1) vanishes.

It means that the rank of matrix A is the highest ord of its non-vanishing minor and it is denoted as ρ (A).

Note :

(i) If the order of matrix A is m × n, then ρ (A) ≤ min (m,n)

(ii) If there exist a non-vanishing minor of order r, the ρ (A) ≥ r

(iii) If all minors of order (r + 1) are zero, then ρ (A) ≤ r.

(iv) Elementary transformation does not alter rank c matrix.

(v) The rank of a product of two matrices is not excee the rank of either matrix.
i.e. ρ (AB) ≤ ρ (A) and ρ (AB) ≤ ρ (B).

(vi) Rank of null matrix is zero i.e. ρ (Z) = 0.

(vii) Rank of non null matrix is greater than or equal to 1

(viii) If A is non singular matrix of order n then ρ (A) = n.

(ix) If A is singular matrix of order n then ρ(A) < n.

(x) Rank of the identity (unit) matrix is equal to order c matrix.

(xi) The rank matrix A and its transpose is same.

(xii) The rank of matrix A and A^{-1} (if exist) is same

1.17 Echelon or Cannonical form of Matrix

The Echelon or Cannonical form of matrix A is r equivalent matrix C of rank r with the following propertie

(i) At least one element in each of the first r rows is n zero and the elements in the remaining rows are z
and

(ii) In the first r rows the first non-zero element in e row is 1 and it appears in the column right to the f non-zero element of the preceding row.

$$\text{e.g.:}\quad A = \begin{bmatrix} \cdot & 5 & 7 & -9 & 2 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

> **Note :** (i) To solve the system of linear equations we can use cannonical (or Echelon) form of a matrix. It is not necessary to obtain first non zero element in the first r rows as unity.
> (ii) The rank of matrix $A_{m \times n}$ is equal to number of non-zero rows in its echelon form.
> (iii) Rank of matrix A = Total numbers of rows – Number of rows containing all zeros.

1.18 Examples on to Find the Rank of Matrix by Echelon or Cannonical Form

Example 1.18.1

Find the rank of the matrix by Echelon form.

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solution : Given matrix is,

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$\text{Comparing with } \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Operate R_{12} (To obtain 1 at a_{11})

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Operate $R_2 - 2R_1$; $R_3 - 3R_1$; $R_4 - 6R_1$

(To obtain all zeros below at a_{11})

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

Operate $R_2 - R_3$ (To obtain 1 at a_{22})

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

Operate $R_3 - 4R_2$; $R_4 - 9R_2$

(To obtain all zeros below at a_{22})

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

Operate $R_4 - 2R_3$ (To obtain all zeros below at a_{33})

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(Note here below a_{33} are multiple of a_{33})

Operate $\frac{R_3}{33}$ (To obtain 1 at a_{33})

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is Echelon form of Matrix A

Rank of matrix A is,

$$\begin{pmatrix}\text{Rank of} \\ \text{matrix}\end{pmatrix} = \rho\,(A) = \begin{pmatrix}\text{Total} \\ \text{number of} \\ \text{rows}\end{pmatrix} - \begin{pmatrix}\text{Number of} \\ \text{rows containing} \\ \text{all zeros}\end{pmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \rightarrow \text{Row 1} \\ \rightarrow \text{Row 2} \\ \rightarrow \text{Row 3} \\ \rightarrow \text{Row 4 (Containing all zero's)} \end{array}$$

$= 4 - 1$

$\therefore$ **Rank of Matrix A = ρ (A) = 3** ✓ ...**Ans.**

Example 1.18.2

Find the rank of matrix by Echelon form.

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Solution : Given matrix is,

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\text{Comparing with } \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Operate $R_2 - 4R_1$ (To obtain all zeros at a_{11})

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

...(At a_{11} already 1)

Operate R_{24} (To obtain 1 at a_{22})

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 5 & -8 & 14 \end{bmatrix}$$

Operate $R_3 - 3R_2$; $R_4 - 5R_2$

(To obtain all zeros below at a_{22})

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -8 & 4 \end{bmatrix}$$

Operate $R_4 + 8R_3$ (To obtain all zeros below at a_{33})

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -12 \end{bmatrix} \quad \text{...(Already 1 at } a_{33}\text{)}$$

Operate $\frac{R_4}{-12}$ (To obtain 1 at a_{44})

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is echelon form of matrix A.

Rank of matrix

$$\begin{pmatrix}\text{Rank of} \\ \text{matrix}\end{pmatrix} = \rho\,(A) = \begin{pmatrix}\text{Total} \\ \text{number of} \\ \text{rows}\end{pmatrix} - \begin{pmatrix}\text{Number of} \\ \text{rows containing} \\ \text{all zeros}\end{pmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \rightarrow \text{Row 1} \\ \rightarrow \text{Row 2} \\ \rightarrow \text{Row 3} \\ \rightarrow \text{Row 4} \end{array}$$

$\rho\,(A) = 4 - 0 = 4$

$\therefore$ **Rank of matrix = ρ (A) = 4** ✓ ...**Ans.**

Example 1.18.3

Reduce the following matrix to echelon form and find its rank,

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

Solution : Given matrix is,

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

$$\text{Comparing with } \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Operate $R_2 - 2R_1$; $R_3 - 3R_1$; $R_4 - 6R_1$

(To obtain all zeros below at a_{11})

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix} \quad \text{...(Already 1 at } a_{11}\text{)}$$

Operate R_{23}

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

...(Need not to obtain 1 at a_{22} of all zeros below a_{22})

Operate $R_4 - R_2$ (To obtain all zeros below at a_{22})

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

Operate $R_4 - R_3$ (To obtain all zeros below at a_{33})

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operate $\frac{R_2}{-4}$; $\frac{R_3}{-3}$ (To obtain 1 at a_{22} and a_{33})

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -3/4 \\ 0 & 0 & 1 & -2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is Echelon form of the matrix A.

$$\begin{pmatrix}\text{Rank of}\\ \text{matrix}\end{pmatrix} = \rho(A) = \begin{pmatrix}\text{Total}\\ \text{number of}\\ \text{rows}\end{pmatrix} - \begin{pmatrix}\text{Number of}\\ \text{rows containing}\\ \text{all zeros}\end{pmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -3/4 \\ 0 & 0 & 1 & -2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \rightarrow \text{Row 1} \\ \rightarrow \text{Row 2} \\ \rightarrow \text{Row 3} \\ \rightarrow \text{Row 4 (Containing all zero's)} \end{array}$$

$\rho(A) = 4 - 1$

Rank of Matrix = ρ (A) = 3 ✓ ...Ans.

Note :

(I) If $A = \begin{bmatrix} a & b & c \\ a+m & b+m & c+m \\ a+n & b+n & c+n \end{bmatrix}$

where a, b, c not all zeros and not all equals. m, n are any integers, then $\rho(A) = 2$

Explanation:

Operate $R_2 - R_1$; $R_3 - R_1$

$$\sim \begin{bmatrix} a & b & c \\ m & m & m \\ n & n & n \end{bmatrix}$$

Operate $\frac{R_2}{m}$; $\frac{R_3}{n}$

$$\sim \begin{bmatrix} a & b & c \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Operate $R_3 - R_2$

$$\sim \begin{bmatrix} a & b & c \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Operate R_{12}

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ 0 & 0 & 0 \end{bmatrix}$$

Operate $R_2 - aR_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$

(II) In above if a = b = c then $\rho(A) = 1$

(III) If $A = \begin{bmatrix} a & b & c \\ ma & mb & mc \\ na & nb & nc \end{bmatrix}$ where m,n are any non zero integers and a, b, c not all equals. then $\rho(A) = 1$

Explanation :

Operate $\frac{R_2}{m}$; $\frac{R_3}{n}$

$$\sim \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$$

Operate $R_2 - R_1$; $R_3 - R_1$

$$\sim \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \therefore \rho(A) = 1$$

1.19 Normal Form of a Matrix

1.19.1 Definition

A matrix $A_{m \times n}$ of rank r can be reduced by using elementary row and column transformation in one of the following four forms :

$$[I_r],\ [I_r\ 0],\ \begin{bmatrix} I_r \\ 0 \end{bmatrix},\ \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

It is known as normal of a matrix A. Where I_r is a unit matrix of order r.

Note : To find echelon form of a matrix we can use only row transformations but to find normal form of a matrix we can use both row and column transformations.

1.19.2 Steps to Find Normal Form of a Matrix

(i) Make the elements a_{11} position equal to 1 by using row or column transformation.

(ii) Make all elements below a_{11} position equal to zero using I^{st} row and row transformation only.

(iii) Make all elements right of a_{11} position equal to all zero using I^{st} column and column transformation only.

(iv) Make the element a_{22} position equal to 1 using row or column transformation without using I^{st} row and I^{st} column.

(v) Use steps (ii) and (iii) for second row and second column.

(vi) If it is in normal form terminate this process otherwise make element a_{33} position equal to 1 using row or column transformation without using first two rows and first two columns.

(vii) Continue this process till the normal form is obtained.

1.20 Examples on to Find Rank of Matrix by Normal Form

Example 1.20.1

Reduce the following matrix to its normal form and hence find its rank.

$$\begin{bmatrix} 3 & 2 & 1 & 3 & 2 \\ 1 & 0 & 2 & 1 & 4 \\ 2 & 3 & 3 & 4 & 6 \\ 1 & 2 & 4 & 1 & 8 \end{bmatrix}$$

Solution :

To find normal form we can use row and column transformation.

Given matrix,

$$A = \begin{bmatrix} 3 & 2 & 1 & 3 & 2 \\ 1 & 0 & 2 & 1 & 4 \\ 2 & 3 & 3 & 4 & 6 \\ 1 & 2 & 4 & 1 & 8 \end{bmatrix}$$

Comparing with

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix}$$

Operate R_{12} (To obtain 1 at a_{11})

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 & 4 \\ 3 & 2 & 1 & 3 & 2 \\ 2 & 3 & 3 & 4 & 6 \\ 1 & 2 & 4 & 1 & 8 \end{bmatrix}$$

Operate $R_2 - 3R_1$, $R_3 - 2R_1$, $R_4 - R_1$

(To obtain all zeros below at a_{11})

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 & 2 \\ 0 & 2 & -5 & 0 & -10 \\ 0 & 3 & -1 & 2 & -2 \\ 0 & 2 & 2 & 0 & 4 \end{bmatrix}$$

Operate $C_3 - 2C_1$; $C_4 - C_1$; $C_5 - 4C_1$

(To obtain all zeros on R.H.S. of a_{11})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -5 & 0 & -10 \\ 0 & 3 & -1 & 2 & -2 \\ 0 & 2 & 2 & 0 & 4 \end{bmatrix}$$

Operate R_{23}

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 2 & -2 \\ 0 & 2 & -5 & 0 & -10 \\ 0 & 2 & 2 & 0 & 4 \end{bmatrix}$$

...(**Target :** Obtain 1 at a_{12})

Operate $R_2 - R_3$ (To obtain 1 at a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0. \\ 0 & 1 & 4 & 2 & 8 \\ 0 & 2 & -5 & 0 & -10 \\ 0 & 2 & 2 & 0 & 4 \end{bmatrix}$$

Operate $R_3 - 2R_2$; $R_4 - 2R_2$

(To obtain all zeros below at a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 4 & 2 & 8 \\ 0 & 0 & -13 & -4 & -26 \\ 0 & 0 & -6 & -4 & -12 \end{bmatrix}$$

Operate $C_3 - 4C_2$; $C_4 - 2C_2$; $C_5 - 8C_2$

(To obtain all zeros on R.H.S. of a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -13 & -4 & -26 \\ 0 & 0 & -6 & -4 & -12 \end{bmatrix}$$

Operate $R_3 - 2R_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & -2 \\ 0 & 0 & -6 & -4 & -12 \end{bmatrix}$$

...(**Target :** Obtain 1 at a_{33})

Operate, $-R_3$ and, $-R_4$ (To obtain 1 at a_{33})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 6 & 4 & 12 \end{bmatrix}$$

Operate $R_4 - 6R_3$ (To obtain all zeros below at a_{33})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 28 & 0 \end{bmatrix}$$

Operate $C_4 + 4C_3$; $C_5 - 2C_3$
(To obtain all zeros on R.H.S. of a_{33})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 28 & 0 \end{bmatrix}$$

Operate $\frac{R_4}{28}$ (To obtain 1 at a_{44})

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Unit matrix I_4 Null matrix

$$\sim \left[\begin{array}{c|c} I_4 & 0 \end{array}\right]$$

Which is a normal form of Matrix A,

Hence rank of matrix A is,

$\rho(A) = 4$ ✓ ...Ans.

Example 1.20.2

Reduce the following matrix to the normal form and hence find its rank.

$$\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & -4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$$

Solution : To find normal form we can use both (Row and Column) transformations.

Given matrix is,

$$A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & -4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$$

Comparing with $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

Operate R_{12} (To obtain 1 at a_{11})

$$\sim \begin{bmatrix} 1 & -4 & -2 & 1 \\ 2 & -1 & 3 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$$

Operate $R_2 - 2R_1$; $R_3 - 5R_1$
(To obtain all zeros below at a_{11})

$$\sim \begin{bmatrix} 1 & -4 & -2 & 1 \\ 0 & 7 & 7 & -1 \\ 0 & 22 & 14 & -2 \end{bmatrix}$$

Operate $C_2 + 4C_1$; $C_3 + 2C_1$; $C_4 - C_1$
(To obtain all zeros on R.H.S. of a_{11})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & 7 & -1 \\ 0 & 22 & 14 & -2 \end{bmatrix}$$

Operate C_{24}

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 7 & 7 \\ 0 & -2 & 14 & 22 \end{bmatrix}$$

(**Target :** Obtain 1 at a_{22})

Operate $-C_2$ (To obtain 1 at a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 7 \\ 0 & 2 & 14 & 22 \end{bmatrix}$$

Operate $R_3 - 2R_2$ (To obtain all zeros below at a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Operate $C_3 - 7C_2$; $C_4 - 7C_2$
(To obtain all zeros on R.H.S. of a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Operate $\left(\frac{1}{8}R_3\right)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(**Target :** Obtain 1 at a_{33})

Operate C_{34} (To obtain 1 at a_{33})

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Unit matrix I_3 Null matrix

$$\sim [I_3 \ 0]$$

Which is a normal form of matrix A.

$\therefore$ Rank of matrix $= \rho(A) = 3$ ✓ ...Ans.

Example 1.20.3

Reduce the following matrix to its normal form and hence find its rank.

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Solution : To find normal form we can use both (Row and Column) transformations.

Given matrix is,

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Comparing with $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$

Operate R_{12},

$$\sim \begin{bmatrix} 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

(**Target :** Obtain 1 at a_{11})

Operate $R_1 - R_2$ (To obtain 1 at a_{11})

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Operate $R_2 - 2R_1$; $R_3 - 4R_1$; $R_4 - 9R_1$
(To obtain all zeros below at a_{11})

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Operate $C_2 - C_1$; $C_3 - C_1$; $C_4 - C_1$
(To obtain all zeros on R.H.S. of a_{11})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Operate $R_3 - R_2$; $R_4 - R_2$ (To obtain all zeros below at a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(**Note :** Already 1 at a_{22})

$C_3 - 2C_2$; $C_4 - 3C_2$ (To obtain all zeros on R.H.S. of a_{22})

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Unit matrix I_2 Null matrix

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Null matrix Null matrix

Which is a normal form of matrix A.

Hence, Rank of matrix A is,

$\rho(A) = 2$ ✓ ...Ans.

Example 1.20.4

Reduce the following matrix to its normal form and find its rank.

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solution : To find normal form we can use both (Row and Column) transformations.

Given matrix is,

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Comparing with $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$

Operate R_{12} (To obtain 1 at a_{11})

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Operate $R_2 - 2R_1$; $R_3 - 3R_1$; $R_4 - 6R_1$

(To obtain all zeros below at a_{11})

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

Operate $C_2 + C_1$; $C_3 + 2C_1$; $C_4 + 4C$

(To obtain all zeros on R.H.S. of a_{11})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

~ Operate $R_2 - R_3$ (To obtain 1 at a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

Operate $R_3 - 4R_2$; $R_4 - 9R_2$

(To obtain all zeros below at a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

Operate $C_3 + 6C_2$; $C_4 + 3C_2$

(To obtain all zeros on R.H.S. of a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

Operate $\frac{C_2}{33}$ and $\frac{C_4}{22}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

(**Target :** Obtain 1 at a_{33})

Operate $R_2 - 2R_3$ (To obtain 1 at a_{33})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operate $C_4 - C_3$ (To obtain all zeros on R.H.S. of a_{33})

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

Unit matrix I_3

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Null matrix Null matrix Null matrix

Which is normal form of a matrix A,

Hence rank is, $\rho(A) = 3$ ✓ ...Ans.

Example 1.20.5 Dr. BATU : Nov. 2016

Reducing the following matrix to normal form find the rank of the given matrix.

$$\begin{bmatrix} 4 & 2 & -1 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -2 & 0 \end{bmatrix}$$

Solution :

To find normal form we can use both (Row and Column) transformations

Given matrix is,

$$A = \begin{bmatrix} 4 & 2 & -1 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -2 & 0 \end{bmatrix}$$

Comparing with $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

Operate R_{12} (To obtain 1 at a_{11})

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$$

Operate $R_2 - 4R_1$; $R_3 - 2R_1$

(To obtain all zeros below at a_{11})

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 6 & -9 & -2 \\ 0 & 4 & -6 & -2 \end{bmatrix}$$

Operate $C_2 + C_1$; $C_3 - 2C_1$; $C_4 - C_1$

(To obtain all zeros on R.H.S. of a_{11})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & -9 & -2 \\ 0 & 4 & -6 & -2 \end{bmatrix}$$

Operate $\frac{1}{2}C_2$; $\frac{-1}{3}C_3$; $\frac{-1}{2}C_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & 1 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

(**Target :** Obtain 1 at a_{22})

Operate $R_2 - R_3$ (To obtain 1 at a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

Operate $R_3 - 2R_2$ (To obtain all zeros below at a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $C_3 - C_2$

(To obtain all zeros on R.H.S. of a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate C_{34} (To obtain 1 at a_{33})

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Unit matrix I_3 Null matrix

$$\sim [I_3 \mid 0]$$

Which is a normal form of matrix A.

$\therefore$ Rank of the matrix is,

$\rho(A) = 3$ ✓ ...Ans.

Example 1.20.6

Reduce the following matrix to normal form and hence find its rank.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

Solution :

To find normal form we can use both (Row and Column) transformations.

Given matrix is,

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

Comparing with $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

Operate $R_2 + 2R_1$; $R_3 - R_1$ (To obtain all zeros below at a_{11})

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

(**Note:** Already 1 at a_{11})

Operate $C_2 - 2C_1$; $C_3 - C_1$

(To obtain all zeros on R.H.S. of a_{11})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

Operate R_{23}

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & -8 \\ 0 & 8 & 5 & 0 \end{bmatrix}$$

(**Target :** Obtain 1 at a_{22})

Operate $\frac{C_2}{-2}$ (To obtain 1 at a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -8 \\ 0 & -4 & 5 & 0 \end{bmatrix}$$

Operate $R_3 + 4R_1$ (To obtain all zeros below at a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -8 \\ 0 & 0 & 9 & -32 \end{bmatrix}$$

Operate $C_3 - C_2$; $C_4 + 8C_2$

(To obtain all zeros on R.H.S. of a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & -32 \end{bmatrix}$$

Operate $\frac{C_3}{9}$; $\frac{C_4}{-32}$ (To obtain 1 at a_{33})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Operate $C_4 - C_3$ (To obtain all zeros on R.H.S. of a_{33})

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Unit matrix I_3 Null matrix

$\sim [I_3 \; 0]$

Which is a normal form of matrix A.

$\therefore$ **Rank of matrix is,** $\rho(A) = 3$ ✓ **...Ans.**

Example 1.20.7 **Dr. BATU : May 2015**

Reduce into normal form and find its rank.

$$\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

Solution :

To find normal form we can use both (Row and Column) transformations.

Consider,

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

Comparing with $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$

Operate C_{12} (To obtain 1 at a_{11})

$$\sim \begin{bmatrix} 1 & 6 & 3 & 8 \\ 2 & 4 & 6 & -1 \\ 3 & 10 & 9 & 7 \\ 4 & 16 & 12 & 15 \end{bmatrix}$$

Operate $R_2 - 2R_1$; $R_3 - 3R_1$; $R_4 - 4R_1$

(To obtain all zeros below at a_{11})

$$\sim \begin{bmatrix} 1 & 6 & 3 & 8 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \end{bmatrix}$$

Operate $C_2 - 6C_1$; $C_3 - 3C_1$; $C_4 - 8C_1$

(To obtain all zeros on R.H.S. of a_{11})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \end{bmatrix}$$

Operate $\frac{C_2}{-8}$; $\frac{C_4}{-17}$ (To obtain 1 at a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Operate $R_3 - R_2$; $R_4 - R_2$ (To obtain all zeros below at a_{22})

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operate $C_4 - C_2$ (To obtain all zeros on R.H.S. of a_{22})

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Unit matrix I_2 Null matrix

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Null matrix   Null matrix

$$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Which is a normal form of matrix A.

$\therefore$ $\rho(A) = 2$ ✓ **...Ans.**

Exercise 2

Ex. 1 Find the rank of the matrix by echelon (cannonical) form.

(i) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ **Ans. : $\rho(A) = 2$**

(ii) $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$ **Ans. : $\rho(A) = 2$**

(iii) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$ **Ans. : $\rho(A) = 2$**

(iv) $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 8 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ **Ans. : $\rho(A) = 2$**

(v) $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & -6 \end{bmatrix}$ **Ans. : $\rho(A) = 4$**

(vi) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ **Ans. : $\rho(A) = 3$**

(vii) $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -3 \end{bmatrix}$ **Ans. : $\rho(A) = 4$**

(viii) $\begin{bmatrix} 3 & -2 & -0 & -1 \\ 0 & 2 & 2 & 7 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ **Ans. : $\rho(A) = 4$**

(ix) $\begin{bmatrix} 1 & 2 & 0 & 5 \\ 2 & 4 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 9 & 8 & 0 & 1 \end{bmatrix}$ **Ans. : $\rho(A) = 4$**

(x) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 0 & 2 & 1 \end{bmatrix}$ **Ans. : $\rho(A) = 3$**

(xi) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ **Ans. : $\rho(A) = 2$**

(xii) $\begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \\ 6 & 13 & 21 & 20 \end{bmatrix}$ **Ans. : $\rho(A) = 2$**

Ex. 2 Reduce into normal form and find its rank.

1. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ **Ans. : $\rho(A) = 3$**

2. $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$ **Ans. : $\rho(A) = 2$**

3. $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ **Ans. : $\rho(A) = 2$**

4. $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ **Ans. : $\rho(A) = 3$**

5. $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ **Ans. : $\rho(A) = 3$**

6. $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 2 & -4 & 2 & 4 \\ 1 & -1 & 0 & 3 \end{bmatrix}$ **Ans. : $\rho(A) = 3$**

7. $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ **Ans. : $\rho(A) = 2$**

8. $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ **Ans. : $\rho(A) = 3$**

9. $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$ **Ans. : $\rho(A) = 4$**

10. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ **Ans. : $\rho(A) = 3$**

11. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$ **Ans. : $\rho(A) = 2$**

12. $\begin{bmatrix} 3 & 2 & 5 & 7 & 12 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$

Ans. : $\rho(A) = 2$

1.21 PAQ Form

Steps to find two non-singular matrices P and Q such that PAQ is in normal form.

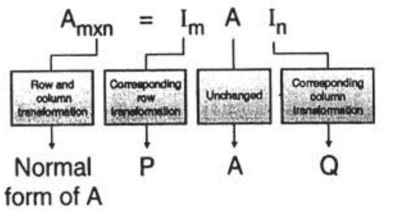
(i) If A is matrix of order $m \times n$.

(ii) Consider, $A_{m \times n} = I_m \, A \, I_n$. Where I_m (pre-matrix) and I_n (post-matrix) are the unit matrices of order m and n respectively.

(iii) Then obtain normal form of L.H.S. matrix A by operating row and column transformations, same transformations operate on R.H.S. matrices stepwise as only row transformations operate on pre-matrix I_m and keep post matrix I_n as it is and only column transformations operate on post matrix I_n and keep pre-matrix I_m as it is.

(iv) Lastly we get L.H.S. is in normal form i.e. R.H.S. PAQ also in normal form which gives matrices P and Q.

$$A_{mxn} = I_m \; A \; I_n$$



> **Note :** (i) If A non-singular is square matrix, then $A^{-1} = QP$.
> (ii) For any matrix A, P and Q are not unique.

1.22 Examples on PAQ Form :

Example 1.22.1

Find non-singular matrices P and Q such that PAQ is in normal form hence find the rank of A where :

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix}$$

Solution :

Step I : Given Matrix is,

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix}$$

The order of matrix A is 3×4

Step II : Consider,

$$A_{3 \times 4} = I_3 \, A \, I_4$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \underbrace{\overset{\text{Unit matrix } I_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}}_{\text{Pre matrix}} A \underbrace{\overset{\text{Unit matrix } I_4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}}_{\text{Post matrix}}$$

Target : Obtain normal form of L.H.S. matrix

Operate, $R_2 - 3R_1$ and $R_3 - 2R_1$

(To obtain all zeros below a_{11} of L.H.S. Matrix)

$$\begin{bmatrix} 1 & -4 & 0 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Same row transformations operate on prematrix of R.H.S.)

Operate $C_2 - C_1$; $C_3 - C_1$ and $C_4 - 2C_1$

(To obtain all zeros on R.H.S. of a_{11} of L.H.S. matrix)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Same column transformations operate on post matrix of R.H.S.)

Operate R_{23} (Target : Obtain 1 at a_{22})

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & -10 \\ 0 & -6 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Same row transformations operate on prematrix of R.H.S.)

Operate $-R_2$ and $-R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 6 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 3 & -1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Same row transformations operate on prematrix of R.H.S.

Operate $R_3 - 6R_2$)

(To obtain all zeros below a_{22} of L.H.S. matrix)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & -28 & -56 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ -9 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Same row transformations operate on prematrix of R.H.S.)

Operate, $C_3 - 5C_2$ and $C_4 - 10\,C_2$

(To obtain all zeros on R.H.S. of a_{22} of L.H.S. matrix)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -28 & -56 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ -9 & -1 & 6 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 8 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Same column transformations operate on post matrix of R.H.S.)

Operate, $\frac{R_3}{-28}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ \frac{9}{28} & \frac{1}{28} & \frac{3}{14} \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 8 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Same row transformations operate on prematrix of R.H.S.)

Operate, $C_4 - 2C_3$

(To obtain all zeros on R.H.S. of a_{33} of L.H.S. matrix)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ \frac{9}{28} & \frac{1}{28} & \frac{3}{14} \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Same column transformations operate on post matrix of R.H.S.)

$$\left[\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{\text{Unit matrix } I_3} \Bigg| \underbrace{\begin{array}{c} 0 \\ 0 \\ 0 \end{array}}_{\text{Null matrix}}\right] = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ \frac{9}{28} & \frac{1}{28} & \frac{3}{14} \end{bmatrix}}_{P} A \underbrace{\begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Q}$$

$$\underbrace{[\,I_3 : 0\,]}_{\text{Normal form}} = PAQ$$

i.e. PAQ is in normal form.

Where, $\quad P = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ \frac{9}{28} & \frac{1}{28} & \frac{3}{14} \end{bmatrix}$

and $\quad Q = \begin{bmatrix} 1 & -1 & 4 & 8 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore$ **Rank of A = $\rho(A) = 3$** ✓ **...Ans.**

Example 1.22.2

Find non-singular matrices P and Q such that PAQ is in normal form where,

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

Solution : Order of given matrix A is 3×4

$\therefore$ Consider, $A = I_3 \, A \, I_4$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 2 & 0 & 2 & -8 \end{bmatrix} = \underbrace{\overset{\text{Unit matrix } I_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}}_{\text{Pre matrix}} A \underbrace{\overset{\text{Unit matrix } I_4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}}_{\text{Post matrix}}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Target : Obtain normal form of L.H.S. matrix

Operate $R_2 + 2R_1$; $R_3 - R_1$

(To obtain all zeros below a_{11} of L.H.S. matrix)

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Same row transformations operate on prematrix of R.H.S.)

Operate $C_2 - 2C_1$; $C_3 - C_1$

(To obtain all zeros on R.H.S. of a_{11} of L.H.S. matrix)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Same column transformations operate on post matrix of R.H.S.)

Operate R_{23} (Target : Obtain 1 at a_{22} of L.H.S. matrix)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & -8 \\ 0 & 8 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Same row transformations operate on prematrix of R.H.S.)

Operate C_{23}

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -8 \\ 0 & 5 & 8 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Same column transformations operate on post matrix of R.H.S.)

Operate $R_3 - 5R_2$

(To obtain all zeros below a_{22} of L.H.S. matrix)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -8 \\ 0 & 0 & 18 & 40 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 7 & 1 & -5 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Same row transformations operate on prematrix of R.H.S.)

Operate $C_3 + 2C_2$; $C_4 + 8C_2$

(To obtain all zeros on R.H.S. of a_{22} of L.H.S. matrix)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 18 & 40 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 7 & 1 & -5 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4 & -8 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Same column transformations operate on post matrix of R.H.S.)

Operate $\frac{R_3}{18}$

(To obtain 1 at a_{33} of L.H.S. matrix)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 20/9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ \frac{7}{18} & \frac{1}{18} & \frac{-5}{18} \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4 & -8 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Same row transformations operate on prematrix of R.H.S.)

Operate $C_4 - \frac{20}{9} C_3$

(To obtain a_{11} zeros R.H.S. of a_{33} of L.H.S. matrix)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ \frac{7}{18} & \frac{1}{18} & \frac{-5}{18} \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(Same column transformations operate on post matrix of R.H.S.)

$$\left[\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{\text{Unit matrix } I_3} \Bigg| \underbrace{\begin{array}{c} 0 \\ 0 \\ 0 \end{array}}_{\text{Null matrix}}\right] = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ \frac{7}{18} & \frac{1}{18} & \frac{-5}{18} \end{bmatrix}}_{P} A \underbrace{\begin{bmatrix} 1 & -1 & -4 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}_{Q}$$

$$\underbrace{[I_3 : 0]}_{\text{Normal form}} = PAQ$$

i.e. PAQ is in normal form

$$\text{Where,}\quad P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ \frac{7}{18} & \frac{1}{18} & \frac{-5}{18} \end{bmatrix}$$

$$\text{and}\quad Q = \begin{bmatrix} 1 & -1 & -4 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ **Rank of matrix A = $\rho(A) = 3$** ✓ **...Ans.**

Example 1.22.3

Find non-singular matrices P and Q such that PAQ is in normal form where $A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$ and hence find A^{-1}

Solution : Order of given matrix A is 3×3

$\therefore$ Consider, $A = I_3 \, A \, I_3$

$$\begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} = \underbrace{\overset{\text{Unit matrix } I_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}}_{\text{Pre matrix}} A \underbrace{\overset{\text{Unit matrix } I_4}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}}_{\text{Post matrix}}$$

$$\begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Target : Obtain normal form of L.H.S. matrix

Operate R_{13} (To obtain 1 at a_{11} of L.H.S. matrix)

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Same row transformations operate on prematrix of R.H.S.)

Operate $R_2 - 3R_1$; $R_3 - 2R_1$

(To obtain a_{11} zeros below a_{11} of L.H.S. matrix)

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Same row transformations operate on prematrix of R.H.S.)

Operate $C_2 - 2C_1$; $C_3 + C_1$ (To obtain all zeros on R.H.S of a_{11} of L.H.S. matrix)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Same column transformations operate on post matrix of R.H.S.)

Operate $R_2 - R_3$ (**Target :** Obtain 1 at a_{22})

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Same row transformations operate on prematrix of R.H.S.)

Operate $(-C_2)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Same column transformations operate on post matrix of R.H.S.)

Operate $R_3 - 6R_2$

(To obtain all zeros below a_{22} of L.H.S. matrix)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 7 & -6 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $\frac{R_3}{5}$ (To obtain 1 at a_{33} on L.H.S. matrix)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ \frac{7}{5} & \frac{-6}{5} & \frac{4}{5} \end{bmatrix} A \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Same row transformations operate on prematrix of R.H.S.)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & -1 \\ \frac{7}{5} & \frac{-6}{5} & \frac{4}{5} \end{bmatrix}}_{P} A \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q}$$

$\underbrace{[I_3 : 0]}_{\text{Normal form}}$ = PAQ is in normal form.

$$\text{Where}\quad P = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ \frac{7}{5} & \frac{4}{5} & \frac{4}{5} \end{bmatrix}$$

$$\text{and}\quad Q = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, we have

$$A^{-1} = QP$$

$$A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ \frac{7}{5} & \frac{-6}{5} & \frac{4}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_2 \end{matrix} \quad \begin{array}{ccc} \begin{bmatrix} 0 & 2 & 1 \\ -1 & 1 & -1 \\ \frac{7}{5} & \frac{-6}{5} & \frac{4}{5} \end{bmatrix} \\ \uparrow \;\; \uparrow \;\; \uparrow \\ C_1 \;\; C_2 \;\; C_3 \end{array}$$

$$\begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix} \quad \text{...(Standard form)}$$

$$R_1c_1 = (1 \times 0) + (2 \times (-1)) + \left(1 \times \frac{7}{5}\right) = 0 - 2 + \frac{7}{5} = \frac{-3}{5}$$

$$R_1c_2 = (0 \times 0) + (2 \times 1) + \left(1 \times \left(\frac{-6}{5}\right)\right) = 0 + 2 - \frac{6}{5} = \frac{4}{5}$$

$$R_1c_3 = (1 \times 1) + (2 \times (-1)) + \left(1 \times \frac{4}{5}\right) = 1 - 2 + \frac{4}{5} = -\frac{1}{5}$$

$$R_1c_1 = (0 \times 0) + (-1 \times (-1)) + \left(0 \times \frac{7}{5}\right) = 0 + 1 + 0 = 1$$

$$R_1c_2 = (0 \times 0) + (-1 \times 1) + \left(0 \times \left(\frac{-6}{5}\right)\right) = 0 - 1 + 0 = -1$$

$$R_1c_3 = (0 \times 1) + (-1 \times (-1)) + \left(0 \times \frac{4}{5}\right) = 0 + 1 + 0 = 1$$

$$R_3c_1 = (0 \times 0) + (0 \times (-1)) + \left(1 \times \frac{7}{5}\right) = 0 + 0 + \frac{7}{5} = \frac{7}{5}$$

$$R_3c_2 = (0 \times 0) + (0 \times 1) + \left(1 \times \left(\frac{-6}{5}\right)\right) = 0 + 0 - \frac{6}{5} = -\frac{6}{5}$$

$$R_3c_3 = (0 \times 1) + (0 \times (-1)) + \left(1 \times \frac{4}{5}\right) = 0 + 0 + \frac{4}{5} = \frac{4}{5}$$

$$\therefore\quad A^{-1} = \begin{bmatrix} \frac{-3}{5} & \frac{4}{5} & \frac{-1}{5} \\ 1 & -1 & 1 \\ \frac{7}{5} & \frac{-6}{5} & \frac{4}{5} \end{bmatrix} ✓ \quad \text{...Ans.}$$

Exercise 3

Find non-singular matrices P and Q such that PAQ is in normal form, and find it's rank where,

(i) $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

and hence find A^{-1} $\left(\textbf{Ans. : } A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}\right)$

(iv) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$ **(Ans. : ρ (A) = 2)**

(vi) $\begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

(viii) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ **(Ans. : ρ (A) = 3)**

(ix) $\begin{bmatrix} 2 & 6 & 5 \\ 2 & 5 & 4 \\ 5 & 16 & 13 \end{bmatrix}$ **(Ans. : ρ (A) = 3)**

(x) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

(xi) $\begin{bmatrix} 2 & 0 & -1 & 1 \\ 4 & -1 & -2 & 4 \\ 3 & 2 & 3 & -2 \\ 6 & 3 & 0 & -5 \end{bmatrix}$ **(Ans. : ρ (A) = 4)**

1.23 University Questions and Answers

→ May 18

Q. 1 Find the rank of the matrix $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ by reducing it to normal form. **(6 Marks)**

Ans. :

Here $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

(by $R_2 - R_1, R_3 - 3R_1$)

$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix}$

(by $C_2 + C_1, C_3 + C_1$)

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix}$

$\frac{R_2}{2}, \frac{R_3}{2}$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

$R_3 - R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$C_2 - C_3$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Which is of the normal form $[I_2]$

Hence $\rho(A) = 2$.

Chapter Ends...

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CHAPTER 2

UNIT I

System of Linear Equations

Syllabus

Consistency of non- homogeneous and homogeneous system of linear equations

2.1 Introduction

The solution of simultaneous linear equations is a task frequently occurring in engineering. In engineering the analysis gives the simultaneous equations for that we want to study two things :

(a) how to represent large systems of linear equations

(b) how to find the solution of such equations.

It is found that knowledge of the theory of matrices is an essential mathematical tool in this area.

2.2 System of Linear Equation

Consider system of m equations in n unknowns

$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \text{--------} + a_{1n} x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \text{--------} + a_{2n} x_n = b_2$

$a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + \text{--------} + a_{mn} x_n = b_m$

This system can be written as in matrix form AX = B

Where, $A = \begin{bmatrix} a_{11} & a_{12} & -- & -- & a_{1n} \\ a_{21} & a_{22} & -- & -- & a_{2n} \\ -- & -- & -- & -- & -- \\ -- & -- & -- & -- & -- \\ a_{m1} & a_{m2} & -- & -- & a_{mn} \end{bmatrix}$,

This Matrix is known as matrix of coefficient or coefficient matrix

$X = \begin{bmatrix} x_1 \\ x_2 \\ -- \\ -- \\ -- \\ x_n \end{bmatrix}$ Matrix of unknown variables

and $B = \begin{bmatrix} b_1 \\ b_2 \\ -- \\ -- \\ -- \\ b_m \end{bmatrix}$ Matrix of R.H.S. constants

Matrix 'A' is called coefficient matrix.

The matrix [A | B] is called augmented matrix.

$[A \mid B] = \left[\begin{array}{cccccc|c} a_{11} & a_{12} & -- & -- & a_{1n} & & b_1 \\ a_{21} & a_{22} & -- & -- & a_{2n} & & b_2 \\ -- & -- & -- & -- & -- & & -- \\ -- & -- & -- & -- & -- & & -- \\ a_{m1} & a_{m2} & -- & -- & a_{mn} & & b_m \end{array}\right]$ is augmented matrix

AX = B, B = Z (Z - null matrix) then system of equation is known as **system of homogeneous equations**.

If AX ≠ Z **(null matrix)**, (i.e. B ≠ Z) system of equation is known as **system of non-homogeneous equations**.

If the system has solution (may be unique or infinite many), it is said to be **consistent**. Otherwise system is **inconsistent**.

2.2.1 Rouche's Theorem

The system AX = B is consistent if rank of the coefficient matrix A and augmented matrix [A | B] are same ;

i.e. $\rho(A) = \rho(A|B)$,

If $\rho(A) \neq \rho(A|B)$ then system is inconsistent.

2.2.2 Non-Homogeneous System of Equations

Consider the system of equations, AX = B

If $\rho(A) \neq \rho(A|B)$, the system is inconsistent i.e. system has no solution.

If $\rho(A) = \rho(A|B) = r$, the system is consistent and there is one of the following case :

(i) If r = n, [n – number of unknowns] the system has unique solution.

(If m = n and $|A| \neq 0$ then this unique solution is $X = A^{-1}B$). OR

(ii) If r < n, the system have infinite many solutions and r unknowns can be expressed as linear combination of remaining (n – r) unknowns.

Note :

(i) For m = n, if $|A| \neq 0$ (i.e. A is nonsingular matrix) then the system possess unique solution.

(ii) If $|A| = 0$ (i.e. A is singular matrix) then the non homogeneous system may or may not be consistent. If consistent then it has infinitely many solution.

2.2.3 Homogeneous System of Equations

Consider the homogeneous system of equations, AX = Z.

Since the $\rho(A) = \rho(A|Z)$, the system is always consistent i.e. the system have always a solution, this solution is X = 0, i.e. $x_1 = x_2 = \cdots = x_n = 0$, is called trivial solution.

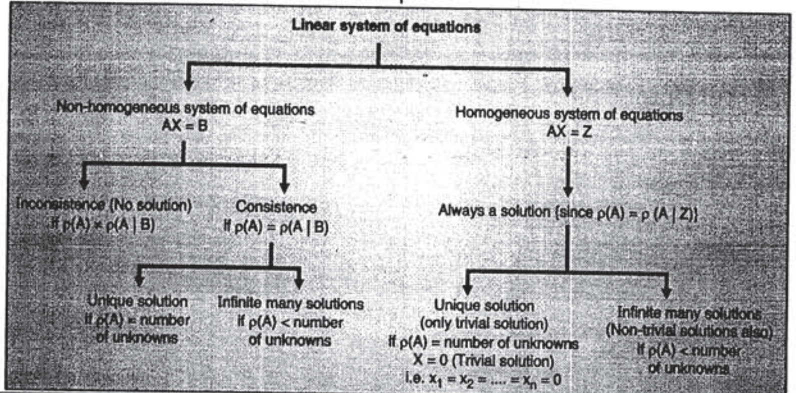
If $\rho(A) = \rho(A|Z) = r$, then there are following cases,

Case I :

If r = n the system has unique solution which is trivial solution. OR

Case II :

If r < n the system have infinite many solutions and r unknowns can be expressed as a linear combination of (n – r) remaining unknowns.



Note :

(i) In Homogeneous system if there are three equations and three unknowns, then $\rho(A) = 3$ if $|A| \neq 0$ and $\rho(A) < 3$ if $|A| = 0$

(ii) $a^3 + b^3 + c^3 - 3abc = 0 \Rightarrow (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc) = 0$

(iii) If there are n unknown and n equations, in the system of linear equations AX = B with $|A| \neq 0$ then $X = A^{-1}B$.

(iv) In the system of linear equations AX = B with n unknowns and n equations if $|A| = 0$ and $\rho(A) = \rho(A|B)$ then there are infinite many solutions.

2.2.4 Working Rule to Find Consistency of System

Non Homogeneous	Homogeneous
Write the given system of equations, in matrix form as, $AX = B$ Consider augmented matrix [A | B] Find the rank by using only row transformation i.e. by echelon form. In this system there is one of the following case	Write the given system of equations, in matrix form as $AX = Z$ Consider augmented matrix [A | Z] Find the rank by using only row transformation i.e. by echelon form. $\rho(A) = \rho[A\|Z] = r$
(1) If $\rho(A) \neq \rho[A\|B]$ then there is no solution i.e. system is inconsistence	(1) If r = n (n = number of unknowns) then there is a unique solution $x_1 = x_2 = x_3 = x_4 = \ldots = x_n = 0$
(2) If $\rho(A) = \rho[A\|B] = r$ and • r = n (n = number of unknowns) then there is a unique solution • r < n then the system have infinitely many solutions and r unknowns can be expressed as linear combination of remaining (n-r) unknowns	(2) If r < n, the system have infinitely many solutions and r unknowns can be expressed as linear combination of remaining (n-r) unknowns

Note : If number of equations are equal to number of unknowns and $|A| \neq 0$ the solution of the system we can find by $X = A^{-1}B$.

2.3 Examples on Linear System of Equations

Type I : Non-Homogeneous Systems

2.3.1 Examples on Non-Homogeneous Linear Systems

Example 2.3.1 :

Solve the equations using matrix method

x + 3y + 2z = 6, 3x – 2y + 5z = 5, 2x – 3y + 6z = 7

Solution : Given equations are,

$x + 3y + 2z = 6 ;\ 3x - 2y + 5z = 5,$

$2x - 3y + 6z = 7$

These we can write in matrix form as,

$$AX = B$$

Where,

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & -3 & 6 \end{bmatrix}$$

Compare with $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ ...(1)

known as coefficient matrix coefficients of x, y, z.

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{...(Matrix of unknown variables)}$$

and $$B = \begin{bmatrix} 6 \\ 5 \\ 7 \end{bmatrix} \quad \text{...(Matrix of R.H.S. constants)}$$

As, $AX = B \qquad \therefore X = A^{-1}B$ ...(2)

We know, $A^{-1} = \frac{1}{|A|} \text{Adj } A$ ...(3)

Now, $|A| = \begin{vmatrix} 1 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & -3 & 6 \end{vmatrix}$

Here order of determinant is 3

$$A = \begin{vmatrix} 1 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & -3 & 6 \end{vmatrix}$$

= 1 × [Minor of 1] – 3 × [Minor of 3] + 2 × [Minor of 2]

$$= 1 \begin{vmatrix} -2 & 5 \\ -3 & 6 \end{vmatrix} - 3 \begin{vmatrix} 3 & 5 \\ 2 & 6 \end{vmatrix} - 2 \begin{vmatrix} 3 & -2 \\ 2 & -3 \end{vmatrix}$$

$$= 1[\underbrace{(-2)(6)}_{-12} - \underbrace{(-3)(5)}_{-15}] - 3\,[\underbrace{(3)(6)}_{18} - \underbrace{(2)(5)}_{10}]$$

$$+ 2\,[\underbrace{(3)(-3)}_{-9} - \underbrace{(2)(-2)}_{-4}]$$

$$= 1\,\underbrace{[-12+15]}_{3} - 3\,\underbrace{[18-10]}_{8} + (2)\,\underbrace{[-9+4]}_{-5}$$

$$= 1\,(3) - 3\,(8) + 2\,(-5) = 3 - 24 - 10 = -31$$

$|A| = -31$ ...(4)

i.e. $|A| \neq 0 \quad \therefore A^{-1}$ exist

To find Adj. A

Minors of elements	Cofactors of elements
$a_{11} (= 1) = \begin{vmatrix} -2 & 5 \\ -3 & 6 \end{vmatrix}$ $= (-2)(6) - (-3)(5)$ $= -12 + 15 = 3 = M_{11}$	$a_{11} = (-1)^{1+1} M_{11}$ $= (1)(3)$ $= \mathbf{3 = C_{11}}$
$a_{12} (= 3) = \begin{vmatrix} 3 & 5 \\ 2 & 6 \end{vmatrix}$ $= (3)(6) - (2)(5)$ $= 18 - 10 = 8 = M_{12}$	$a_{12} = (-1)1 + 2\,M_{12}$ $= (-1)(8)$ $= \mathbf{-8 = C_{12}}$
$a_{13} (= 2) = \begin{vmatrix} 3 & -2 \\ 2 & -3 \end{vmatrix}$ $= (3)(-3) - (2)(-2)$ $= -9 + 4 = -5 = M_{13}$	$a_{13} = (-1)^{1+3} M_{13} = (1)(5)$ $= \mathbf{-5 = C_{13}}$
$a_{21} (= 3) = \begin{vmatrix} 3 & 2 \\ -3 & 6 \end{vmatrix}$ $= (3)(6) - (-3)(2)$ $= 18 + 6 = 24 = M_{21}$	$a_{21} = (-1)^{2+1} M_{21} = (-1)(24)$ $= \mathbf{-24 = C_{21}}$
$a_{22} (= -2) = \begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix}$ $= (1)(6) - (2)(2)$ $= 6 - 4 = 2 = M_{22}$	$a_{22} = (-1)^{2+2} M_{22} = (1)(2)$ $= \mathbf{2 = C_{22}}$
$a_{23} (= 5) = \begin{vmatrix} 1 & 3 \\ 2 & -3 \end{vmatrix}$ $= (1)(-3) - (2)(3)$ $= -3 - 6 = -9 = M_{23}$	$a_{23} = (-1)^{2+3} M_{23} = (-1)(-9)$ $= \mathbf{9 = C_{23}}$
$a_{31} (= 2) = \begin{vmatrix} 3 & 2 \\ -2 & 5 \end{vmatrix}$ $= (3)(5) - (-2)(2)$ $= 15 + 4 = 19 = M_{31}$	$a_{31} = (-1)^{3+1} M_{31} = (1)(19)$ $= \mathbf{19 = C_{31}}$
$a_{32} (= -3) = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$ $= (1)(5) - (3)(2)$ $= 5 - 6 = -1 = M_{32}$	$a_{32} = (-1)^{3+2} M_{32}$ $= (-1)(-1)$ $= \mathbf{1 = C_{32}}$
$a_{33} (= 6) = \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix}$ $= (1)(-2) - (3)(3)$ $= -2 - 9 = -11 = M_{33}$	$a_{33} = (-1)^{3+3} M_{33} = (1)(-11)$ $= \mathbf{-11 = C_{33}}$

By these values,

$$\text{Matrix of cofactors} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -8 & -5 \\ -24 & 2 & 9 \\ 19 & 1 & 11 \end{bmatrix} \quad \text{...(5)}$$

Adjoint of matrix A = Transpose of matrix of cofactors

$$= \begin{bmatrix} 3 & -24 & 19 \\ -8 & 2 & 1 \\ -5 & 9 & -11 \end{bmatrix}$$

...[Interchange rows and columns of matrix (5)]

$$\text{Adj A} = \begin{bmatrix} 3 & -24 & 19 \\ -8 & 2 & 1 \\ -5 & 9 & -11 \end{bmatrix} \quad \text{...(6)}$$

Substitute values from Equations (4) and (6) in Equation (3)

$$\therefore \quad A^{-1} = \frac{1}{-31} \begin{bmatrix} 3 & -24 & 19 \\ -8 & 2 & 1 \\ -5 & 9 & -11 \end{bmatrix}$$

Substitute this value in Equation (2), it gives,

$$X = \frac{-1}{31} \begin{bmatrix} 3 & -24 & 19 \\ -8 & 2 & 1 \\ -5 & 9 & -11 \end{bmatrix} \times B$$

$$= \frac{-1}{31} \begin{bmatrix} 3 & -24 & 19 \\ -8 & 2 & 1 \\ -5 & 9 & -11 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix} \begin{bmatrix} 6 \\ 5 \\ 7 \end{bmatrix}$$

C_1

$$= \begin{bmatrix} R_1\,C_1 \\ R_2\,C_1 \\ R_3\,C_1 \end{bmatrix} \quad [\because \text{ Multiplication of two matrices}]$$

$R_1C_1 = (3 \times 6) + (-24 \times 5) + (19 \times 7) = 18 - 120 + 133 = 31$

$R_2C_1 = (-8 \times 6) + (2 \times 5) + (1 \times 7) = -48 + 10 + 7 = -31$

$R_3C_1 = (-5 \times 6) + (9 \times 5) + (-11 \times 7) = -30 + 45 - 77$

$= -62$

$$X = \frac{-1}{31} \begin{bmatrix} 31 \\ -31 \\ -62 \end{bmatrix}$$

$$X = \begin{bmatrix} -\frac{1}{31} \times 31 \\ -\frac{1}{31} \times (-31) \\ -\frac{1}{31} \times (-62) \end{bmatrix} \quad \text{...(by scalar multiplication)}$$

$$= \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \qquad \left\{ \because X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\}$$

By equating corresponding elements of matrices of both sides, it gives,

$\therefore$ **x = −1 , y = 1 , z = 2✓** ...**Ans.**

Which is required solution.

Example 2.3.2 :

Examine for consistency and solve, if consistence

x + y + z = 3 ; 2x − y + 3z = 1 ; 4x + y + 5z = 2 ;

3x − 2y + z = 4

Solution :

Step I : Given system of equations is,

$1x + 1y + 1z = 3$; $\quad 2x - 1y + 3z = 1$;

$4x + 1y + 5z = 2$; $\quad 3x - 2y + 1z = 4$

This system can be written as,

AX = B

$$\text{where } A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 4 & 1 & 5 \\ 3 & -2 & 1 \end{bmatrix}$$

(...Coefficient matrix : Coefficients of, x, y, z from given equations)

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{...(Matrix of unknown variables)}$$

$$B = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix} \quad \text{...(Matrix of R.H.S. constants)}$$

To check consistency : Find ρ(A) and ρ (A | B)

Step II : Consider, [A|B]

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 1 \\ 4 & 1 & 5 & 2 \\ 3 & -2 & 1 & 4 \end{array}\right] \quad \text{...(1)}$$

Comparing with

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Find Echelon form of matrix (1) : [Use only row transformation]

Operate $R_2 - 2R_1$; $R_3 - 4R_1$; $R_4 - 3R_1$

(To obtain all zeros below a_{11})

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -5 \\ 0 & -3 & 1 & -10 \\ 0 & -5 & -2 & -5 \end{array}\right]$$

Operate R_{24}

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & -5 & -2 & -5 \\ 0 & -3 & 1 & -10 \\ 0 & -3 & 1 & -5 \end{array}\right] \text{(Target : Obtain 1 at } a_{22}\text{)}$$

Operate $R_2 - 2R_3$ (To obtain 1 at a_{22})

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & -4 & 15 \\ 0 & -3 & 1 & -10 \\ 0 & -3 & 1 & -5 \end{array}\right]$$

Operate $R_2 + 3R_2$; $R_4 + 3R_2$

(To obtain all zeros below a_{22})

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -4 & 15 \\ 0 & 0 & -11 & 35 \\ 0 & 0 & -11 & 40 \end{bmatrix}$$

Operate $R_4 - R_3$ (To obtain all zeros below a_{33})

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -4 & 15 \\ 0 & 0 & -11 & 35 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

Operate, $\frac{R_3}{-11}$ and $\frac{R_4}{5}$ (To obtain 1 at a_{33})

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -4 & 15 \\ 0 & 0 & 1 & -35/11 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \text{...(2)}$$

Step III :

We know,

$$\text{Rank of matrix} = \begin{pmatrix}\text{Total number} \\ \text{of rows}\end{pmatrix} - \begin{pmatrix}\text{Number of Rows} \\ \text{containing} \\ \text{all zeros}\end{pmatrix}$$

From matrix (2)

$\rho(A) = 3$ and $\rho(A|B) = 4$

$\therefore$ $\rho(A) \neq \rho(A|B)$ **...(condition of consistency fails)**

Hence system is inconsistence ✓ ...Ans.

Example 2.3.3 :

Examine for consistency the system of equations

$1x - 1y - 1z = 2$; $1x + 2y + z = 2$;

$4x - 7y - 5z = 2$ and solve them if found consistence.

Solution :

Step I : Given system of equations,

$1x - 1y - 1z = 2$; $\quad 1x + 2y + 1z = 2$;

$4x - 7y - 5z = 2$

This system can be written as,

Method I :

$$AX = B$$

$$\text{Where} \quad A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix}$$

...(Coefficient matrix : Coefficients of x, y, z from given equations)

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{...(Matrix of unknown variables)}$$

$$B = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \text{...(Matrix of R.H.S. constants)}$$

To check consistency : Find $\rho(A)$ and $\rho(A|B)$

Step II : Consider, [A|B]

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{array}\right]$$

Comparing with

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \quad \text{...(1)}$$

Operate, $R_2 - R_1$; $R_3 - 4R_1$ (To obtain all zeros below a_{11})

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 3 & 2 & 0 \\ 0 & -3 & -1 & -6 \end{array}\right]$$

Operate, $R_3 + R_2$ (To obtain all zeros below a_{22})

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

Operate, $\frac{R_2}{3}$ (To obtain 1 at a_{22})

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -6 \end{array}\right] \quad \text{...(2)}$$

Step III :

We know,

$$\text{Rank of matrix} = \begin{pmatrix}\text{Total number} \\ \text{of rows}\end{pmatrix} - \begin{pmatrix}\text{Number of Rows} \\ \text{containing} \\ \text{all zeros}\end{pmatrix}$$

From matrix (2)

$\rho(A) = \rho(A|B) = 3$. ...(condition of consistency)

Hence system is consistent.

Also, $\rho(A|B) = r = 3$ and here,

number of unknowns $= n = 3$ $\therefore$ **r = n**

...(condition of unique solution)

Hence system has a unique solution.

From Last Matrix

From R_1 : $x - y - z = 2$;

From R_2 : $y + \frac{2}{3} z = 0$;

From R_3 : $z = -6$

$y = \frac{-2}{3} z = \frac{-2}{3}(-6) = 4$ $\quad \therefore$ $\quad$ **y = 4**

$x = 2 + y + z = 2 + 4 - 6 = 0$ $\therefore$ **x = 0**

$\therefore$ **x = 0, y = 4, z = −6 ✓** **...Ans.**

Alternative Method :

Step I : AX = B

Since here 3 Equations and 3 unknowns :

so first check | A |

$$\therefore |A| = \begin{vmatrix} 1 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{vmatrix} \quad \text{...(Order of determinant is 3)}$$

$= 1 \times$ [minor of 1] $-$ $(-1) \times$ [minor of (-1)]

$+ (-1) \times$ [minor of (-1)]

$$= 1 \times \begin{vmatrix} 2 & 1 \\ -7 & -5 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 4 & -5 \end{vmatrix} - 1 \times \begin{vmatrix} 1 & 2 \\ 4 & -7 \end{vmatrix}$$

$$= 1 \times [\underbrace{(2)(-5)}_{-10} - \underbrace{(-7)(1)}_{-7}] + 1 \times [\underbrace{(1)(-5)}_{-5} - \underbrace{(4)(1)}_{4}]$$

$$- 1 \times [\underbrace{(1)(-7)}_{-7} - \underbrace{(4)(2)}_{8}]$$

$$= 1 \times [\underbrace{-10 + 7}_{-3}] + 1 \times [\underbrace{-5 - 4}_{-9}] - 1 \times [\underbrace{-7 - 8}_{-15}]$$

$= 1 [-3] + 1 [-9] - 1 [-15] = 3$

$\therefore$ $|A| \neq 0$ i.e. A^{-1} exist.

As AX = B

$\therefore$ $X = A^{-1} B$ $\quad$ ...(1)

Now,

$$A^{-1} = \frac{1}{|A|} \text{ adj. A.} \quad \text{...(2)}$$

Cofactor of each element of

$$A = \begin{bmatrix} -3 & 9 & -15 \\ 2 & -1 & 3 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\therefore \text{Adj. A} = \begin{bmatrix} -3 & 2 & 1 \\ 9 & -1 & -2 \\ -15 & 3 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{3} \begin{bmatrix} -3 & 2 & 1 \\ 9 & -1 & -2 \\ -15 & 3 & 3 \end{bmatrix}$$

[From Equation (2)]

Step II : Put this in Equation (1)

$$\therefore X = \frac{1}{3} \begin{bmatrix} -3 & 2 & 1 \\ 9 & -1 & -2 \\ -15 & 3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 0 \\ 12 \\ -18 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -6 \end{bmatrix}$$

$\therefore$ **x = 0, y = 4, z = −6** is a solution. ✓ **...Ans.**

Example 2.3.4 :

Solve the system of linear Equations :

$2x + y - z + 3w = 8$; $x + y + z - w = -2$;

$3x + 2y - z = 6$; $4y + 3z + 2w = -8$

Solution :

Step I : Given system of linear equations can be written as,

$$AX = B$$

$$\text{where } A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 1 & 1 & -1 \\ 3 & 2 & -1 & 0 \\ 0 & 4 & 3 & 2 \end{bmatrix}$$

Coefficient matrix : Coefficients of x, y, z from given equations;

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \text{...(Matrix of unknown variables)}$$

$$B = \begin{bmatrix} 8 \\ -2 \\ 6 \\ -8 \end{bmatrix} \quad \text{...(Matrix of R.H.S. constants)}$$

To check consistency : Find ρ(A) and ρ (A|B)

Step II : Consider, [A|B]

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 3 & 8 \\ 1 & 1 & 1 & -1 & -2 \\ 3 & 2 & -1 & 0 & 6 \\ 0 & 4 & 3 & 2 & -8 \end{array}\right]$$

Comparing with

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix} \quad \text{...(1)}$$

Find Echelon form of matrix (1) :

[Use only row transformation]

Operate R_{12} (To obtain 1 at a_{11})

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 2 & 1 & -1 & 3 & 8 \\ 3 & 2 & -1 & 0 & 6 \\ 0 & 4 & 3 & 2 & -8 \end{array}\right]$$

Operate $R_2 - 2R_1$; $R_3 - 3R_1$ (To obtain all zeros below a_{11})

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & -1 & -3 & 5 & 12 \\ 0 & -1 & -4 & 3 & 12 \\ 0 & 4 & 3 & 2 & -8 \end{array}\right]$$

Operate $-R_2$ (To obtain 1 at a_{22})

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & -3 & -5 & -12 \\ 0 & -1 & -4 & 3 & 12 \\ 0 & 4 & 3 & 2 & -8 \end{array}\right]$$

Operate $R_3 + R_2$; $R_4 - 4R_2$

(To obtain all zeros below a_{22})

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 3 & -5 & -12 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & -9 & 22 & 40 \end{array}\right]$$

Operate $R_4 - 9R_3$ (To obtain all zeros below a_{33})

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 3 & -5 & -12 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 40 & 40 \end{array}\right]$$

Operate $(-R_3)$ and $\frac{1}{40}R_4$ (To obtain 1 at a_{33} and a_{44})

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 3 & -5 & -12 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad \text{...(2)}$$

Step III :

We know,

$$\text{Rank of matrix} = \begin{pmatrix} \text{Total number} \\ \text{of rows} \end{pmatrix} - \begin{pmatrix} \text{Number of Rows} \\ \text{containing} \\ \text{all zeros} \end{pmatrix}$$

From matrix (2),

ρ(A) = ρ(A|B) = 4 ...(condition of consistency)

(i.e. number of unknowns)

∴ r = n ...(condition of unique solution)

∴ The system have unique solution.

From last matrix,

Row 1: $1x + 1y + 1z - 1w = -2$;

Row 2 : $0x + y + 3z - 5w = -12$;

Row 3 : $0x + 0y + 1z + 2w = 0$;

Row 4 : $0x + 0y + 0z + 1 + w = 1$

i.e. $x + y + z - w = -2$

$y + 3z - 5w = -12$

$z + 2w = 0$

$w = 1$

$\therefore$ **w = 1** and $z = -2w = -2(1)$ $\therefore$ **z = −2**

$y = -12 - 3z + 5w = -12 - 3(-2) + 5(1)$

$= -12 + 6 + 5 \qquad \therefore y = -1$

$x = -2 - y - z + w = -2 + 1 + 2 + 1 \therefore x = 2$

$\therefore$ **x = 2, y = −1, z = −2 ; w = 1** ✓ **...Ans.**

Example 2.3.5 :

Examine for the consistency and if consistent.

Solve the system

2x − y − z = 2 ; x + 2y + z = 2 ; 4x − 7y − 5z = 2.

Solution :

Step I : Given system of equation,

$2x - 1y - 1z = 2$; $\quad 1x + 2y + 1z = 2$;

$4x - 7y - 5z = 2$

It can be written as,

$AX = B$

$$\text{Where} \quad A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix}$$

...(Coefficient matrix : Coefficients of x, y, z from given equations)

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{...(Matrix of unknown variables)}$$

$$B = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \text{...(Matrix of R.H.S. constants)}$$

To check consistency : Find ρ(A) and ρ (A|B)

Step II :

$$\text{Consider, } [A|B] = \left[\begin{array}{ccc|c} 2 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{array}\right]$$

Comparing with

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \quad \text{...(1)}$$

Find Echelon form of matrix (1) : [Use only row transformations]

Operate, R_{12} (To obtain 1 at a_{11})

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & -1 & -1 & 2 \\ 4 & -7 & -5 & 2 \end{array}\right]$$

Operate, $R_2 - 2R_1$; $R_3 - 4R_1$

(To obtain all zeros below a_{11})

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & -15 & -9 & -6 \end{array}\right]$$

Operate, $R_3 - 3R_2$

(To obtain all zeros below a_{22})

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{...(2)}$$

Step III :

We know,

$$\text{Rank of matrix} = \begin{pmatrix} \text{Total number} \\ \text{of rows} \end{pmatrix} - \begin{pmatrix} \text{Number of Rows} \\ \text{containing} \\ \text{all zeros} \end{pmatrix}$$

From matrix (2),

ρ(A) = ρ(A|B) = 2 ...(condition of consistency)

i.e. ∴ r = 2 ; n = number of unknowns = 3

∴ r < n, ...(condition of infinite many solutions)

Therefore the system have infinite many solutions,

From the last matrix,

$1x + 2y + 1z = 2$

$0x - 5y - 3z = -2$

$x + 2y + z = 2$; $-5y - 3z = -2$

Put **z = t** $\therefore -5y = -2 + 3z$

$y = \frac{-2 + 3z}{-5} = \frac{2 - 3z}{5}$

$\therefore \quad$ **$y = \frac{2 - 3t}{5}$** and

$x = 2 - 2y - z = 2 - 2\left(\frac{2 - 3t}{5}\right) - t$

$x = 2 - \left(\frac{4 - 6t}{5}\right) - t$

$= 2 - \frac{4}{5} + \frac{6t}{5} - t$

$= \left(2 - \frac{4}{5}\right) + \left(\frac{6t}{5} - t\right) = \frac{6}{5} + \frac{t}{5}$

$\therefore x = \frac{t + 6}{5}$

$\therefore$ **$x = \frac{t + 6}{5}$, $y = \frac{2 - 3t}{5}$, $z = t$** ✓ **...Ans.**

Example 2.3.6 :

Examine for the following system of equations for consistency and if consistent then solve it :

$2x - 3y + 5z = 1$; $3x + y - z = 2$; $x + 4y - 6z = 1$.

Solution :

Step I : Given system of equations,

$$2x - 3y + 5z = 1$$
$$3x + 1y - 1z = 2$$
$$1x + 4y - 6z = 1$$

It can be written in matrix form as,

$$AX = B$$

Where, $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -1 \\ 1 & 4 & -6 \end{bmatrix}$;

Coefficient matrix :

Coefficients of x, y, z from given equations;

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ...(Matrix of unknown variables)

$B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ ...(Matrix of R.H.S. constants)

Step II : Consider,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 3 & 1 & -1 & 2 \\ 1 & 4 & -6 & 1 \end{array}\right]$$

Comparing with

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \quad ...(1)$$

Find Echelon form of matrix (1) : [Use only row transformations]

Operate, R_{13} (To obtain 1 at a_{11})

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 3 & 1 & -1 & 2 \\ 2 & -3 & 5 & 1 \end{array}\right]$$

Operate, $R_2 - 3R_1$; $R_3 - 2R_1$ (To obtain all zeros below a_{11})

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & -11 & 17 & -1 \end{array}\right]$$

Operate, $R_3 - R_2$ (To obtain all zeros below a_{22})

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Step III :

We know,

This shows that, $\rho(A) = \rho(A|B) = 2$

...(condition of consistency)

(i.e. $r = 2$)

$\therefore$ System is consistent

Number of unknowns = n = 3

...(condition of infinite many solutions)

$\therefore$ $r < n$

The system have infinite number of solutions.

From the last matrix,

$$1x + 4y - 6z = 1$$
$$0x - 11y + 17z = 1$$

$\therefore$ $x + 4y - 6z = 1$; $-11y + 17z = -1$;

Put $z = t$

$$-11y = -1 - 17z$$
$$\therefore \quad 11y = 1 + 17z$$
$$\therefore \quad y = \frac{1+17z}{11}$$
$$\therefore \quad y = \frac{1+17t}{11}$$
$$x = 1 - 4y + 6z$$
$$\therefore \quad x = 1 - 4\left(\frac{1+17t}{11}\right) + 6t$$
$$= 1 - \frac{4+68t}{11} + 6t$$
$$= 1 - \frac{4}{11} - \frac{68t}{11} + 6t$$
$$= \left(1 - \frac{4}{11}\right) + \left(\frac{-68t}{11} + 6t\right)$$
$$= \left(\frac{11-4}{11}\right) + \left(\frac{-68t + 66t}{11}\right)$$
$$= \frac{7}{11} - \frac{2t}{11}$$
$$\therefore \quad x = \frac{7-2t}{11}$$

$\therefore$ $x = \frac{7-2t}{11}$, $y = \frac{1+17t}{11}$, $z = t$ ✓ **...Ans.**

Type II : Non-Homogeneous Systems Containing One or Two Constants

2.3.2 Examples on Non-Homogeneous System of Equations Containing One or Two Constants

Example 2.3.7 :

Investigate the values of λ and μ so that the equations $2x + 3y + 5z = 9$; $7x + 3y - 2z = 8$; $2x + 3y + \lambda z = \mu$ have (i) No Solution (ii) Unique Solution (iii) An infinite number of solutions.

Solution :

Step I : Given system of the equations,

$$2x + 3y + 5z = 9; \quad 7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$

We can write this in matrix form,

$$AX = B$$

Where $A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}$

...(Coefficient matrix : Coefficients of x, y, z from given equations)

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ...(Matrix of unknown variables)

$B = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$ ...(Matrix of R.H.S. constants)

To check consistency : Find ρ(A) and ρ (A|B)

Step II : Consider,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right]$$

Comparing with

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \quad ...(1)$$

Find Echelon form of matrix (1) :

[Use only row transformation]

Operate R_{12}

$$\sim \left[\begin{array}{ccc|c} 7 & 3 & -2 & -8 \\ 2 & 3 & 5 & 9 \\ 2 & 3 & \lambda & \mu \end{array}\right] \quad ...\text{[Target : Obtain 1 at } a_{11}]$$

Operate $R_1 - 3R_2$ (To obtain 1 at a_{11})

$$\sim \left[\begin{array}{ccc|c} 1 & -6 & -17 & -19 \\ 2 & 3 & 5 & 9 \\ 2 & 3 & \lambda & \mu \end{array}\right]$$

Operate $R_2 - 2R_1$; $R_3 - 2R_1$

(To obtain all zeros below a_{11})

$$\sim \left[\begin{array}{ccc|c} 1 & -6 & -17 & -19 \\ 0 & 15 & 39 & 47 \\ 0 & 15 & \lambda + 34 & \mu + 38 \end{array}\right]$$

Operate $R_3 - R_2$ (To obtain all zeros below a_{22})

$$\sim \left[\begin{array}{ccc|c} 1 & -6 & -17 & -19 \\ 0 & 15 & 39 & 47 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right] \quad ...(2)$$

Step III :

(i) **No solution :** If $\rho(A) \neq \rho(A|B)$

...(condition of inconsistency)

then system have no solution

From matrix (1), This is possible only if $\lambda = 5$ and $\mu \neq 9$.

(ii) **Unique solution :** If $\rho(A) = \rho(A|B) = r$ (condition of consistency) and $r = n$ (number of unknowns) ... (condition of unique solution) then system have unique solution. i.e. $r = 3$

$\therefore$ This is possible only if $\lambda \neq 5$ and μ can have any value.

(iii) **Infinite number of solutions :**

If $\rho(A) = \rho(A|B) = r$ ...(condition of consistency)

$r < n$ (number of unknowns)

...(condition of infinite many solutions)

i.e. $r = 3$

Then system have infinite many solutions.

This is possible only if $\lambda = 5$ and $\mu = 9$

Example 2.3.8 :

Determine the value of λ for which the equations $3x_1 + 2x_2 + 4x_3 = 3$; $x_1 + x_2 + x_3 = \lambda$; $5x_1 + 4x_2 + 6x_3 = 15$ are consistent. Find also the corresponding solution.

Solution :

Step I : Given system of equations,

$$3x_1 + 2x_2 + 4x_3 = 3$$
$$1x_1 + 1x_2 + 1x_3 = \lambda$$
$$5x_1 + 4x_2 + 6x_3 = 15$$

It can be written in matrix form as,

$$AX = B$$

where $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 1 & 1 \\ 5 & 4 & 6 \end{bmatrix}$;

Coefficient matrix : Coefficients of x, y, z from given equations;

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{...(Matrix of unknown variables)}$$

$$B = \begin{bmatrix} 3 \\ \lambda \\ 15 \end{bmatrix} \quad \text{...(Matrix of R.H.S. constants)}$$

To check consistency : Find $\rho(A)$ and $\rho(A|B)$

Step II : Consider,

$$[A|B] \left[\begin{array}{ccc:c} 3 & 2 & 4 & 3 \\ 1 & 1 & 1 & \lambda \\ 5 & 4 & 6 & 15 \end{array}\right]$$

$$\text{Comparing with} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \quad \text{...(1)}$$

Find Echelon form of matrix (1) : [Use only row transformation]

Operate R_{12} (To obtain 1 at a_{11})

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & \lambda \\ 3 & 2 & 4 & 3 \\ 5 & 4 & 6 & 15 \end{array}\right]$$

Operate $R_2 - 3R_1$; $R_3 - 5R_1$

(To obtain all zeros below a_{11})

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & \lambda \\ 0 & -1 & 1 & 3-3\lambda \\ 0 & -1 & 1 & 15-5\lambda \end{array}\right]$$

Operate $R_3 - R_2$

(To obtain all zeros below a_{22})

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & \lambda \\ 0 & -1 & 1 & 3-3\lambda \\ 0 & 0 & 0 & 12-2\lambda \end{array}\right] \quad \text{...(2)}$$

Step III : We know, system of equations is consistent only if $\rho(A) = \rho(A|B)$

Also,

$$\text{Rank of matrix} = \begin{pmatrix}\text{Total number} \\ \text{of rows}\end{pmatrix} - \begin{pmatrix}\text{Number of Rows} \\ \text{containing} \\ \text{all zeros}\end{pmatrix}$$

It is possible only if $12 - 2\lambda = 0 \quad \therefore \quad \boldsymbol{\lambda = 6}$

Now, we have to find solution

$\therefore$ For $\lambda = 6$ matrix (2) as,

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & -15 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{...(3)}$$

Here $\rho(A) = \rho(A|B) = 2$ and $r < n$ where n = number of unknowns = 3

$\therefore$ system have infinite number of solutions.

From matrix (3)

$$1x_1 + 1x_2 + 1x_3 = 6$$
$$0x_1 - 1x_2 + 1x_3 = -15$$
$$x_1 + x_2 + x_3 = 6$$
$$-x_2 + x_3 = -15$$

Since 3 unknowns and 2 equations.

$\therefore$ Put $x_3 = t \Rightarrow x_2 = 15 + x_3 \Rightarrow x_2 = 15 + t$

and $x_1 = 6 - x_2 - x_3 = 6 - 15 - t - t \Rightarrow x_1 = -9 - 2t$

$\therefore$ $\boldsymbol{x_1 = -9 - 2t ; x_2 = 15 + t ; x_3 = t}$ ✓ **...Ans.**

Example 2.3.9 :

Use matrix method to determine the values of λ for which the equations, x + 2y + z = 3 ; x + y + z = λ; 3x + y + 3z = λ² are consistent and solve them for these value of λ.

Solution :

Step I : Given system of equations,

$Ix + 2y + Iz = 3; \quad Ix + Iy + Iz = \lambda; \quad 3x + Iy + 3z = \lambda^2$

It can be written in matrix from as,

$$AX = B$$

where, $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}$

...(Coefficient matrix : Coefficients of x, y, z from given equations)

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{...(Matrix of unknown variables)}$$

$$B = \begin{bmatrix} 3 \\ \lambda \\ \lambda^2 \end{bmatrix} \quad \text{...(Matrix of R.H.S. constants)}$$

Step II : Consider,

$$[A|B] = \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & \lambda \\ 3 & 1 & 3 & \lambda^2 \end{array}\right] \quad \text{...From matrix}$$

$$\text{Comparing with} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \quad \text{...(1)}$$

Find Echelon form of matrix (1) : [Use only row transformation]

Operate $R_2 - R_1$; $R_3 - 3R_1$

(To obtain all zeros below a_{11})

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda - 3 \\ 0 & -5 & 0 & \lambda^2 - 9 \end{array}\right] \quad \text{...(Here at } a_{11} \text{ already 1)}$$

Operate $R_3 - 5R_2$ (To obtain all zeros below a_{22})

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda - 3 \\ 0 & 0 & 0 & \lambda^2 - 5\lambda + 6 \end{array}\right] \quad \text{...(2)}$$

Step III : The system is consistent only if,

$$\rho(A) = \rho(A|B)$$

$$\text{Rank of matrix} = \begin{pmatrix}\text{Total number} \\ \text{of rows}\end{pmatrix} - \begin{pmatrix}\text{Number of Rows} \\ \text{containing} \\ \text{all zeros}\end{pmatrix}$$

We know,

From above matrix (2) it is possible only if $\lambda^2 - 5\lambda + 6 = 0$

$\Rightarrow (\lambda - 3)(\lambda - 2) = 0 \Rightarrow \lambda = 2, 3$

Step IV : For λ = 2 :

$$\left.\begin{aligned} x + 2y + z &= 3 \\ -y &= -1 \\ \Rightarrow y &= 1 \end{aligned}\right\} \text{...From matrix}$$

Put, $z = k_1$

$\therefore x = 3 - 2y - z \Rightarrow x_1 = 1 - k_1$

$\therefore$ $\boldsymbol{x = 1 - k_1 ; y = 1 ; z = k_1}$

Step V : For λ = 3 :

$$\left.\begin{aligned} x + 2y + z &= 3 \\ y &= 0 \end{aligned}\right\} \text{...From matrix}$$

$\therefore x + z = 3$ put $z = k_2 \Rightarrow \therefore x = 3 - k_2$

$\therefore$ $\boldsymbol{x_1 = 3 - k_2; \quad y = 0; \quad z = k_2}$ ✓ ...

Example 2.3.10 :

Show that the system 3x + 4y + 5z = α ; 4x + 5y + 6 5x + 6y + 7z = γ is the consistence only when α, β arithmetic progression i.e. 2β = α + γ.

Solution :

Note :

Arithmetic progression (A. P)

The numbers a, a + d, a + 2d, a + 3d, ... are k as Arithmetic progression.

Where d is common difference.

Here, a, a + d, a + 2d we can write as,

$$a + d = \frac{(a) + (a + 2d)}{2}$$

Step I : Given system of equations,

$3x + 4y + 5z = \alpha; \quad 4x + 5y + 6z = \beta;$

$5x + 6y + 7z = \gamma$

It can be written in matrix form as,

$$AX = B$$

where, $A = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}$

Coefficient matrix : Coefficients of x, y, z from equations;

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{...(Matrix of unknown vari}$$

$$B = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \quad \text{...(Matrix of R.H.S. cons}$$

To check consistency : Find $\rho(A)$ and $\rho(A|B)$

Step II : Consider,

$$[A|B] = \left[\begin{array}{ccc:c} 3 & 4 & 5 & \alpha \\ 4 & 5 & 6 & \beta \\ 5 & 6 & 7 & \gamma \end{array}\right] \quad \text{...}$$

Find Echelon form of matrix (1) : [Use only row transformation]

Operate R_{12}

$$\sim \left[\begin{array}{ccc:c} 4 & 5 & 6 & \beta \\ 3 & 4 & 5 & \alpha \\ 5 & 6 & 7 & \gamma \end{array}\right] \quad \text{...[Target : Obtain 1 at } a_{11}]$$

Operate $R_1 - R_2$ (To obtain 1 at a_{11})

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & \beta-\alpha \\ 3 & 4 & 5 & \alpha \\ 5 & 6 & 7 & \gamma \end{array}\right]$$

Operate $R_2 - 3R_1$; $R_3 - 5R_1$

(To obtain all zeros below a_{11})

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & \beta-\alpha \\ 0 & 1 & 2 & 4\alpha-3\beta \\ 0 & 1 & 2 & 5\alpha-5\beta+\gamma \end{array}\right]$$

Operate $R_3 - R_2$

(To obtain all zeros below a_{22})

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & \beta-\alpha \\ 0 & 1 & 2 & 4\alpha-3\beta \\ 0 & 0 & 0 & \alpha-2\beta+\gamma \end{array}\right] \quad ...(2)$$

Step III : The system is consistent only if $\rho(A) = \rho(A|B)$

We know,

$$\text{Rank of matrix} = \begin{pmatrix}\text{Total number} \\ \text{of rows}\end{pmatrix} - \begin{pmatrix}\text{Number of Rows} \\ \text{containing} \\ \text{all zeros}\end{pmatrix}$$

From above matrix (2) this is possible only if

$\alpha - 2\beta + \gamma = 0$

which gives $\beta = \dfrac{\alpha+\gamma}{2}$

This shows that α, β, γ are in arithmetic progression. ✓

...Hence Proved.

Example 2.3.11 :

Find what values of k, the set of equations $2x - 3y + 6z = 3$; $y - 4z = 1$; $4x - 5y + 8z = k$ have infinite number of solutions. Hence find solutions.

Solution :

Step I : Given system of equations,

$2x - 3y + 6z = 3$; $y - 4z = 1$; $4x - 5y + 8z = k$

It can be written in matrix form as,

$$AX = B$$

where, $A = \begin{bmatrix} 2 & -3 & 6 \\ 0 & 1 & -4 \\ 4 & -5 & 8 \end{bmatrix}$;

Coefficient matrix : Coefficients of x, y, z from given equations;

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{...(Matrix of unknown variables)}$$

$$B = \begin{bmatrix} 3 \\ 1 \\ k \end{bmatrix} \quad \text{...(Matrix of R.H.S. constants)}$$

To check consistency : Find $\rho(A)$ and $\rho(A|B)$

Step II :

Consider, $[A|B] = \left[\begin{array}{ccc:c} 2 & -3 & 6 & 3 \\ 0 & 1 & -4 & 1 \\ 4 & -5 & 8 & k \end{array}\right]$

$$\text{Comparing with } \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \quad ...(1)$$

Find Echelon form of matrix (1) : [Use only row transformations] [diagonally 1 is not necessary]

Operate $R_3 - 2R_1$

(To obtain all zeros below a_{11})

$$\sim \left[\begin{array}{ccc:c} 2 & -3 & 6 & 3 \\ 0 & 1 & -4 & 1 \\ 0 & 1 & -4 & k-6 \end{array}\right]$$

Operate $R_3 - R_2$

(To obtain all zeros below a_{22})

$$\sim \left[\begin{array}{ccc:c} 2 & -3 & 6 & 3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & k-7 \end{array}\right] \quad ...(2)$$

Step III : The system have infinite number of solutions if $\rho(A) = \rho(A|B) <$ number of unknowns.

We know,

$$\text{Rank of matrix} = \begin{pmatrix}\text{Total number} \\ \text{of rows}\end{pmatrix} - \begin{pmatrix}\text{Number of Rows} \\ \text{containing} \\ \text{all zeros}\end{pmatrix}$$

It is possible only if $k - 7 = 0 \Rightarrow k = 7$.

If k = 7;

$\rho(A) = \rho(A|B) = 2 <$ number of unknowns $= 3$

(i.e. $r < n$)

Hence the system have infinite number of solutions

$$\left.\begin{aligned} 2x - 3y + 6z &= 3 \\ y - 4z &= 1 \end{aligned}\right\} \quad \text{...From matrix}$$

Put $z = k \Rightarrow y = 1 + 4k$

$2x = 3 + 3y - 6z$

$\Rightarrow 2x = 3 + 3(1 + 4k) - 6k \;\; 6 + 6k$

$\Rightarrow x = 3 + 3k$

Hence solution is

$x = 3 + 3k$; $y = 1 + 4k$; $z = k$ ✓ **...Ans.**

Type III : Non-Homogeneous Systems

2.3.3 Examples on Homogeneous System

Example 2.3.12 :

Solve the system of equations

$x + y + 3z = 0$; $x - y + z = 0$; $-x + 2y = 0$, $x - y + 2z = 0$

Solution :

Step I : Given system of equations,

$1x + 1y + 3z = 0$; $1x - 1y + 1z = 0$

$-1x + 2y + 0z = 0$; $1x - 1y + 2z = 0$

It can be written as, $AX = Z$

Where, $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

Coefficient matrix :

Coefficients of x, y, z from given equations;

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{...(Matrix of unknown variables)}$$

$$Z = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(Matrix of R.H.S. constants)}$$

To check consistency :

Since $\rho(A) = \rho(A|Z)$ always, find $\rho(A)$

Step II :

Consider, $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

$$\text{Comparing with } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \quad ...(1)$$

Find Echelon form of matrix (1) : [Use only row transformations]

Operate $R_2 - R_1$; $R_3 + R_1$; $R_4 - R_1$

(To obtain all zeros below a_{11})

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 3 & 3 \\ 0 & -2 & -1 \end{bmatrix} \quad \text{... [Here already 1 at } a_{11}]$$

Operate $\dfrac{R_2}{-2}$, (To obtain 1 at a_{22})

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & -2 & -1 \end{bmatrix}$$

Operate $R_3 - 3R_2$; $R_4 + 2R_2$

(To obtain all zeros below a_{22})

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate R_{34} (To obtain 1 at a_{33})

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad ...(2)$$

Step III :

We know,

$$\text{Rank of matrix} = \begin{pmatrix}\text{Total number} \\ \text{of rows}\end{pmatrix} - \begin{pmatrix}\text{Number of Rows} \\ \text{containing} \\ \text{all zeros}\end{pmatrix}$$

From matrix (2),

$\rho(A) = 3 =$ number of unknowns i.e. **r = n**

...(condition of unique solution)

$\therefore$ The system have only trivial solution,

$x = 0$, $y = 0$, $z = 0$ ✓ **...Ans.**

Example 2.3.13

Examine for non-trivial solution, the following set of equations and solve them :

5x + 2y – 3z = 0, 3x + y + z = 0, 2x + y + 6z = 0

Solution :

Step I : Given system of equations,

$$5x + 2y - 3z = 0 ; \qquad 3x + 1y + 1z = 0$$
$$2x + 1y + 6z = 0$$

It can be written as, $AX = Z$

$$\text{Where } A = \begin{bmatrix} 5 & 2 & -3 \\ 3 & 1 & 1 \\ 2 & 1 & 6 \end{bmatrix}$$

...(Coefficient matrix : Coefficients of x, y, z from given equations)

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{...(Matrix of unknown variables)}$$

$$Z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(Matrix of R.H.S. constants)}$$

To check consistency : Since $\rho(A) = \rho(A|Z)$ always,

$\therefore$ Find $\rho(A)$

Step II : Consider,

$$A = \begin{bmatrix} 5 & 2 & -3 \\ 3 & 1 & 1 \\ 2 & 1 & 6 \end{bmatrix}$$

$$\text{Comparing with} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{...(1)}$$

Find Echelon form of matrix (1) : [Use only row transformations]

Operate $R_1 - 2R_3$ (To obtain 1 at a_{11})

$$\sim \begin{bmatrix} 1 & 0 & -15 \\ 3 & 1 & 1 \\ 2 & 1 & 6 \end{bmatrix}$$

Operate $R_2 - 3R_1$; $R_3 - 2R_1$

(To obtain all zeros below a_{11})

$$\sim \begin{bmatrix} 1 & 0 & -15 \\ 0 & 1 & 46 \\ 0 & 1 & 36 \end{bmatrix}$$

Operate $R_3 - R_2$

(To obtain all zeros below a_{22})

$$\sim \begin{bmatrix} 1 & 0 & -15 \\ 0 & 1 & 46 \\ 0 & 0 & -10 \end{bmatrix} \quad \text{...(2)}$$

Step III :

We know,

$$\text{Rank of matrix} = \begin{pmatrix} \text{Total number} \\ \text{of rows} \end{pmatrix} - \begin{pmatrix} \text{Number of Rows} \\ \text{containing} \\ \text{all zeros} \end{pmatrix}$$

From Matrix (2),

$\rho(A) = r = 3$

$\therefore$ **r = n** ...(condition of unique solution)

Here number of unknown = n

$\therefore$ The system has only trivial solution.

x = 0 ; y = 0 ; z = 0 ✓ **...Ans.**

Example 2.3.14

Solve the system of equations :

x + 2y + 3z = 0 ; 2x + 3y + z = 0 ; 4x + 5y + 4z = 0 ; x + y – 2z = 0

Solution :

Step I : Given system of equations,

$$1x + 2y + 3z = 0 ; \qquad 2x + 3y + 1z = 0 ;$$
$$4x + 5y + 4z = 0 ; \qquad 1x + 1y - 2z = 0$$

It can be written as, $AX = Z$

$$\text{Where } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 5 & 4 \\ 1 & 1 & -2 \end{bmatrix}$$

...(Coefficient matrix : Coefficients of x, y, z from given equations)

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{...(Matrix of unknown variables)}$$

$$Z = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(Matrix of R.H.S. constants)}$$

To check consistency : Since $\rho(A) = \rho(A|Z)$ always.

$\therefore$ Find $\rho(A)$

Step II : Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 5 & 4 \\ 1 & 1 & -2 \end{bmatrix} \quad \text{...(1)}$$

$$\text{Comparing with} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

Operate $-R_2 - 2R_1$; $R_3 - 4R_1$; $R_4 - R_1$

(To obtain all zeros below a_{11})

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -3 & -8 \\ 0 & -1 & -5 \end{bmatrix}$$

Operate R_2 ; $-R_3$; $-R_4$ (To obtain 1 at a_{22})

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 3 & 8 \\ 0 & 1 & 5 \end{bmatrix}$$

(Operate $R_3 - 3R_2$; $R_4 - R_2$)

(To obtain all zeros below a_{22})

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -7 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{...(2)}$$

Step III :

$$\text{Rank of matrix} = \begin{pmatrix} \text{Total number} \\ \text{of rows} \end{pmatrix} - \begin{pmatrix} \text{Number of Rows} \\ \text{containing} \\ \text{all zeros} \end{pmatrix}$$

We know, $r = \rho(A) = \rho(A \mid Z) = 3$

Here n = number of unknowns = 3

i.e. **r = n** **...(condition of unique solution)**

$\therefore$ The system has only trivial solution.

$\therefore$ **x = 0, y = 0, z = 0** ✓ **...Ans.**

Example 2.3.15

Examine for non trivial solutions, the following set of equations and solve them

x + 3y + 4z – 6w = 0 ; y + 6z = 0 ; 2x + 2y + 2z – 3w = 0 ; x + y – 4z – 4w = 0.

Solution :

Step I : Given set of equations,

$$1x + 3y + 4z - 6w = 0 ;$$
$$0x + 1y + 0z + 6z = 0 ;$$
$$2x + 2y + 2z - 3w = 0 ;$$
$$1x + 1y - 4z - 4w = 0$$

It can be written as,

$$AX = Z$$

$$A = \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 2 & 2 & 2 & -3 \\ 1 & 1 & -4 & -4 \end{bmatrix} ;$$

...(Coefficient matrix : Coefficients of x, y, z from given equations)

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \text{...(Matrix of unknown variables)}$$

$$Z = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(Matrix of R.H.S. constants)}$$

To check consistency : Since $\rho(A) = \rho(A|Z)$ always, find $\rho(A)$

Step II : Consider

$$A = \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 2 & 2 & 2 & -3 \\ 1 & 1 & -4 & -4 \end{bmatrix}$$

$$\text{Comparing with} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad \text{...(1)}$$

Find Echelon form of matrix (1) : [Use only row transformations] [diagonally 1 is not necessary.]

Operate $R_3 - 2R_1$; $R_4 - R_1$

(To obtain all zeros below a_{11})

$$\sim \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & -4 & -6 & 9 \\ 0 & -2 & -8 & 2 \end{bmatrix}$$

Operate $R_3 + 4R_2$; $R_4 + 2R_2$

(To obtain all zeros below a_{22})

$$\sim \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 18 & 9 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

Operate $\frac{R_3}{9}$; $\frac{R_4}{4}$

$$\sim \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Operate $R_4 - R_3$

(To obtain all zeros below a_{33})

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \ldots(2)$$

Step III :

We know,

$$\text{Rank of matrix} = \begin{pmatrix} \text{Total number} \\ \text{of rows} \end{pmatrix} - \begin{pmatrix} \text{Number of Rows} \\ \text{containing} \\ \text{all zeros} \end{pmatrix}$$

From Matrix (2),

$\therefore$ $\rho(A) = \rho(A \mid Z) = r = 3$ and here, n = number of unknowns = 4

$\therefore$ **r < n** ...(condition of infinite many solutions)

$\therefore$ The system here infinite many solutions.

$$\left.\begin{aligned} x + 3y + 4z - 6w &= 0 \\ y + 6z &= 0 \\ 2z + w &= 0 \end{aligned}\right\} \text{From matrix}$$

[Since 3 Equations and 4 unknowns]

Put z = k

$w = -2z \Rightarrow w = -2k \quad \therefore \quad \mathbf{w = -2k}$

$y = -6z \Rightarrow y = -6k \quad \therefore \quad \mathbf{y = -6k}$

$x = -3y - 4z + 6w \Rightarrow \quad x = 18k - 4k - 12k \quad \mathbf{x = 2k}$

$\therefore$ Solution is, $\mathbf{x = 2k, \; y = -6k, \; z = k, \; w = -2k}$ ✓

...Ans.

Example 2.3.16

Solve the equations 4x + 2y + z + 3w = 0 ; 6x + 3y + 4z + 7w = 0 ; 2x + y + w = 0.

Solution :

Step I : Given equations.

$4x + 2y + 1z + 3w = 0$; $\quad 6x + 3y + 4z + 7w = 0$;

$2x + 1y + 0z + 1w = 0$

It can be written as, AX = Z

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

Coefficient matrix : Coefficients of x, y, z from given equations;

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \ldots\text{(Matrix of unknown variables)}$$

$$Z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots\text{(Matrix of R.H.S. constants)}$$

To check consistency : Since $\rho(A) = \rho(A|Z)$ always.

$\therefore$ Find $\rho(A)$

Step II : Consider

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

$$\text{Comparing with} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \quad \ldots(1)$$

Find Echelon form of matrix (1) : [Use only row transformations]

$$\text{Operate, } R_{13} \quad \sim \begin{bmatrix} 2 & 1 & 0 & 1 \\ 6 & 3 & 4 & 7 \\ 4 & 2 & 1 & 3 \end{bmatrix}$$

Operate $R_2 - 3R_1$; $R_3 - 2R_1$

(To obtain all zeros below a_{11})

$$\sim \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{Operate } R_{23} \quad \sim \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\text{Operate } R_3 - 4R_2 \quad \sim \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \ldots(2)$$

Step III :

We know,

$$\text{Rank of matrix} = \begin{pmatrix} \text{Total number} \\ \text{of rows} \end{pmatrix} - \begin{pmatrix} \text{Number of Rows} \\ \text{containing} \\ \text{all zeros} \end{pmatrix}$$

From matrix (2),

$\therefore$ $r = \rho(A) = \rho(A \mid Z) = 2$ and here, number of unknowns = n = 4

$\therefore$ **r < n** ...(condition of infinite many solutions)

$\Rightarrow$ There are infinite many solutions.

$$\therefore \left.\begin{aligned} 2x + y + w &= 0 \\ z + w &= 0 \end{aligned}\right\} \text{From matrix}$$

Since here 4 unknowns and 2 Equations.

Put $\mathbf{w = k_1}$, $\quad \mathbf{y = k_2}$

$z = -w$ i.e. $\mathbf{z = -k_1}$

$$x = \left(\frac{y+w}{2}\right)$$

$$\text{i.e.} \quad x = -\left(\frac{\mathbf{k_1 + k_2}}{\mathbf{2}}\right)$$

Hence solution is,

$$\mathbf{x = -\left(\frac{k_1 + k_2}{2}\right) \; ; \; y = k_2, \; z = -k_1 \; ; \; w = k_1} \checkmark$$

...Ans.

Type IV : Homogeneous Systems of Equations Containing Constants

2.3.4 Examples on Homogeneous System of Equations Containing Constants

Example 2.3.17

Show that the system of equations ax + by + cz = 0 ; bx + cy + az = 0 ; cx + ay + bz = 0 has a non-trivial solution only if a + b + c = 0 or if a = b = c and hence find its solution.

Solution :

Step I : Given system of equations,

$$\begin{aligned} ax + by + cz &= 0 ; \\ bx + cy + az &= 0 ; \\ cx + ay + bz &= 0 \end{aligned}$$

It can be written as,

AX = Z

$$A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

(coefficient matrix : coefficients of x, y, z from given equations)

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \ldots\text{(Matrix of unknown variables)}$$

$$Z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots\text{(Matrix of R.H.S. constants)}$$

Step II : This is homogeneous system, this system has a non-trivial solution only if r < n

i.e. $\rho(A)$ < number of unknowns = 3.

We known, $\rho(A) < 3$ only if $|A| = 0$

$$\text{i.e.} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

Order of determinant is 3

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

a × [minor of a] − b × [Minor of b] + c × [Minor of c] = 0

$$a \times \begin{vmatrix} c & a \\ a & b \end{vmatrix} - b \times \begin{vmatrix} b & a \\ c & b \end{vmatrix} + c \times \begin{vmatrix} b & c \\ c & a \end{vmatrix} = 0$$

$$\begin{aligned} a(bc - a^2) - b(b^2 - ac) + c(ab - c^2) &= 0 \\ abc - a^3 - b^3 + abc + abc - c^3 &= 0 \end{aligned}$$

$$-a^3 - b^3 - c^3 + 3abc = 0$$

Throughout multiply by (– 1)

$$\therefore\ a^3 + b^3 + c^3 - 3abc = 0 \qquad ...(1)$$

→ **Use factors of eqn. (1)**

$$[(a+b+c)(a^2+b^2+c^2-ab-ac-bc) = a^3+b^3+c^3-3abc]$$

$$\therefore\ (a+b+c)(a^2+b^2+c^2-ab-ac-bc) = 0$$

$$\therefore\ a+b+c = 0 \quad \text{or}$$

$$a^2+b^2+c^2-ab-ac-bc = 0$$

$\therefore$ The system has non trivial solutions only if

$$a+b+c = 0 \quad \text{or}$$

$$a^2+b^2+c^2-ab-ac-bc = 0$$

Multiply throughout by 2

$$2a^2+2b^2+2c^2-2ab-2ac-2bc = 0$$

Rearrange as : $a^2+a^2+b^2+b^2+c^2+c^2-2ab-2ac-2bc = 0$

$$(a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ac+a^2) = 0$$

$$(a-b)^2+(b-c)^2+(c-a)^2 = 0$$

This is possible only if

$a - b = 0,\ b - c = 0,\ c - a = 0$

...(squares always positive and addition of positive is zero means all terms are equal to zero)

$$\Rightarrow\ a = b = c$$

Now we have to find solutions,

Step III : Case (i) : a + b + c = 0

$$A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

Operate, $R_3 + (R_2 + R_1)$

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c+a+b & a+b+c & b+a+c \end{bmatrix}$$

$$\sim \begin{bmatrix} a & b & c \\ b & c & a \\ 0 & 0 & 0 \end{bmatrix} \qquad (\because a+b+c=0)$$

Matrix gives equations as

$$\therefore\ ax + by + cz = 0; \qquad bx + cy + az = 0$$

By Cramer's Rule :

$$\therefore\ \frac{x}{\begin{vmatrix} b & c \\ c & a \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a & c \\ b & a \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & b \\ b & c \end{vmatrix}} = k$$

$$\frac{x}{ba-c^2} = \frac{-y}{a^2-bc} = \frac{z}{ac-b^2} = k$$

$$\frac{x}{ba-c^2} = \frac{y}{bc-a^2} = \frac{z}{ac-b^2}$$

$\therefore$ $\mathbf{x = (ab - c^2)k\ ;\ y = (bc - a^2)k\ ;\ z = (ac - b^2)k}$

is a solution.

Case (ii) : $\quad \mathbf{a = b = c}$

$$A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ b-a & c-b & a-c \\ c-a & a-b & b-c \end{bmatrix}$$

Operate, $R_2 - R_1\ ;\ R_3 - R_1$ $\sim \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Matrix gives equations,

$$\therefore\ ax + by + cz = 0$$

Since 3 unknowns and 1 Equation.

$$\therefore\ \text{Put } z = k_1 \quad \text{and} \quad y = k_2$$

$$ax = -by - cz$$

$$x = \frac{-by - cz}{a}$$

$$\therefore\ x = \frac{-k_2 b - ck_1}{a}$$

$$\Rightarrow x = -(k_1 + k_2) \qquad (\because a = b = c)$$

$\therefore$ Solution is, $\mathbf{x = -(k_1 + k_2)\ ;\ y = k_2\ ;\ z = k_1}$ ✓ **...Ans.**

Example 2.3.18

For different values of k, discuss the following equations : x + 2y – z = 0 ; 3x + (k + 7)y – 3z = 0 ; 2x + 4y + (k – 3)z = 0.

Solution :

Step I : Given equations,

$$1x + 2y - 1z = 0;$$

$$3x + (k+7)y - 3z = 0;$$

$$2x + 4y + (k-3)z = 0$$

It can be written as,

$$AX = Z$$

Where, $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & k+7 & -3 \\ 2 & 4 & k-3 \end{bmatrix}$

Coefficient matrix : Coefficients of x, y, z from given equations;

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{...(Matrix of unknown variables)}$$

$$Z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{...(Matrix of R.H.S. constants)}$$

To check consistency : Since $\rho(A) = \rho(A|Z)$ always, find $\rho(A)$

Step II : Consider,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & k+7 & -3 \\ 2 & 4 & k-3 \end{bmatrix} \qquad ...(1)$$

Find Echelon form of matrix (1) : [Use only row transformations]

Operate, $R_2 - 3R_1$ and $R_3 - 2R_1$

(To obtain all zeros below a_{11})

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & k+1 & 0 \\ 0 & 0 & k-1 \end{bmatrix} \qquad ...(2)$$

Step III : $\therefore$ The system have infinite number of solutions if $r < n$ i.e. $\rho(A) < 3$ (number of unknowns = n = 3)

Rank of matrix = (Total number of rows) – (Number of Rows containing all zeros)

We know,

From the last matrix it is possible only if

$k + 1 = 0$ or $k - 1 = 0$ i.e. $k = -1$, or $k = 1$

(I) For k = 1 :

$$[A \mid Z] = \left[\begin{array}{ccc:c} 1 & 2 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{...[from matrix (1)]}$$

Matrix gives equations as,

$$x + 2y - z = 0$$

$$2y = 0$$

$$\Rightarrow \quad \mathbf{y = 0}$$

Put, $\quad z = k_1$

$$\mathbf{x = z = k_1}$$

i.e. $\quad \mathbf{x = k_1, y = 0, z = k_1}$

(II) For k = – 1 :

$$[A \mid Z] = \left[\begin{array}{ccc:c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \qquad \text{...[from matrix (1)]}$$

Matrix gives equations as,

$$x + 2y - z = 0$$

$$-2z = 0$$

$$\Rightarrow \quad \mathbf{z = 0}$$

$$x = -2y \quad \text{put } y = k_2 \Rightarrow x = -2k_2$$

i.e $\mathbf{x = -2k_2,\ y = k_2,\ z = 0}$ ✓ **...Ans.**

Example 2.3.19

Show that the system of equations ; $2x_1 - 2x_2 + x_3 = \lambda x_1$; $2x_1 - 3x_2 + 2x_3 = \lambda x_2$; $-x_1 + 2x_2 = \lambda x_3$ can posesses a non trivial solution only if λ = 1 ; λ = – 3. Obtain the general solution in each case

Solution :

Step I : Given system of Equations,

$$(2-\lambda)x_1 - 2x_2 + 1x_3 = 0$$

$$2x_1 - (3+\lambda)x_2 + 2x_3 = 0$$

$$-1x_1 + 2x_2 - \lambda x_3 = 0$$

It can be written as,

$$AX = Z$$

$$A = \begin{bmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{bmatrix}$$

Coefficient matrix : Coefficients of x, y, z from given equations;

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \text{...(Matrix of unknown variables)}$$

$$Z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{...(Matrix of R.H.S. constants)}$$

Step II : This system has a non trivial solution only if r < n

r = rank of [A | Z] and n = number of unknowns r < 3 only if | A | = 0 (Here n = 3)

i.e. $\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} = 0$

For simplicity

To Solve determinant we can use elementary transformations. Since by elementary transformation value of determinant is not alter.

Operate, $R_3 + R_1$;
$$\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ 1-\lambda & 0 & -\lambda+1 \end{vmatrix} = 0$$

Operate, $C_3 - C_1$;
$$\begin{vmatrix} 2-\lambda & -2 & \lambda-1 \\ 2 & -3-\lambda & 0 \\ 1-\lambda & 0 & 0 \end{vmatrix} = 0$$

$$(2-\lambda)[0] - 2[0] + (\lambda-1)[-(-3-\lambda)(1-\lambda)] = 0$$
$$(\lambda-1)(\lambda+3)(1-\lambda) = 0$$
i.e. $\lambda = 1, -3$

Step III : This shows that system has a nontrivial solution only if $\lambda = 1, -3$

For $\lambda = 1$:

$$\begin{vmatrix} 1 & -2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$x - 2y = 0$ ...[From determinant]

$2x - 4y = 0$

i.e. $x - 2y = 0 \Rightarrow x = 2y$

Put $\mathbf{y = k_1} \Rightarrow \mathbf{x = 2k_1}$

z is any value i.e. $z = k_2$

For $\lambda = -3$:

$$\begin{vmatrix} 5 & -2 & -4 \\ 2 & 0 & 0 \\ 4 & 0 & 0 \end{vmatrix} = 0 \text{ ...(From determinant)}$$

$5x - 2y - 4z = 0$

$2x = 0$

$2x = 0 \Rightarrow \mathbf{x = 0}$

$\therefore 2y = -4z \Rightarrow y = -2z$

Put $\mathbf{z = \alpha}$ $\quad \therefore \mathbf{y = -2\alpha}$

$\therefore \mathbf{x = 0,\ y = -2\alpha,\ z = \alpha}$ ✓ **...Ans.**

Exercise

Type I :

Ex. 1 Is the following system of equations consistent ? If so solve it

$x + y + z = 6$; $x - y + 2z = 5$; $3x + y + z = 8$;

$2x - 2y + 3z = 7$ **Ans. : $x = 1, y = 2, z = 3$**

Ex. 2 Test for consistency and if consistent solve it.

$x + 3y + 7z = 4$; $3x + 26y + 2z = 9$; $7x + 2y + 10z = 5$

Ans. : $x = \frac{7-16k}{11}$; $y = \frac{3+k}{11}$; $z = k$

Ex. 3 Test for consistency and if consistent solve it.

$4x - 2y + 6z = 8$; $x + y - 3z = -1$; $15x - 3y + 9z = 21$

Ans. : $x = 1$; $y = 3k - 2$; $z = k$

Ex. 4 Test for consistency and if consistent solve it.

$2x - 3y + 7z = 5$; $3x + y - 3z = 13$;

$2x + 19y - 47z = 32$

Ans. : System is inconsistent.

Ex. 5 Test for consistency and if consistent solve it.

$x_1 - x_2 + x_3 + x_4 = -2$; $x_1 + x_2 - x_3 + x_4 = -4$;

$x_1 + x_2 + x_3 - x_4 = 4$; $x_1 + x_2 + x_3 + x_4 = 0$

Ans. : $x_1 = 1$; $x_2 = -1$; $x_3 = 2$; $x_4 = -2$

Ex. 6 Test for consistency and if consistent solve it.

$2x - 3y + 5z = 1$; $3x + y - z = 2$; $x + 4y - 6z = 1$

Ans. : $x = \frac{7-2k}{11}$, $y = \frac{1+17k}{11}$, $z = k$

Type II :

Ex. 1 Investigate for what values of μ and λ the system of equations : $x + y + z = 6$; $x + 2y + 3z = 10$;

$x + 2y + \lambda z = \mu$ have

(i) no solution (ii) unique solution (iii) Infinite many solutions.

Ans. : (i) $\lambda = 3$; $\mu \neq 10$ (ii) $\lambda \neq 3$ and μ can have any value. (iii) $\lambda = 3$; $\mu = 10$

Ex. 2 Investigate the values of λ and μ so that the equations

$2x + 3y + 5z = 9$; $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$ has (i) no solution (ii) unique solution (iii) Infinite number of solutions.

Ans. : (i) $\lambda = 5, \mu \neq 9$

(ii) $\lambda \neq 5$, μ can have any value

(iii) $\lambda = 5, \mu = 9$.

Ex. 3 Solve the equations for all values of λ

$x + y - z = 1$, $x + y + z = 2$; $x + 2y + \lambda z = 3\lambda$

Ans. : No solutions for all values of λ.

Type III :

Ex. 1 Examine for non trivial solution the following set of equations and solve them

$4x - y + 2z + t = 0$; $2x + 3y - z - 2t = 0$;

$7y - 4z - 5t = 0$; $2x - 11y + 7z + 8t = 0$

Ans. : $x = \frac{-5k_2 - k_1}{14}$; $y = \frac{4k_2 + 5k_1}{7}$; $z = k_2$; $t = k_1$

Ex. 2 Solve the Equations : $x + 2y + 3z = 0$;

$3x + 4y + 4z = 0$; $7x + 10y + 12z = 0$

Ans. : Only trivial solution i.e. $x = y = z = 0$

Ex. 3 Solve the system of Equations : $x + y - 2z + 3w = 0$; $x - 2y + z - w = 0$; $4x + y - 5z + 8w = 0$; $5x - 7y + 2z - w = 0$.

Ans. : $x = \frac{k_2 - 5k_1}{3}$; $y = \frac{k_2 - 4k_1}{3}$; $z = k_2$; $w = k_1$

Ex. 4 Examine for non-trivial solutions ; the following system of equations and solve them :

$3x + 4y - z - 6w = 0$; $2x + 3y + 2z - 3w = 0$; $2x + y - 14z - 9w = 0$; $x + 3y + 13z + 3w = 0$

Ans. : $x = 11k_2 + 6k_1$; $y = -8k_2 - 3k_1$; $z = k_2$; $w = k_1$

Ex. 5 Solve the system : $x + y + 2z = 0$; $x + 2y + 3z = 0$; $x + 3y + 4z = 0$; $3x + 4y + 7z = 0$

Ans. : $x = -k$; $y = -k$; $z = k$

Type IV :

Ex. 1 Show that the system of equations

$x_1 + 2x_2 + 3x_3 = \lambda x_1$; $3x_1 + x_2 + 2x_3 = \lambda x_2$;

$2x_1 + 3x_2 + x_3 = \lambda x_3$

can posses a nontrivial solution only if $\lambda = 6$.

Obtain the general solution for real values of λ.

Ans. : For $\lambda = 6$; $x_1 = x_2 = x_3 = k$

Ex. 2 Determine the values of λ for which the following set of equations may possess nontrivial solutions.

$3x_1 + x_2 - \lambda x_3 = 0$; $4x_1 - 2x_2 - 3x_3 = 0$;

$2\lambda x_1 + 4x_2 + \lambda x_3 = 0$

Ans. : $\lambda = 1, -9$

For $\lambda = 1$; $x_1 = k$; $x_2 = -k$; $x_3 = 2k$

$\lambda = -9$; $x_1 = 3k$; $x_2 = 9k$; $x_3 = -2k$

2.4 University Questions and Answers

➔ Dec 17

Q. 1 For what value of λ, the following system of linear equations is consistent and solve it completely in each case : $x + y + z = 1$; $x + 2y + 4z = \lambda$; $x + 4y + 10z = \lambda^2$.

Ans. :

Step I : Given system of equations,

$1x + 1y + 1z = 1$; $1x + 2y + 4z = \lambda$; $1x + 4y + 10z = \lambda^2$

It can be written in matrix from as,

$$AX = B$$

where, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}$

...(Coefficient matrix : Coefficients of x, y, z from given equations)

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ...(Matrix of unknown variables)

$B = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$ ...(Matrix of R.H.S. constants)

Step II : Consider,

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{array}\right] \text{ ...From matrix}$$

Comparing with $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$ ...(1)

Find Echelon form of matrix (1) : [Use only row transformation]

Operate $R_2 - R_1$; $R_3 - R_1$

(To obtain all zeros below a_{11})

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 3 & 9 & \lambda^2 - 1 \end{array}\right] \text{ ...(Here at } a_{11} \text{ already 1)}$$

Operate $R_3 - 3R_2$ (To obtain all zeros below a_{22})

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 0 & 0 & \lambda^2 - 3\lambda + 2 \end{array}\right] \quad \ldots(2)$$

Step III : The system is consistent only if,

$$\rho(A) = \rho(A|B)$$

$$\text{Rank of matrix} = \begin{pmatrix}\text{Total number} \\ \text{of rows}\end{pmatrix} - \begin{pmatrix}\text{Number of Rows} \\ \text{containing} \\ \text{all zeros}\end{pmatrix}$$

We know,

From above matrix (2) it is possible only if $\lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow (\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda = 1, 2$

Step IV : **For $\lambda = 1$:**

$$\left.\begin{aligned} x + y + z &= 1 \\ y + 3z &= 0 \end{aligned}\right\} \quad \text{...From matrix}$$

Put, $z = k_1$

$\therefore y + 3k_1 = 0 \Rightarrow y = -3k_1$

$\therefore$ $\mathbf{x + (-3k_1) + k_1} = 1 \Rightarrow x - 2k_1 = 1 \Rightarrow x = 1 + 2k_1$

$\therefore$ $\mathbf{x_1 = 1 + 2k_1; \quad y = -3k_1; \; z = k_1}$

Step V : For $\lambda = 2$:

$$\left.\begin{aligned} x + y + z &= 1 \\ y + 3z &= 1 \end{aligned}\right\} \quad \text{...From matrix}$$

Put, $z = k_2$

$\therefore y + 3k_2 = 1 \Rightarrow y = 1 - 3k_2$

$\therefore$ $\mathbf{x + (1 - 3k_2) + k_2} = 1 \Rightarrow x + 1 - 2k_2 = 1 \Rightarrow x = \mathbf{2k_2}$

$\therefore$ $\mathbf{x_1 = 2k_2; \; y = 1 - 3k_2; \; z = k_2}$ ✓ **...Ans.**

Chapter Ends...

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CHAPTER 3

Unit I

Eigen Values and Eigen Vectors

Syllabus

Eigen values and eigen vectors; Properties of eigen values and eigen vectors (without proofs); Cayley-Hamilton's theorem (without proof) and its applications.

3.1 Introduction

The concept of eigen value and eigen vector have a considerable theoretical interest and wide-ranging application. This concept is crucial in solving systems of differential equations, analyzing population growth models, and calculating powers of matrices (in order to define the exponential matrix). Other areas such as physics, sociology, biology, economics and statistics have focused considerable attention on "eigenvalues" and "eigenvectors" their applications and computations. Eigen values and eigen vectors also plays an important role in the study of vibration of beams, probability (Markov process), quantum mechanism etc.

3.2 Prerequisite

[I] Cramer's Rule : Consider two different equations,

$$a_{11}x + a_{12}y + a_{13}z = 0$$
$$a_{21}x + a_{22}y + a_{23}z = 0$$

then by Cramer's rule,

$$\frac{x}{\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = k$$

[II] Synthetic Division :

In the process of finding roots of polynomial equation $f(D) = 0$ by synthetic division method, the first root is obtained by trial and error method.

e.g. $D^4 - 2D^3 - 3D^2 + 4D + 4 = 0$

$$\begin{array}{c|ccccc} 1 & 1 & -2 & -3 & 4 & 4 \\ & \times & 1 & -1 & -4 & 0 \\ \hline & 1 & -1 & -4 & 0 & \boxed{4} \end{array}$$

These are the coefficients. Write down the coefficient of the power of D in order. [Adding the missing power of D by zero]

If we get here zero then $D = 1$ is one of the root of equation and if not zero then try for next.

$$\begin{array}{r|rrrrr} -1 & 1 & -2 & -3 & 4 & 4 \\ & \times & -1 & 3 & 0 & -4 \\ \hline & 1 & -3 & 0 & 4 & \boxed{0} \end{array}$$

$\therefore$ D = -1 is one of the root and it gives

$D^3 - 3D^2 + 0D + 4 = 0$ i.e. $D^3 - 3D^2 + 4 = 0$

Again do the same procedure

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & \times & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$ gives again D = -1

And $D^2 - 4D + 4 = 0$ and so on.

3.3 Eigen Values and Eigen Vectors

If A is a square matrix, then a vector X is said to be Eigen vector (or characteristics vector) of the matrix A if there exist a number λ such that,

$$AX = \lambda X$$

If A is a square matrix of order n then X is a column matrix of order n.

The number λ is known as **Eigen values or characteristic roots or latent roots.**

$$AX = \lambda X$$
$$AX = \lambda IX$$
$$AX - \lambda X = 0$$
$$[A - \lambda I] X = 0$$

If A is a square matrix then the characteristic polynomial of matrix A is $|A - \lambda I|$.

If A is a square matrix then the characteristic equation is $|A - \lambda I| = 0$.

e.g. If A = $\begin{bmatrix} a_{11} & a_{12} & -- & -- & a_{1n} \\ a_{21} & a_{22} & -- & -- & a_{2n} \\ -- & -- & -- & -- & -- \\ -- & -- & -- & -- & -- \\ -- & -- & -- & -- & -- \\ a_{n1} & a_{n2} & -- & -- & a_{nn} \end{bmatrix}$

then $|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & -- & -- & a_{1n} \\ a_{21} & a_{22}-\lambda & -- & -- & a_{2n} \\ -- & -- & -- & -- & -- \\ -- & -- & -- & -- & -- \\ -- & -- & -- & -- & -- \\ a_{n1} & a_{n2} & -- & -- & a_{nn}-\lambda \end{vmatrix}$

is a characteristic polynomial of matrix A.

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & -- & -- & a_{1n} \\ a_{21} & a_{22}-\lambda & -- & -- & a_{2n} \\ -- & -- & -- & -- & -- \\ -- & -- & -- & -- & -- \\ -- & -- & -- & -- & -- \\ a_{n1} & a_{n2} & -- & -- & a_{nn}-\lambda \end{vmatrix} = 0$$

Gives characteristic equation of A.

The roots of characteristics equation are known as

3.3.1 Eigen Values or Characteristic Roots or Latent Roots

Note : (i) If A is a square matrix of order 3 then $|A - \lambda I| = 0$ is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

Where S_1 = sum of the principal diagonal elements i.e. $[a_{11} + a_{22} + a_{33}]$

S_2 = sum of minors of the principal diagonal elements.

$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

(ii) spectrum of the matrix is a set of Eigen values

(iii) The trace of a square matrix is the sum of its diagonal elements i.e. Trace of a matrix A = $a_{11} + a_{22} + a_{33}$

(iv) The largest of the absolute values of the Eigen values of A is called the spectral radius of A.

(v) If λ is a Eigen value of the matrix A then matrix $[A - \lambda I]$ is singular.

(vi) Degree of characteristics equation of matrix A is equal to order of matrix A.

(vii) Eigen values of diagonal matrix, upper triangular matrix, and lower triangular matrix are the diagonal elements.

e.g. Diagonal matrix

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $\therefore$ Eigen values are $\lambda = 2, 1, 5$

Upper triangular matrix :

$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}$ $\therefore$ Eigen values are $\lambda = 1, 3, 2$

Lower triangular matrix

$\begin{bmatrix} 2 & 0 & 0 \\ -3 & 5 & 0 \\ 0 & 7 & 4 \end{bmatrix}$ $\therefore$ Eigen values are $\lambda = 2, 5, 4$

3.3.2 Properties of Eigen values

1. The sum of the Eigen values of a matrix is the sum of the elements of the principal diagonal.

   $\lambda_1 + \lambda_2 + --- + \lambda_n = a_{11} + a_{22} + ---- + a_{nn}$

2. The product of the Eigen values of a matrix A is equal to its determinant.

   $|A| = \lambda_1\lambda_2 --- \lambda_n$

3. If λ is the Eigen value of matrix A, then $\frac{1}{\lambda}$ is the Eigen value of A^{-1}

4. If λ is the Eigen value of an orthogonal matrix A, then $\frac{1}{\lambda}$ is also its Eigen value.

5. If $\lambda_1, \lambda_2, ---, \lambda_n$ are the Eigen values of a matrix A, then A^m has the Eigen values $\lambda_1^m, \lambda_2^m, ..., \lambda_n^m$ (m being a positive integer).

6. A square matrix A and its transpose A^T (or A′) have same Eigen values.

7. If $\lambda_1, \lambda_2, \lambda_3, ---, \lambda_n$ are Eigen values of matrix A then the Eigen values of matrix kA are $k\lambda_1, k\lambda_2, k\lambda_3, ---, k\lambda_n$ (k is scalar and non zero)

8. If $\lambda_1, \lambda_2, \lambda_3, ---, \lambda_n$ are Eigen values of matrix A then the Eigen values of matrix $A \pm kI$ are $\lambda_1 \pm k, \lambda_2 \pm k, \lambda_3 \pm k, ---, \lambda_n \pm k$ (k is scalar and non zero).

3.3.3 Properties of Eigen Vectors

If λ_r and λ_s are two distinct Eigen values of an n-square matrix 'A' having Eigen vectors X_r and X_s respectively then

1. Eigen vectors corresponding to distinct Eigen values are always linearly independent.
2. X_r and X_s are orthogonal if A is symmetric.

3.3.4 Steps to Find Eigen Vectors

If Eigen values are $\lambda = \lambda_1, \lambda_2, \lambda_3$

(A) Non-repeated Eigen values for un-symmetric and symmetric matrix

- If Eigen values $\lambda = \lambda_1, \lambda_2, \lambda_3$ are non repeated then find Eigen vectors corresponding to each Eigen values separately by following method :
- Consider a Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and a matrix $[A - \lambda I] X = 0$
- Substitute value $\lambda = \lambda_1$, in a matrix $[A - \lambda I] X = 0$
- This matrix gives three equations and by Cramer's rule find values of x, y, z, gives X_1.
- Similarly and X_2 and X_3 corresponding to $\lambda = \lambda_2$ and $\lambda = \lambda_3$ respectively.

(B) Repeated Eigen values for un-symmetric Matrix

- If Eigen values are $\lambda = \lambda_1, \lambda_2, \lambda_3$ and two Eigen values are repeated i.e. $\lambda_2 = \lambda_3$
- First find Eigen vector X_1 corresponding to λ_1 by the above method.
- To find Eigen vectors corresponding to repeated Eigen values $\lambda_2 = \lambda_3$
- First find the rank of matrix $[A - \lambda I]$ after substituting λ_2.
- If rank = r and 'n' - number of unknowns then (n – r) number of linearly independent Eigen vectors are possible.
- If only one vector is possible then find it by above method given in (A)
- If two linearly independent vectors are possible.
- There is only one equation from the last matrix, in this equation put z = 0 find x and y given Eigen vector X_2
- Again in the same equation put y = 0 and x and z gives Eigen vector X_3.

(C) Repeated Eigen values for symmetric Matrix

- If Eigen values are $\lambda = \lambda_1, \lambda_2, \lambda_3$ and two Eigen values are repeated i.e. $\lambda_2 = \lambda_3$
- First find Eigen vectors X_1 corresponding to λ_1 by the above method.

- To find Eigen vectors corresponding to repeated Eigen values $\lambda_2 = \lambda_3$.
- First find the rank of matrix $[A - \lambda I]$ after substituting λ_2.
- If rank = r and 'n' - number of unknowns then (n – r) number of linearly independent Eigen vectors are possible.
- If only one vector is possible then find it by above method given in (A)
- If two linearly independent vectors are possible.
- Find First Eigen vector X_2 corresponding to $\lambda_2 = \lambda_3$ by above method given in (A)
- To find third Eigen vector X_3, consider $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$
- From the equations from $X_1 X_3^T = 0$ and $X_2 X_3^T = 0$, find values l, m, n. It gives X_3.

To find Eigen vectors

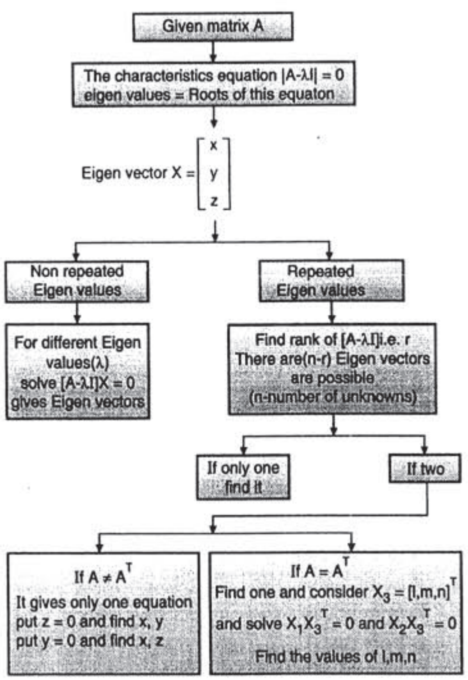

Fig. 3.3.1

Type I : Non-Repeated Eigen Values for Unsymmetric Matrix

Example 3.3.1

Find the Eigen values and Eigen vectors of the matrix.

$$A = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}$$

Solution : Given matrix is,

$$A = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}$$

The characteristic equation is,

$$|A - \lambda I| = 0$$

Since,

$$A - \lambda I = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

→ Using Scalar multiplication

$$= \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

→ Using subtraction of Matrices

$$[A - \lambda I] = \begin{bmatrix} 14-\lambda & -10 \\ 5 & -1-\lambda \end{bmatrix}$$

∴ Characteristic equations is,

$$\therefore \begin{vmatrix} 14-\lambda & -10 \\ 5 & -1-\lambda \end{vmatrix} = 0$$

$$= \begin{vmatrix} 14-\lambda & -10 \\ 5 & -1-\lambda \end{vmatrix} = 0$$

By solving determinant

$$(14 - \lambda)(-1-\lambda) + (-5)(-10) = 0$$

$$\therefore \quad (1+\lambda)(\lambda - 14) + 50 = 0$$

$$\lambda^2 - 13\lambda - 14 + 50 = 0 \Rightarrow \lambda^2 - 13\lambda + 36 = 0$$

Which is a quadratic equation,

$\therefore (\lambda - 4)(\lambda - 9) = 0$ | Factors : $(-4) + (-9) = -13$

$\lambda - 4 = $ and $\lambda - 9 = 0$ | $(-4) \times (-9) = 36$

$$\therefore \quad \lambda = 4, 9$$

Hence **Eigen values are $\lambda_1 = 4, \lambda_2 = 9$** ✓ **...Ans.**

{verification : $\lambda_1 + \lambda_2 = a_{11} + a_{22}$ and $|A| = \lambda_1 \cdot \lambda_2$}

Eigen vectors : To find Eigen vector $X = \begin{bmatrix} x \\ y \end{bmatrix}$ corresponding to Eigen values $\lambda_1 = 4, \lambda_2 = 9$

Consider $[A - \lambda I]\ X = 0$

$$\begin{bmatrix} 14-\lambda & -10 \\ 5 & -1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

For $\lambda_1 = 4$, Put $\lambda = 4$ in Equation (1), it gives,

$$\therefore \begin{bmatrix} 14-4 & -10 \\ 5 & -1-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}$$

$$\begin{array}{l} \begin{bmatrix} 10 & -10 \\ 5 & -5 \end{bmatrix} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \end{array} \end{array} \underset{\uparrow C_1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \mathbf{0}$$

→ Using Matrix Multiplication

$$\therefore \begin{bmatrix} R_1C_1 \\ R_2C_1 \end{bmatrix} = \mathbf{0}$$

$$R_1C_1 = (10)(x) - (-10)(y) = 10x - 10y$$

$$R_1C_1 = (5)(x) - (-5)(y) = 5x + 5y$$

$$\begin{bmatrix} 10x - 10y \\ 5x - 5y \end{bmatrix} = \mathbf{0}$$

$$10x - 10y = \mathbf{0}$$

$$5x - 5y = 0 \Rightarrow x = y$$

∴ For Eigen value $\lambda_1 = 4$, Eigen vector $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

...(choose x = 1)

For $\lambda_2 = 9$, Put $\lambda = 9$ in Equation (1), it gives,

$$5x - 10y = 0$$

$$5x - 10y = 0 = 5x = 10y \Rightarrow x = \frac{10}{5}y$$

$$x = 2y$$

For Eigen value $\lambda_2 = 9$ Eigen vector $X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

...(Choose y = 1)

Hence Eigen vectors are

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\lambda_1 = 4} ; X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\lambda_2 = 9} ✓$$ **...Ans.**

Ex. 3.3.2

Find Eigen values and corresponding Eigen vectors for the matrix

$$A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

Soln. : Given Matrix is,

$$A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

The characteristic equation is,

$$|A - \lambda I| = 0$$

Since,

$$A - \lambda I = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Using Scalar Multiplication

$$= \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

→ Using Subtraction of matrices

$$[A - \lambda I] = \begin{bmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{bmatrix}$$

∴ Characteristic equations is,

$$\begin{vmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{vmatrix} = 0$$

→ Using Standard formula

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \quad ...($$

Where S_1 and S_2 are sum of the minors of order 1 a
2 along the principal diagonal respectively.

$$S_1 = \begin{vmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{vmatrix} \leftarrow S_1 = \text{Sum of Diago elements}$$

$$= 4 + 3 + 1 = 8$$

(Addition of diagonal Element

$S_2 = $ [Minor of 4] + [Minor of 3] +[Minor of 1]

$$= \begin{vmatrix} \cancel{4} & \cancel{2} & \cancel{-2} \\ \cancel{-5} & 3 & 2 \\ \cancel{-2} & 4 & 1 \end{vmatrix} + \begin{vmatrix} 4 & \cancel{2} & -2 \\ \cancel{-5} & \cancel{3} & \cancel{2} \\ -2 & \cancel{4} & 1 \end{vmatrix} + \begin{vmatrix} 4 & 2 & - \\ -5 & 3 & \\ \cancel{-2} & \cancel{4} & \end{vmatrix}$$

(Sum of Minors of diagonal eleme

$$= \underbrace{[(3)(1)}_{3} - \underbrace{(4)(2)]}_{8} + \underbrace{[(4)(1)}_{4} - \underbrace{(-2)(-2)]}_{4} + \underbrace{[(4)(3)}_{12} - \underbrace{(-5)(2)]}_{-10}$$

$= (3-8)+(4-4)+(12+10)$

$= -5+0+22 = 17$

$$|A| = \begin{vmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{vmatrix}$$

$= 4 \times [\text{Minor of } 4] - 2 \times [\text{Minor of } 2] + (-2) \times [\text{Minor of } (-2)]$

$$= 4 \times \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} - (2) \times \begin{vmatrix} -5 & 2 \\ -2 & 1 \end{vmatrix} + (-2) \times \begin{vmatrix} -5 & 3 \\ -2 & 4 \end{vmatrix}$$

$$= 4 \times \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} - (2) \times \begin{vmatrix} -5 & 2 \\ -2 & 1 \end{vmatrix} + (-2) \times \begin{vmatrix} -5 & 3 \\ -2 & 4 \end{vmatrix}$$

$$= 4 \times [\underbrace{(3)(1)}_{3} - \underbrace{(4)(2)}_{8}] - 2 \times [\underbrace{(-5)(1)}_{-5} - \underbrace{(-2)(2)}_{-4}]$$

$$+ (-2) \times [\underbrace{(-5)(4)}_{-20} - \underbrace{(-2)(3)}_{-6}]$$

$= 4 \times [3-8] - 2 \times [-5+4] + (-2) \times [-20+6]$

$= 4(-5) - 2(-1) - 2(-14)$

$= -20+2+28 = 10$

Substitute values of S_1, S_2 and $|A|$ in Equation (1), it gives

$$\lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0$$

Which is quadratic equation,

→ Using synthetic Division

1	1	−8	17	−10
	–	1	−7	10
	1	−7	10	$\boxed{0}$

$\therefore \ \lambda = 1, \lambda^2 - 7\lambda + 10 = 0$

$(\lambda-5)(\lambda-2) = 0$ | Factors : $(-5)+(-2) = -7$

$\lambda - 5 = 0$ and $\lambda - 2 = 0$ | $(-5)\times(-2) = 10$

$\therefore \ \lambda = 1, 2, 5$

$\therefore$ **Eigen values are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 5$ ✓ ...Ans.**

Verification : $S_1 = \lambda_1 + \lambda_2 + \lambda_3$ and $|A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

$S_1 = 1+2+5 = 8$ and $|A| = 1 \times 2 \times 5 = 10$

Eigen vectors : To Find Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

corresponding to Eigen values, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 5$

Consider, $[A - \lambda I]X = 0$

$$\begin{bmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\because \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad ...(2)$$

For $\lambda_1 = 1$, Put $\lambda = 1$ in Equation (2), it gives,

$$\begin{bmatrix} 4-1 & 2 & -2 \\ -5 & 3-1 & 2 \\ -2 & 4 & 1-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_2 \end{array} \underset{\uparrow C_1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

→ Using Matrix Multiplication

$$\begin{bmatrix} R_1C_1 \\ R_2C_1 \\ R_3C_1 \end{bmatrix} = 0 \ ... \text{(Standard form)}$$

$$\begin{bmatrix} 3x+2y-2z \\ -5x+2y+2z \\ -2x+4y+0z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Equating corresponding elements of both sides matrices,

$3x + 2y - 2z = 0$

$-5x + 2y + 2z = 0$

$-2x + 4y + 0z = 0$

→ Using cramer's rule

$$\frac{x}{\begin{vmatrix} 2 & 2 \\ 4 & 0 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -5 & 2 \\ -2 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -5 & 2 \\ -2 & 4 \end{vmatrix}} = k$$

$$\frac{x}{(2)(0)-(4)(2)} = \frac{-y}{(-5)(0)-(-2)(2)} = \frac{z}{(-5)(4)-(-2)(2)} = k$$

$$\frac{x}{0-8} = \frac{-y}{0+4} = \frac{z}{-20+4} = k$$

$$\frac{x}{-8} = \frac{y}{-4} = \frac{z}{-16} = k$$

$x = -8k$; $y = -4k$; $z = -16k$

...$\left(\text{use } k = -\frac{1}{4} \text{ to get smallest integers}\right)$

For Eigen value $\lambda_1 = 1$, Eigen vector is $X_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

For $\lambda_2 = 2$, Put $\lambda = 2$ in Equation (2), it gives,

$$\begin{bmatrix} 4-2 & 2 & -2 \\ -5 & 3-2 & 2 \\ -2 & 4 & 1-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array} \underset{\uparrow C_1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = 0$$

→ Using Matrix Multiplication

$$\begin{bmatrix} R_1C_1 \\ R_2C_1 \\ R_3C_1 \end{bmatrix} = 0 \qquad ... \text{(standard form)}$$

$$\begin{bmatrix} 2x+2y+2z \\ -5x+1y+2z \\ -2x+4y-1z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Equating corresponding elements of both sides matrices,

$2x + 2y - 2z = 0$

$-5x + y + 2z = 0$

$-2x + 4y - z = 0$

→ Using cramer's rule

$$\frac{x}{\begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -5 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -5 & 1 \\ -2 & 4 \end{vmatrix}} = k$$

$$\frac{x}{(1)(-1)-(4)(2)} = \frac{-y}{(-5)(-1)-(-2)(2)}$$

$$= \frac{z}{(-5)(4)-(-2)(1)} = k$$

$$\frac{x}{-1-8} = \frac{-y}{5+4} = \frac{z}{-20+2} = k$$

$$\frac{x}{-9} = \frac{y}{-9} = \frac{z}{-18} = k$$

$x = -9k$; $y = -9k$; $z = -18k$

$\therefore$ For Eigen value $\lambda_2 = 2$, Eigen vector is $X_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

...$\left(\text{use } k = \frac{-1}{9} \text{ to get smallest integers}\right)$

For $\lambda_3 = 5$, Put $\lambda = 5$ in Equation (2), it gives,

$$\begin{bmatrix} 4-5 & 2 & -2 \\ -5 & 3-5 & 2 \\ -2 & 4 & 1-5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_2 \end{array} \underset{\uparrow C_1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} R_1C_1 \\ R_2C_1 \\ R_3C_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -x+2y-2z \\ -5x-2y+2z \\ -2x+4y-4z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Equating corresponding elements of both sides matrices,

$-x + 2y - 2z = 0$

$-5x - 2y + 2z = 0$

$-2x + 4y - 4z = 0$

→ Using cramer's rule

$$\frac{x}{\begin{vmatrix} -2 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -5 & 2 \\ -2 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -5 & -2 \\ -2 & 4 \end{vmatrix}} = k$$

$$\frac{x}{(-2)(-4)-(4)(2)} = \frac{-y}{(-5)(-4)-(-2)(2)}$$

$$= \frac{z}{(-5)(4)-(-2)(-2)} = k$$

$$\frac{x}{8-8} = \frac{-y}{20+4} = \frac{z}{-20-4} = k$$

$$\frac{x}{0} = \frac{y}{-24} = \frac{z}{-24} = k$$

$x = 0$; $y = -24k$; $z = -24k$

...$\left(\text{Choose } k = -\frac{1}{24}, \text{ to get smallest integer}\right)$

$\therefore$ For Eigen value $\lambda_3 = 5$, Eigen vector is $X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Hence Eigen vectors are

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}_{\lambda_1 = 1} ; X_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{\lambda_2 = 2} ;$$

$$X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_{\lambda_3 = 5} \checkmark \qquad \text{...Ans.}$$

Example 3.3.3

Find Eigen values and Eigen vector corresponding to lowest Eigen value for the matrix.

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 3 & -2 & 3 \\ 0 & 3 & 0 \end{bmatrix}$$

Solution :

Step I : The characteristic equation of matrix A is,

$$|A - \lambda I| = 0$$

Since,

$$A - \lambda I = \begin{bmatrix} 0 & 2 & 0 \\ 3 & -2 & 3 \\ 0 & 3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Using Scalar Multiplication

$$[A - \lambda I] = \begin{bmatrix} 0 & 2 & 0 \\ 3 & -2 & 3 \\ 0 & 3 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

→ Using Subtraction of Matrices

$$[A - \lambda I] = \begin{bmatrix} 0-\lambda & 2 & 0 \\ 3 & -2-\lambda & 3 \\ 0 & 3 & 0-\lambda \end{bmatrix}$$

$\therefore$ Characteristic equations is,

$$\begin{vmatrix} 0-\lambda & 2 & 0 \\ 3 & -2-\lambda & 3 \\ 0 & 3 & 0-\lambda \end{vmatrix} = 0$$

→ Using Standard formula

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - |A| = 0 \qquad \text{...(1)}$$

Where S_1 and S_2 are sum of minors of order one and two along diagonal elements respectively.

$$S_1 = \begin{vmatrix} 0 & 2 & 0 \\ 3 & -2 & 3 \\ 0 & 3 & 0 \end{vmatrix} \leftarrow S_1 = \text{Sum of Diagonal elements}$$

$= 0 - 2 + 0$ (Addition of diagonal Elements)

$= -2$

S_2 = Minor of 0 + Minor of (– 2) + Minor of 0

$$= \begin{vmatrix} \cancel{0} & \cancel{2} & \cancel{0} \\ \cancel{3} & -2 & 3 \\ \cancel{0} & 3 & 0 \end{vmatrix} + \begin{vmatrix} 0 & \cancel{2} & 0 \\ \cancel{3} & \cancel{-2} & \cancel{3} \\ 0 & \cancel{3} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 2 & \cancel{0} \\ 3 & -2 & \cancel{3} \\ \cancel{0} & \cancel{3} & \cancel{0} \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 3 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix}$$

(Sum of Minors of diagonal elements)

$$= [\underbrace{(-2)(0)}_{0} - \underbrace{(3)(3)}_{9}] + [\underbrace{(0)(0)}_{0} - \underbrace{(0)(0)}_{0}] + [\underbrace{(0)(-2)}_{-2} - \underbrace{(3)(2)}_{6}]$$

$$= (0-9) + 0 + (0-6) = -9 - 6 = -15$$

$S_2 = -15$

$$A| = \begin{vmatrix} 0 & 2 & 0 \\ 3 & -2 & 3 \\ 0 & 3 & 0 \end{vmatrix}$$

$= 0 \times$ [Minor of 0] – (2) × [Minor of 2] + 0 × [Minor of 0]

$$= -(2) \times [\text{Minor of } 2] = 2 \times \begin{vmatrix} 0 & \cancel{2} & 0 \\ \cancel{3} & \cancel{-2} & \cancel{3} \\ 0 & \cancel{3} & 0 \end{vmatrix}$$

$$= 2 \times \begin{vmatrix} 3 & 3 \\ 0 & 0 \end{vmatrix} = 2 \times [\underbrace{(3)(0)}_{0} - \underbrace{(0)(3)}_{0}] = 2 \times 0 = 0$$

$|A| = 0$

Substitute values of S_1, S_2 and $|A|$ in Equation (1)

$\therefore \lambda^3 + 2\lambda^2 - 15\lambda = 0$

$\lambda(\lambda^2 + 2\lambda - 15) = 0$

$\lambda = 0;\ \lambda^2 + 2\lambda - 15 = 0$ Which is quadratic equation,

$\lambda = 0;\ (\lambda + 5)(\lambda - 3) = 0$

$\lambda = 0;\ \lambda + 5 = 0$ and $\lambda - 3 = 0$

$\lambda = 0, \lambda = -5, \lambda = 3$

Eigen values for matrix A.

$$\therefore \boldsymbol{\lambda = -5, 0, 3} \checkmark \qquad \text{...Ans.}$$

[Verification : $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = (-5)(0)(3) = 0 = |A|$ and

$\lambda_1 + \lambda_2 + \lambda_3 = -5 + 0 + 3 = -2 = a_{11} + a_{22} + a_{33}$]

Step II : Eigen vectors :

Consider Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $[A - \lambda I] X = 0$

$$\begin{bmatrix} 0-\lambda & 2 & 0 \\ 3 & -2-\lambda & 3 \\ 0 & 3 & 0-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\because \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{...(2)}$$

For $\lambda = -5$ (lowest Eigen value),

Put $\lambda = -5$ in Equation (1), it gives,

$$\begin{bmatrix} 0-(-5) & 2 & 0 \\ 3 & -2-(-5) & 3 \\ 0 & 3 & 0-(-5) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} \begin{bmatrix} 5 & 2 & 0 \\ 3 & 3 & 3 \\ 0 & 3 & 5 \end{bmatrix} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \uparrow C_1 \end{array}$$

$$\begin{bmatrix} R_1 C_1 \\ R_2 C_1 \\ R_3 C_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5x + 2y + 0z \\ 3x + 3y + 3z \\ 0x + 3y + 5z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By equating corresponding elements of both sides matrices.

$\therefore 5x + 2y + 0z = 0;$

$3x + 3y + 3z = 0;$

$0x + 3y + 5z = 0$

→ Using Cramer's rule

$$\frac{x}{\begin{vmatrix} 3 & 3 \\ 3 & 5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 3 \\ 0 & 5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 3 \\ 0 & 3 \end{vmatrix}} = k$$

$$\frac{x}{(3)(5) - (3)(3)} = \frac{-y}{(3)(5) - (0)(3)} = \frac{z}{(3)(3) - (0)(3)} = k$$

$$\frac{x}{15-9} = \frac{-y}{15-0} = \frac{z}{9-0} = k$$

$$\frac{x}{6} = \frac{-y}{15} = \frac{z}{9} = k$$

$x = 6k,\ y = -15k,\ z = 9k$

Choose $k = \frac{1}{3}$, to get smallest integer

$$\therefore X_1 = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}_{\lambda = -5} \text{ is Eigen vector.} \checkmark \qquad \text{...Ans.}$$

Example 3.3.4

Find Eigen values and Eigen vector corresponding to highest Eigen value for the matrix.

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Solution :

Step I : The characteristic of given matrix A is

$$|A - \lambda I| = 0$$

Since,

$$A - \lambda I = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Using Scalar Multiplication

$$= \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

→ Using subtraction of matrices

$$[A - \lambda I] = \begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix}$$

$\therefore$ Characteristic equations is,

$$\begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

→ Using Standard formula

$$\therefore \lambda^3 - S_1 \lambda^2 + S_2 \lambda - |A| = 0 \qquad \text{...(1)}$$

Where S_1 and S_2 are sum of minors of order one and two along diagonal elements respectively.

$$\therefore S_1 = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \leftarrow S_1 = \text{Sum of Diagonal elements}$$

$= 1 + 2 - 1$ (Addition of diagonal Elements)

$= 2$

S_2 = [Minor of 1] + [Minor of 2] + [Minor of (– 1)]

$$= \begin{vmatrix} \cancel{1} & \cancel{1} & \cancel{-2} \\ \cancel{-1} & 2 & 1 \\ \cancel{0} & 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & \cancel{1} & -2 \\ \cancel{-1} & \cancel{2} & \cancel{1} \\ 0 & \cancel{1} & -1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & \cancel{-2} \\ -1 & 2 & \cancel{1} \\ \cancel{0} & \cancel{1} & \cancel{-1} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$$

(Sum of Minors of diagonal elements)

$$= [\underbrace{(2)(-1)}_{-2} - \underbrace{(1)(1)}_{1}] + [\underbrace{(1)(-1)}_{-1} - \underbrace{(0)(-2)}_{0}] + [\underbrace{(1)(2)}_{2} - \underbrace{(-1)(1)}_{-1}]$$

$= (-2-1) + (-1-0) + (2+1)$

$= -3 - 1 + 3 = -1$

$S_2 = -1$

$$|A| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 2 & 1 \\ 0 & 3 & -1 \end{vmatrix}$$

$= 1 \times$ [Minor of 1] $- (1) \times$ [Minor of 1] $+ (-2) \times$ [Minor of (– 2)]

$$= 1 \times \begin{vmatrix} \cancel{1} & \cancel{1} & \cancel{-2} \\ \cancel{-1} & 2 & 1 \\ \cancel{0} & 1 & -1 \end{vmatrix} - (1) \times \begin{vmatrix} \cancel{1} & \cancel{1} & \cancel{-2} \\ -1 & \cancel{2} & 1 \\ 0 & \cancel{1} & -1 \end{vmatrix} + (-2) \times \begin{vmatrix} \cancel{1} & \cancel{1} & \cancel{-2} \\ -1 & 2 & \cancel{1} \\ 0 & 1 & \cancel{-1} \end{vmatrix}$$

$$= 1 \times \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} - (1) \times \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} + (-2) \times \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= [\underbrace{(2)(-1)}_{-2} - \underbrace{(1)(1)}_{1}] - (1) \times [\underbrace{(-1)(-1)}_{1} - \underbrace{(0)(1)}_{0}] + (-2) \times [\underbrace{(-1)(1)}_{-1} - \underbrace{(0)(2)}_{0}]$$

$= (1)[-2-1] - (1)[1-0] + (-2)[-1-0]$

$|A| = (1)(-3) - 1 - 2(-1) = -3 - 1 + 2 = -2$

$|A| = -1$

Substitute the values of S_1, S_2 and $|A|$ in equation (1),

$\therefore \lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

$\lambda^2(\lambda - 2) - (\lambda - 2) = 0 \Rightarrow (\lambda - 2)(\lambda^2 - 1) = 0$

$(\lambda - 2)(\lambda - 1)(\lambda + 1) = 0 \quad ...[\because a^2 - b^2 = (a-b)(a+b)]$

$\therefore \lambda = 2, 1, -1$

$\therefore$ **Eigen values are $\lambda = -1, 1, 2$ ✓** ...**Ans.**

$\left\{ \begin{array}{l} \textbf{Verification}: \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = (-1)(1)(2) = -2 = |A| \text{ and} \\ \lambda_1 + \lambda_2 + \lambda_3 = -1 + 1 + 2 = 2 = a_{11} + a_{22} + a_{33} \end{array} \right\}$

Step II : Eigen vectors : Consider the Eigen vector

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } [A - \lambda I]X = 0$$

$$\begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\because \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ...(2)$$

For $\lambda = 2$ [Highest Eigen value], Put $\lambda = 2$ in Equation (2), it gives,

$$\begin{bmatrix} 1-2 & 1 & -2 \\ -1 & 2-2 & 1 \\ 0 & 1 & -1-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{bmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

C_1

→ Using matrix multiplication

$$\begin{bmatrix} R_1C_1 \\ R_2C_1 \\ R_3C_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ...\text{(Standard form)}$$

$R_1C_1 = (-1)(x) + (1)(y) + (-2)(z) = (-x) + y - 2z$

$R_2C_1 = (-1)(x) + (0)(y) + (1)(z) = (-x) + 0y + 1z$

$R_3C_1 = (0)(x) + (1)(y) + (-3)(z) = 0x + 1y - 3z$

$$\begin{bmatrix} -1x + 1y - 2z \\ -1x + 0y + 1z \\ 0x + 1y - 3z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By equating corresponding elements of both sides matrices.

$-x + y - 2z = 0; -x + 0y + z$

$= 0; 0x + y - 3z = 0$

→ Using Cramer's rule

$$\frac{x}{\begin{vmatrix} 0 & 1 \\ 1 & -3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -1 & 1 \\ 0 & -3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}} = k$$

$$\frac{x}{(0)(-3)-(1)(1)} = \frac{-y}{(-1)(-3)-(0)(1)} = \frac{z}{(-1)(1)-(0)(0)} = k$$

$$\frac{x}{0-1} = \frac{-y}{3-0} = \frac{z}{-1-0} = k$$

$$\frac{x}{-1} = \frac{-y}{3} = \frac{z}{-1} = k$$

$x = -k, y = -3k, z = -k$

choose $k = -1$ to get smallest integer

$$\therefore \; x_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}_{\lambda = 2} \checkmark \quad ...\textbf{Ans.}$$

Example 3.3.5

Determine the Eigen values and Eigen vectors of A :

$$\mathbf{A} = \begin{bmatrix} \mathbf{4} & \mathbf{6} & \mathbf{6} \\ \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{-1} & \mathbf{-4} & \mathbf{-3} \end{bmatrix}$$

Solution : Given matrix is,

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

The characteristic equation is,

$$|A - \lambda I| = 0$$

Since,

$$A - \lambda I = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Using scalar multiplication

$$= \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

→ Using subtraction of matrices

$$[A - \lambda I] = \begin{bmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{bmatrix}$$

$\therefore$ Characteristic equations is,

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \quad ...(1)$$

Where S_1 and S_2 are sum of the minors of order one and two along the principal diagonal respectively.

$$S_1 = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} \leftarrow S_1 = \text{Sum of Diagonal elements}$$

$= 4 + 3 - 3 = 4$

(Addition of diagonal Elements)

$S_2 = 4 \times$ [Minor of 4] + [Minor of 3] + [Minor of (– 3)]

$$= \begin{vmatrix} \cancel{4} & \cancel{6} & \cancel{6} \\ \cancel{1} & 3 & 2 \\ \cancel{-1} & -4 & -3 \end{vmatrix} + \begin{vmatrix} 4 & \cancel{6} & 6 \\ \cancel{1} & \cancel{3} & \cancel{2} \\ -1 & \cancel{-4} & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 & \cancel{6} \\ 1 & 3 & \cancel{2} \\ \cancel{-1} & \cancel{-4} & \cancel{-3} \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 2 \\ -4 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}$$

(Sum of Minors of diagonal elements)

$= [(3)(-3) - (-4)(2)] + [(4)(-3) - (-1)(6)] + [(4)(3) - (1)(6)]$

$= [-9 + 8] + [-12 + 6] + [12 - 6]$

$S_2 = -1 - 6 + 6 = -1$

$$|A| = \begin{vmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{vmatrix} \quad ...\textbf{Ans.}$$

$= 4 \times$ [Minor of 4] $- (6) \times$ [Minor of 6] $+ 6 \times$ [Minor of 6]

$$= 4 \times \begin{vmatrix} \cancel{4} & \cancel{6} & \cancel{6} \\ \cancel{1} & 3 & 2 \\ \cancel{-1} & -4 & -3 \end{vmatrix} - (6) \times \begin{vmatrix} \cancel{4} & \cancel{6} & \cancel{6} \\ 1 & \cancel{3} & 2 \\ -1 & \cancel{-4} & -3 \end{vmatrix} + 6 \times \begin{vmatrix} \cancel{4} & \cancel{6} & \cancel{6} \\ 1 & 3 & \cancel{2} \\ -1 & -4 & \cancel{-3} \end{vmatrix}$$

$$= 4 \times \begin{vmatrix} 3 & 2 \\ -4 & -3 \end{vmatrix} - 6 \times \begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix} + 6 \times \begin{vmatrix} 1 & 3 \\ -1 & -4 \end{vmatrix}$$

$= 4 \times [(3)(-3) - (-4)(2)] - 6 \times [(1)(-3) + (-1)(2)] + 6 \times [(1)(-4) - (-1)(3)]$

$$= 4\times[-9+8]-6\times[-3+2]+6\times[-4+3]$$
$$= 4(-1)-6(-1)+6(-1)$$
$$= -4+6-6=-4$$

Substitute values of S_1, S_2 and $|A|$ in equation (1),

$$\therefore \quad \lambda^3-4\lambda^2-\lambda+4 = 0 \Rightarrow \lambda^2(\lambda-4)-(\lambda-4)=0$$
$$\therefore \quad (\lambda-4)(\lambda^2-1) = 0 \Rightarrow (\lambda-4)(\lambda-1)(\lambda+1)=0$$

➔ **Using standard formula :** $\quad ...[a^2-b^2=(a-b)(a+b)]$

$$\Rightarrow \lambda = 4, 1, -1$$

Hence Eigen values are $\boldsymbol{\lambda_1=1, \lambda_2=-1, \lambda_3=4}$ ✓ **...Ans.**

{**Verification :** $S_1=\lambda_1+\lambda_2+\lambda_3$

$$S_1 = 4+1-1=4$$

and $|A| = \lambda_1\cdot\lambda_2\cdot\lambda_3$

$$|A| = 4\times(1)\times(-1)=-4\}$$

Eigen vectors : To find Eigen vector $X = \begin{bmatrix} x\\ y\\ z\end{bmatrix}$ corresponding to Eigen values $\lambda_1=1, \lambda_2=-1, \lambda_3=4$

Consider, $[A-\lambda I]\,X = 0$

$$\begin{bmatrix} 4-\lambda & 6 & 6\\ 1 & 3-\lambda & 2\\ -1 & -4 & -3-\lambda\end{bmatrix}\begin{bmatrix} x\\ y\\ z\end{bmatrix} = \mathbf{0}$$

$$\because \mathbf{0} = \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix} \qquad ...(2)$$

For $\lambda_1=1$, Put $\lambda=1$ in Equation (2), it gives,

$$\begin{bmatrix} 4-1 & 6 & 6\\ 1 & 3-1 & 2\\ -1 & -4 & -3-1\end{bmatrix}\begin{bmatrix} x\\ y\\ z\end{bmatrix} = 0$$

$$\begin{array}{c} \\ \begin{bmatrix} 3 & 6 & 6\\ 1 & 2 & 2\\ -1 & -4 & -4\end{bmatrix}\begin{array}{l} \leftarrow R_1\\ \leftarrow R_2\\ \leftarrow R_3\end{array} \end{array} \begin{array}{c} \begin{bmatrix} x\\ y\\ z\end{bmatrix} \\ \uparrow \\ C_1 \end{array} = \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}$$

➔ **Using matrix multiplication**

$$\begin{bmatrix} R_1C_1\\ R_2C_1\\ R_3C_1\end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix} \qquad ...\text{(Standard form)}$$

$$R_1C_1 = (3)(x)+(6)(y)+(6)(z) = 3x+6y+6z$$
$$R_2C_1 = (1)(x)+(2)(y)+(2)(z) = 1x+2y+2z$$
$$R_3C_1 = (-1)(x)+(-4)(y)+(-4)(z) = -1x-4y-4z$$

$$\begin{bmatrix} 3x+6y+6z\\ 1x+2y+2z\\ -1x-4y-4z\end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}$$

By equating corresponding elements of both sides matrices

$$3x+6y+6z = 0$$
$$x+2y+2z = 0$$
$$-x-4y-4z = 0$$

➔ **Using Cramer's rule**

$$\frac{x}{\begin{vmatrix} 2 & 2\\ -4 & -4\end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 2\\ -1 & -4\end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2\\ -1 & -4\end{vmatrix}} = k$$

$$\frac{x}{(-2)(-4)-(4)(2)} = \frac{-y}{(1)(-4)-(-1)(2)}$$
$$= \frac{z}{(1)(-4)-(-1)(2)} = k$$
$$\frac{x}{-8+8} = \frac{-y}{-4+2} = \frac{z}{-4+2} = k$$
$$\frac{x}{0} = \frac{-y}{-2} = \frac{z}{-2} = k$$

$$x=0; \quad y=2k; \quad z=-2k$$

$\therefore$ For eigen value $\lambda_1 = 1$,

$$...\left(\text{Choose } k=\frac{1}{2} \text{ to get lowest integers}\right)$$

Eigen vector is $X_1 = \begin{bmatrix} 0\\ 1\\ -1\end{bmatrix}$

For $\lambda_2=-1$, Put $\lambda=-1$ in Equation (2), it gives,

$$\begin{bmatrix} 4-(-1) & 6 & 6\\ 1 & 3-(-1) & 2\\ -1 & -4 & -3-(-1)\end{bmatrix}\begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}$$

$$\begin{array}{c} \\ \begin{bmatrix} 5 & 6 & 6\\ 1 & 4 & 2\\ -1 & -4 & -2\end{bmatrix}\begin{array}{l} \leftarrow R_1\\ \leftarrow R_2\\ \leftarrow R_3\end{array} \end{array} \begin{array}{c} \begin{bmatrix} x\\ y\\ z\end{bmatrix} \\ \uparrow \\ C_1 \end{array} = \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix} R_1C_1\\ R_2C_1\\ R_3C_1\end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}$$

$$R_1C_1 = (5)(x)+6(y)+(6)(z) = 5x+6y+6z$$
$$R_2C_1 = (1)(x)+(4)(y)+(2)(z) = 1x+4y+2z$$
$$R_3C_1 = (-1)(x)+(-4)(y)+(-2)(z) = -1x-4y-2z$$

$$\begin{bmatrix} 5x+6y+6z\\ 1x+4y+2z\\ -1x-4y-2z\end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}$$

By equating corresponding elements of both sides matrices

$$5x+6y+6z = 0$$
$$x+4y+2z = 0$$
$$x-4y-2z = 0$$

➔ **Using Cramer's rule**

$$\frac{x}{\begin{vmatrix} 6 & 6\\ 4 & 2\end{vmatrix}} = \frac{-y}{\begin{vmatrix} 5 & 6\\ 1 & 2\end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & 6\\ 1 & 4\end{vmatrix}} = k$$

($\because$ Last two equations are identical)

$$\frac{x}{(6)(2)-(4)(6)} = \frac{-y}{(5)(2)-(1)(6)}$$
$$= \frac{z}{(5)(4)-(1)(6)} = k$$
$$\frac{x}{12-24} = \frac{-y}{10-6} = \frac{z}{20-6} = k$$
$$\frac{x}{-12} = \frac{-y}{4} = \frac{z}{14} = k$$

$$x = -12k; \quad y = -4k; \quad z = 14k$$

$$...\left(\text{Choose } k=\frac{1}{2} \text{ to get lowest integers}\right)$$

For Eigen value $\lambda_2=-1$, Eigen vector is $X_2 = \begin{bmatrix} 6\\ 2\\ -7\end{bmatrix}$

For $\lambda_3=4$, Put $\lambda=4$ in Equation (2), it gives,

$$\begin{bmatrix} 4-4 & 6 & 6\\ 1 & 3-4 & 2\\ -1 & -4 & -3-4\end{bmatrix}\begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}$$

$$\begin{array}{c} \\ \begin{bmatrix} 0 & 6 & 6\\ 1 & -1 & 2\\ -1 & -4 & -7\end{bmatrix}\begin{array}{l} \leftarrow R_1\\ \leftarrow R_2\\ \leftarrow R_3\end{array} \end{array} \begin{array}{c} \begin{bmatrix} x\\ y\\ z\end{bmatrix} \\ \uparrow \\ C_1 \end{array} = \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}$$

➔ **Using matrix multiplication**

$$\begin{bmatrix} R_1C_1\\ R_2C_1\\ R_3C_1\end{bmatrix} = \begin{bmatrix} C\\ C\\ C\end{bmatrix} \qquad ...\text{(Standard form)}$$

$$R_1C_1 = (0)(x)+(6)(y)+(6)(z) = 0x+6y+6z$$
$$R_2C_1 = (1)(x)+(-1)(y)+(2)(z) = 1x-1y+2z$$
$$R_3C_1 = (-1)(x)+(-4)(y)+(-7)(z) = -1x-4y-7z$$

$$\begin{bmatrix} 0x+6y+6z\\ 1x-1y+2z\\ -1x-4y-7z\end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}$$

By equating corresponding elements of both sides matrices

$$0x+6y+6z = 0$$
$$x-y+2z = 0$$
$$-x-4y-7z = 0$$

➔ **Using Cramer's rule**

$$\frac{x}{\begin{vmatrix} -1 & 2\\ -4 & -7\end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 2\\ -1 & -7\end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1\\ -1 & -4\end{vmatrix}} = k$$

$$\frac{x}{(-1)(-7)-(-4)(2)} = \frac{y}{(1)(-7)-(-1)(2)}$$
$$= \frac{z}{(1)(-4)-(-1)(-1)} = k$$
$$\frac{x}{7+8} = \frac{y}{-7+2} = \frac{z}{-4-1} = k$$
$$\frac{x}{15} = \frac{-y}{-5} = \frac{z}{-5} = k$$

$$x = 15k; \quad y = 5k; \quad z = -5k$$

$\therefore$ For Eigen value $\lambda_3=4$ Eigen vector is $X_3 = \begin{bmatrix} 3\\ 1\\ -1\end{bmatrix}$

$$...\left(\text{Choose } k=\frac{1}{2} \text{ to get lowest integers}\right)$$

Hence Eigen vectors are

$$\mathbf{X_1} = \begin{bmatrix} 0\\ 1\\ -1\end{bmatrix}_{\lambda_1=1};$$

$$\mathbf{X_2} = \begin{bmatrix} 6\\ 2\\ -7\end{bmatrix}_{\lambda_2=-1};$$

$$\mathbf{X_3} = \begin{bmatrix} 3\\ 1\\ -1\end{bmatrix}_{\lambda_3=4} \checkmark \qquad \textbf{...Ans.}$$

Example 3.3.6 **Dr. BATU - May 15, Nov. 16**

Find Eigen values and Eigen vectors of the matrix.

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Solution :

Step I : Given matrix A is,

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

A is upper triangular matrix

[**Upper triangular matrix :** All elements below diagonal elements are zero]

Use : [Eigen values of triangular (upper or lower matrix are diagonal elements)]

$\therefore$ Eigen values are 3, 2, 5

$\therefore$ **λ = 2, 3, 5 are the eigen values of matrix A** ✓ **...Ans.**

Step II : Eigen vectors

Consider Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $[A - \lambda I]\, X = 0$

$$A - \lambda I = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} - \lambda \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 1-0 & 4-0 \\ 0-0 & 2-\lambda & 6-0 \\ 0-0 & 0-0 & 5-\lambda \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

To find Eigen vector consider $[A - \lambda I]\, X = 0$

$$\therefore \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\because \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad ...(1)$$

For $\lambda = 2$ Put $\lambda = 2$ Equation (1),

$$\therefore \begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{C_1} = \mathbf{0}$$

➔ Using matrix multiplication

$$\therefore \begin{bmatrix} R_1C_1 \\ R_2C_1 \\ R_3C_1 \end{bmatrix} = 0 \quad \text{...(Standard form)}$$

$$\begin{bmatrix} 1x + 1y + 4z \\ 0x + 0y + 6z \\ 0x + 0y + 3z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Equating corresponding elements of both sides matrices,

$\therefore 1x + 1y + 4z = 0$

$0x + 0y + 6z = 0$

$0x + 0y + 3z = 0$

➔ Using Cramer's Rule

$$\frac{x}{\begin{vmatrix} 1 & 4 \\ 0 & 6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 4 \\ 0 & 6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}}$$

$$\frac{x}{(1\times 6)-(4\times 0)} = \frac{-y}{(1\times 6)-(6\times 4)} = \frac{z}{(1\times 0)-(1\times 0)}$$

$$\frac{x}{6-0} = \frac{-y}{6-0} = \frac{z}{0-0}$$

$$\frac{x}{6} = \frac{-y}{6} = \frac{z}{0}$$

$$\frac{x}{6} = \frac{y}{-1} = \frac{z}{0} \quad \text{(Divide denominators by 6)}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \left(\textbf{First Eigen vector coresponding to Eigen Value } \boldsymbol{\lambda = 2}\right) ✓ \quad \textbf{...Ans.}$$

For $\lambda = 3$

Put $\lambda = 3$ in Equation (1),

$$\therefore \begin{bmatrix} 3-3 & 1 & 4 \\ 0 & 2-3 & 6 \\ 0 & 0 & 5-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

➔ Using matrix multiplication

$$\begin{bmatrix} 0x + 1y + 4z \\ 0x + 1y + 6z \\ 0x + 0y + 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Equating corresponding elements of both sides matrices,

$\therefore 0x + 1y + 4z = 0$

$0x - 1y + 6z = 0$

$0x + 0y + 2z = 0$

➔ Using Cramer's Rule

$$\frac{x}{\begin{vmatrix} -1 & 6 \\ 0 & 2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 0 & 6 \\ 0 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix}}$$

$$\frac{x}{(-1\times 2)-(0\times 6)} = \frac{-y}{(0\times 2)-(0\times 6)}$$

$$= \frac{z}{(0\times 0)-(0\times(-1))}$$

$$\frac{x}{-2-0} = \frac{-y}{0-0} = \frac{z}{0-0}$$

$$\frac{x}{-2} = \frac{y}{0} = \frac{z}{0}$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0} \quad \left(\text{Divide denominators by } (-2)\right)$$

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \left(\textbf{Second Eigen vector coresponding to Eigen Value } \boldsymbol{\lambda = 3}\right) ✓ \quad \textbf{...Ans.}$$

For $\lambda = 5$

Put $\lambda = 5$, in Equation (1),

$$\begin{bmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2x + 1y + 4z \\ 0x - 3y + 6z \\ 0x + 0y + 0z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Equating corresponding elements of both sides matrices,

$\therefore -2x + 1y + 4z = 0$

$0x - 3y + 6z = 0$

$0x + 0y + 0z = 0$

➔ Using Cramer's Rule

$$\frac{x}{\begin{vmatrix} 1 & 4 \\ -3 & 6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & 4 \\ 0 & 6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix}}$$

$$\frac{x}{(1\times 6)-(-3\times 4)} = \frac{-y}{(-2\times 6)-(0\times 4)}$$

$$= \frac{z}{(-2\times(-3))-(0\times 1)}$$

$$\frac{x}{6+12} = \frac{-y}{-12-0} = \frac{z}{6-0}$$

$$\frac{x}{18} = \frac{-y}{-12} = \frac{z}{6} \quad \left(\text{Divide denominators by 6}\right)$$

$$\frac{x}{3} = \frac{y}{2} = \frac{z}{1}$$

(Third eigen vector corresponding to $\lambda = 5$)

$$\therefore X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \left(\textbf{Third eigen vector corresponding to } \boldsymbol{\lambda = 5}\right) ✓ \quad \textbf{...Ans.}$$

Type II : Examples on Non-Repeated Eigen Values for Symmetric Matrix

Example 3.3.7

Find the Eigen value and Eigen vectors for the following Matrix :

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Solution : Given matrix is,

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation is,

$$|A - \lambda I| = 0$$

Since, $A - \lambda I = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

→ Using Scalar Multiplication

$$A - \lambda I = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

→ Using Subtraction of Matrices

$$[A - \lambda I] = \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}$$

$\therefore$ Characteristic equations is,

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - |A| = 0$$

Where S_1 and S_2 are sum of the minors of order 1 and 2 along the principal diagonal respectively.

$$S_1 = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

S_1 = sum of diagonal elements

$= 3 + 5 + 3 = 11$ (Addition of diagonal Elements)

S_2 = [Minor of 3] + [Minor of 5] + [Minor of 3]

$$S_2 = \begin{vmatrix} \cancel{3} & \cancel{-1} & \cancel{1} \\ \cancel{-1} & 5 & -1 \\ \cancel{1} & -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & \cancel{-1} & 1 \\ \cancel{-1} & \cancel{5} & \cancel{-1} \\ 1 & \cancel{-1} & 3 \end{vmatrix} + \begin{vmatrix} \cancel{3} & \cancel{-1} & \cancel{1} \\ -1 & 5 & \cancel{-1} \\ 1 & -1 & \cancel{3} \end{vmatrix}$$

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

(Sum of Minors of diagonal elements)

$$= [\underbrace{(5)(3)}_{15} - \underbrace{(-1)(-1)}_{1}] + [\underbrace{(3)(3)}_{9} - \underbrace{(1)(1)}_{1}] + [\underbrace{(3)(5)}_{15} - \underbrace{(1)(-1)}_{1}]$$

$= [15 - 1] + [9 - 1] + [15 - 1]$

$= 14 + 8 + 14 = 36$

$$|A| = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$= 3 \times$ [Minor of 3] $- (-1) \times$ [Minor of (-1)] $+ (1) \times$ [Minor of 1]

$$= 3\times \begin{vmatrix} \cancel{3} & \cancel{-1} & \cancel{1} \\ \cancel{-1} & 5 & -1 \\ \cancel{1} & -1 & 3 \end{vmatrix} + 1\times \begin{vmatrix} \cancel{3} & \cancel{-1} & \cancel{1} \\ -1 & \cancel{5} & -1 \\ 1 & \cancel{-1} & 3 \end{vmatrix} + 1\times \begin{vmatrix} \cancel{3} & \cancel{-1} & \cancel{1} \\ -1 & 5 & \cancel{-1} \\ 1 & -1 & \cancel{3} \end{vmatrix}$$

$$= 3\times \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + 1\times \begin{vmatrix} -1 & -1 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} -1 & 5 \\ 1 & -1 \end{vmatrix}$$

$$= 3\times[\underbrace{(5)(3)}_{15} - \underbrace{(-1)(-1)}_{1}] + 1\times[\underbrace{(-1)(3)}_{-3} - \underbrace{(1)(-1)}_{-1}] + 1\times[\underbrace{(-1)(-1)}_{1} - \underbrace{(1)(5)}_{5}]$$

$= 3 \times [15 - 1] + 1 \times [-3 + 1] + 1 \times [1 - 5]$

$= 3(14) + 1(-2) + 1(-4)$

$= 42 - 2 - 4 = \mathbf{36}$

→ Using Synthetic division

$\therefore \lambda^3 - 11\lambda^2 + 36\lambda - 36$

$\lambda = 2, \quad (\lambda^2 - 9\lambda + 18)$

$(\lambda - 6)(\lambda - 3) = 0$

$\therefore \quad \lambda = 2, 3, 6$

2	1	−11	36	−36
	–	2	−18	36
	1	−9	18	$\boxed{0}$

Hence the Eigen values are $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6$ ✓ ...Ans.

{Verification, $S_1 = \lambda_1 + \lambda_2 + \lambda_3$; $|A| = \lambda_1 \lambda_2 \lambda_3\}$

Eigen vectors :

To find Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to Eigen values $\lambda_1 = 2, \; \lambda_2 = 3; \; \lambda_3 = 6$

Consider, $[A - \lambda I] X = 0$

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\because \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(1)}$$

For $\lambda_1 = 2$, Put $\lambda = 2$ in Equation (1), it gives,

By Equating corresponding elements of both sides matrices

$x - y + z = 0;$

$-x + 3y - z = 0;$

$x - y + z = 0$

→ Using Cramer's Rule

$$\frac{x}{\begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 3 \\ 1 & -1 \end{vmatrix}} = k$$

$\therefore \quad x = 2k, \quad y = 0, \quad z = -2k$

$\therefore$ For eigen values $\lambda_2 = 2$, Eigen vector is $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

For $\lambda_2 = 3$, Put $\lambda = 3$ in Equation (1), it gives,

By Equating corresponding elements of both sides matrices

$0x - y + z = 0; -x + 2y - z = 0; x - y + 0z = 0$

→ Using Cramer's Rule

$$\frac{x}{\begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}} = k$$

$$\frac{x}{-1} = \frac{-y}{1} = \frac{z}{-1} = k \Rightarrow x = -k, \; y = -k, \; z = -k$$

$\therefore$ For eigen values $\lambda_2 = 2$, Eigen vector is $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda_3 = 6$, Put $\lambda = 6$ in Equation (1), it gives,

By Equating corresponding elements of both sides matrices

$-3x - y + z = 0;$

$-x - y - z = 0;$

$x - y - 3z = 0$

→ Using Cramer's Rule

$$\frac{x}{\begin{vmatrix} -1 & -1 \\ -1 & -3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -1 & -1 \\ 1 & -3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix}} = k$$

$$\frac{x}{2} = \frac{-y}{4} = \frac{z}{2} = k \Rightarrow \; x = 2k, \; y = -4k, \; z = 2k$$

$\therefore$ For Eigen values $\lambda_3 = 3$, Eigen vector is $X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Hence Eigen vectors corresponding to Eigen values are ;

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_{\lambda=2} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\lambda=3} \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}_{\lambda=6} \quad ✓ \quad \text{...Ans.}$$

Example 3.3.8 **Dr. BATU - Dec. 17**

Find the Eigen values and corresponding vectors for the following matrix A ;

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$$

Solution : Given matrix is,

$$A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$$

The characteristic equation is,

$$|A - \lambda I| = 0$$

Since, $A - \lambda I = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

→ Using Scalar Multiplication

$$= \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

→ Using Substraction of Matrices

$$[A - \lambda I] = \begin{bmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{bmatrix}$$

$\therefore$ Characteristic equations is,

$$\begin{vmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - |A| = 0$$

Where S_1 and S_2 are sum of the minors of order 1 and 2 along the principal diagonal respectively.

$$S_1 = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$$

S_1 = sum of diagonal elements

$S_1 = 1 + 5 + 3 = 9$ (Addition of diagonal Elements)

S_2 = [Minor of 1] + [Minor of 5] + [Minor of 3]

$$= \begin{vmatrix} \cancel{1} & \cancel{0} & \cancel{-4} \\ \cancel{0} & 5 & 4 \\ \cancel{-4} & 4 & 3 \end{vmatrix} + \begin{vmatrix} 1 & \cancel{0} & -4 \\ \cancel{0} & \cancel{5} & \cancel{4} \\ -4 & \cancel{4} & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 & \cancel{-4} \\ 0 & 5 & \cancel{4} \\ \cancel{-4} & \cancel{4} & \cancel{3} \end{vmatrix}$$

(Sum of Minors of diagonal elements)

$$= \begin{vmatrix} 5 & 4 \\ 4 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 5 \end{vmatrix}$$

$$= [\underbrace{(5)(3)}_{15} - \underbrace{(4)(4)}_{16} + \underbrace{(1)(3)}_{3} - \underbrace{(-4)(-4)}_{-16}] + [\underbrace{(1)(5)}_{5} - \underbrace{(0)(0)}_{0}]$$

$$= [15-16] + [3-16] + [5-0]$$

$$= -1-13+5 = -9$$

$$|A| = \begin{vmatrix} \boxed{1 \quad 0 \quad -4} \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{vmatrix}$$

$$= 1 \times [\text{Minor of } 1] - (0) \times [\text{Minor of } 0] + (-4) \times [\text{Minor of } (-4)]$$

$$= \mathit{1} \times \begin{vmatrix} \cancel{1} & \cancel{0} & \cancel{-4} \\ \cancel{0} & 5 & 4 \\ \cancel{-4} & 4 & 3 \end{vmatrix} - \mathit{0} - \mathit{4} \times \begin{vmatrix} \cancel{1} & \cancel{0} & \cancel{-4} \\ 0 & 5 & \cancel{4} \\ -4 & 4 & \cancel{3} \end{vmatrix}$$

$$= 1 \times \begin{vmatrix} 5 & 4 \\ 4 & 3 \end{vmatrix} - 4 \times \begin{vmatrix} 0 & 5 \\ -4 & 4 \end{vmatrix}$$

$$= 1 \times [\underbrace{(5)(3)}_{15} - \underbrace{(4)(4)}_{16}] - 4 \times [\underbrace{(0)(4)}_{0} - \underbrace{(4)(5)}_{-20}]$$

$$= 1 \times [15-16] - 4 \times [0+20]$$

$$|A| = 1(-1) - 4(20) = -81$$

$$\therefore \lambda^3 - 9\lambda^2 - 9\lambda + 81 = 0 \Rightarrow \lambda^2(\lambda-9) - 9(\lambda-9) = 0$$

$$\therefore (\lambda-9)(\lambda^2-9) = 0 \Rightarrow (\lambda-9)(\lambda+3)(\lambda-3) = 0$$

$$[a^2 - b^2 = (a-b)(a+b)] \Rightarrow \lambda = 9, 3, -3$$

Hence Eigen values are $\lambda_1 = 3, \lambda_2 = -3, \lambda_3 = 9$ ✓ ...Ans.

$$\left\{ \begin{array}{l} \textbf{Verification: } S_1 = \lambda_1 + \lambda_2 + \lambda_3 \text{ and } |A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \\ S_1 = 3 - 3 + 9 = 9 \text{ and } |A| = (3)(-3)(9) = -81 \end{array} \right\}$$

Eigen Vectors

To find Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to Eigen values $\lambda = 3, -3, 9$

Consider, $[A - \lambda I] X = 0$

$$\begin{bmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \because \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ...(1)$$

For $\lambda_1 = 3$, Put $\lambda = 3$ in Equation (1), it gives,

By Equating corresponding elements of both sides matrices

$$-2x + 0y - 4z = 0;$$
$$0x + 2y + 4z = 0;$$
$$-4x + 4y + 0z = 0$$

→ **Using Cramer's Rule**

$$\frac{x}{\begin{vmatrix} 2 & 4 \\ 4 & 0 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 0 & 4 \\ -4 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 2 \\ -4 & 4 \end{vmatrix}} = k$$

$$\therefore x = -16k, \quad y = -16k, \quad z = 8k$$

$\therefore$ For Eigen value $\lambda_1 = 3$; Eigen vector $X_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

For $\lambda_2 = -3$, Put $\lambda = -3$ in Equation (1), it gives,

By Equating corresponding elements of both sides matrices

$$4x + 0y - 4z = 0$$
$$0x + 8y + 4z = 0;$$
$$-4x + 4y + 6z = 0$$

→ **Using Cramer's Rule**

$$\frac{x}{\begin{vmatrix} 8 & 4 \\ 4 & 6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 0 & 4 \\ -4 & 6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 8 \\ -4 & 4 \end{vmatrix}} = k$$

$$x = 32k, \quad y = -16k, \quad z = 32k$$

$\therefore$ For Eigen value $\lambda_2 = -3$; Eigen vector is $X_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

For $\lambda_3 = 9$, Put $\lambda = 9$ in Equation (1), it gives,

By Equating corresponding elements of both sides matrices

$$-8x + 0y - 4z = 0;$$
$$0x - 4y + 4z = 0;$$
$$-4x + 4y - 6z = 0$$

→ **Using Cramer's Rule**

$$\frac{x}{\begin{vmatrix} -4 & 4 \\ 4 & -6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 0 & 4 \\ -4 & -6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & -4 \\ -4 & 4 \end{vmatrix}} = k$$

$$x = 8k, \quad y = -16k, \quad z = -16k$$

$\therefore$ For Eigen value $\lambda_3 = 9$; Eigen vector is $X_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

Hence Eigen vectors are ;

$$X_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}_{\lambda_1 = 3}; \quad X_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}_{\lambda_2 = -3};$$

$$X_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}_{\lambda_3 = 9} \checkmark \quad \textbf{...Ans.}$$

Example 3.3.9

Find Eigen values and corresponding Eigen vectors of the matrix.

$$\mathbf{A} = \begin{bmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{bmatrix}$$

Solution : Given matrix is,

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

The characteristic equation is,

$$|A - \lambda I| = 0$$

Since,

$$[A - \lambda I] = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ **Using Scalar Multiplication**

$$= \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

→ **Using Subtraction of Matrices**

$$[A - \lambda I] = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$$

$\therefore$ Characteristic equations is,

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - |A| = 0$$

Where S_1 and S_2 are sum of the minors of order 1 and 2 along principal diagonal respectively.

$$\therefore \quad S_1 = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \leftarrow S_1 = \text{Sum of Diagonal elements}$$

$$= 8 + 7 + 3 = 18 \quad \text{(Addition of diagonal Elements)}$$

$$S_2 = [\text{Minor of } 8] + [\text{Minor of } 7] + [\text{Minor of } 3]$$

$$= \begin{vmatrix} \cancel{8} & \cancel{-6} & \cancel{2} \\ \cancel{-6} & 7 & -4 \\ \cancel{2} & -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & \cancel{-6} & 2 \\ \cancel{-6} & \cancel{7} & \cancel{-4} \\ 2 & \cancel{-4} & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 & \cancel{2} \\ -6 & 7 & \cancel{-4} \\ \cancel{2} & \cancel{-4} & \cancel{3} \end{vmatrix}$$

(Sum of Minors of diagonal elements)

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= [\underbrace{(7)(3)}_{21} - \underbrace{(-4)(-4)}_{16} + \underbrace{(8)(3)}_{24} - \underbrace{(2)(2)}_{4}] + [\underbrace{(8)(7)}_{56} - \underbrace{(-6)(-6)}_{36}]$$

$$= [21-16] + [24-4] + [56-36]$$

$$= 5 + 20 + 20 = \mathbf{45}$$

$$|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8 \times [\text{Minor of } 8] - (-6) \times [\text{Minor of } (-6)] + 2 \times [\text{Minor of } 2]$$

$$= \mathit{8} \times \begin{vmatrix} \cancel{8} & \cancel{-6} & \cancel{2} \\ \cancel{-6} & 7 & -4 \\ \cancel{2} & -4 & 3 \end{vmatrix} + \mathit{6} \times \begin{vmatrix} \cancel{8} & \cancel{-6} & \cancel{2} \\ -6 & \cancel{7} & -4 \\ 2 & \cancel{-4} & 3 \end{vmatrix} + \mathit{2} \times \begin{vmatrix} \cancel{8} & \cancel{-6} & \cancel{2} \\ -6 & 7 & \cancel{-4} \\ 2 & -4 & \cancel{3} \end{vmatrix}$$

$$= 8 \times \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + 6 \times \begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix} + 2 \times \begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}$$

$$= 8 \times [\underbrace{(7)(3)}_{21} - \underbrace{(-4)(-4)}_{16}] + 6 \times [\underbrace{(-6)(3)}_{-18} - \underbrace{(2)(-4)}_{-8}] + 2 \times [\underbrace{(-6)(-4)}_{24} - \underbrace{(2)(7)}_{14}]$$

$= 8 \times [21 - 16] + 6 \times [-18 + 8] + 2 \times [24 - 14]$

$= 8(5) + 6(-10) + 2(10) = 0$

$\therefore \quad \lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0 \Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda(\lambda^2 - 18\lambda + 45) = 0 \Rightarrow \lambda(\lambda - 15)(\lambda - 3) = 0$

$\Rightarrow \lambda = 0, 3, 15$

Hence Eigen values are $\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 15$ ✓ **...Ans.**

{**Verification** : $S_1 = \lambda_1 + \lambda_2 + \lambda_3$ and $|A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$}

Eigen vectors

To find Eigen vectors $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to Eigen values $\lambda_1, \lambda_2, \lambda_3$

Consider, $[A - \lambda I]\, X = 0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0} \qquad \because \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad ...(1)$$

For $\lambda_1 = 0$, Put $\lambda = 0$ in Equation (1), it gives,

By Equating corresponding elements of both sides matrices

$8x - 6y + 2z = 0;$

$-6x + 7y - 4z = 0$

$2x - 4y + 3z = 0$

➔ Using Cramer's Rule

$$\frac{x}{\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}} = k$$

$$\frac{x}{5} = \frac{-y}{-10} = \frac{z}{10} = k \Rightarrow x = 5k,\ y = 10k,\ z = 10k,$$

$\therefore$ For Eigen value $\lambda_1 = 0$, Eigen vector is $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

For $\lambda = 3$, Put $\lambda = 3$ in Equation (1), it gives,

By Equating corresponding elements of both sides matrices

$5x - 6y + 2z = 0;$

$-6x + 4y - 4z = 0$

$2x - 4y + 0z = 0$

➔ Using Cramer's Rule

$$\frac{x}{\begin{vmatrix} 4 & -4 \\ -4 & 0 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -6 & -4 \\ 2 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -6 & 4 \\ 2 & -4 \end{vmatrix}} = k$$

$$\frac{x}{-16} = \frac{-y}{8} = \frac{z}{16} = k$$

$$\Rightarrow x = -16k,\ y = -8k,\ z = 16k$$

$\therefore$ For Eigen values $\lambda = 3$, Eigen vector is $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

For $\lambda = 15$, Put $\lambda = 15$ in Equation (1), it gives,

By Equating corresponding elements of both sides matrices

$-7x - 6y + 2z = 0;$

$-6x - 8y - 4z = 0;$

$2x - 4y - 12z = 0$

➔ Using Cramer's Rule

$$\frac{x}{\begin{vmatrix} -8 & -4 \\ -4 & -12 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -6 & -4 \\ 2 & -12 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -6 & -8 \\ 2 & -4 \end{vmatrix}} = k$$

$$\frac{x}{80} = \frac{-y}{80} = \frac{z}{40} = k \Rightarrow x = 80k,\ y = -80k,\ z = 40k$$

$\therefore$ For Eigen values $\lambda = 15$, Eigen vector is $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Hence Eigen vectors are;

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}_{\lambda=0} \quad \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}_{\lambda=3} \quad \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}_{\lambda=15}$$ ✓ **...Ans.**

Type III : Examples on Repeated Eigen values for Non-Symmetric Matrix

Example 3.3.10

Find Eigen values and Eigen vectors for the following matrix :

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

Solution : Given matrix is,

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

The characteristic equation is,

$$|A - \lambda I| = 0$$

Since,

$$A - \lambda I = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

➔ Using Scalar Multiplication

$$= \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

➔ Using Subtraction of Matrices

$$[A - \lambda I] = \begin{bmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{bmatrix}$$

$\therefore$ Characteristic equations is,

$$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

Where S_1 and S_2 are sum of the minors of order 1 and 2 along the principal diagonal respectively.

$$S_1 = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix} \leftarrow S_1 = \text{Sum of Diagonal elements}$$

$= -9 + 3 + 7 = 1$ (Addition of diagonal Elements)

$S_2 =$ [Minor of (– 9)] + [Minor of 3] + [Minor of 7]

$$= \begin{vmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{vmatrix}$$

(Sum of Minors of diagonal elements)

$$= \begin{vmatrix} 3 & 4 \\ 8 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -16 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -8 & 3 \end{vmatrix}$$

$$= [\underbrace{(3)(7)}_{21} - \underbrace{(8)(4)}_{32}] + [\underbrace{(-9)(7)}_{-63} - \underbrace{(-16)(4)}_{-64}] + [\underbrace{(-9)(3)}_{-27} - \underbrace{(-8)(4)}_{-32}]$$

$= [21 - 32] + [-63 + 64] + [-27 + 32]$

$= -11 + 1 + 5 = \mathbf{-5}$

$$|A| = \begin{vmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{vmatrix}$$

$= (-9) \times$ [Minor of (– 9)] $- 4 \times$ [Minor of 4] $+ 4 \times$ [Minor of 4]

$$= (-9) \times \begin{vmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{vmatrix} - 4 \times \begin{vmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{vmatrix} + 4 \times \begin{vmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{vmatrix}$$

$$= (-9) \times \begin{vmatrix} 3 & 4 \\ 8 & 7 \end{vmatrix} - 4 \times \begin{vmatrix} -8 & 4 \\ -16 & 7 \end{vmatrix} + 4 \times \begin{vmatrix} -8 & 3 \\ -16 & 8 \end{vmatrix}$$

$= (-9) \times [(3)(7) - (8)(4)] - 4 \times [(-8)(7) - (16)(4)]$
$\quad + 4 \times [(-8)(8) - (-16)(3)]$

$= (-9)[21 - 32] - 4 \times [-56 + 64] + 4 \times [-64 + 48]$

$= -9(-11) - 4(8) + 4(-16) = 3$

➔ Using Synthetic Division

$\therefore \quad \lambda^3 - \lambda^2 - 5\lambda - 3 =$

$\lambda = -1, \quad \lambda^2 - 2\lambda - 3$

$(\lambda - 3)(\lambda + 1) = 0$

– 1	1	– 1	– 5	– 3
	–	– 1	2	3
	1	– 2	– 3	0

$\therefore \quad \lambda = -1, -1, 3$

Hence Eigen value are $\lambda_1 = 3, \lambda_2 = -1, \lambda_3 = -1$ ✓ **...Ans.**

{**Verification :** $S_1 = \lambda_1 + \lambda_2 + \lambda_3$ and $|A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$}

Eigen vectors :

To find Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to Eigen values, $\lambda_1 = 3, \quad \lambda_2 = \lambda_3 = 1$

Consider, $[A - \lambda I]\, X = 0$

$$\begin{bmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0} \qquad \because \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad ...(1)$$

For $\lambda_1 = 3$, Put $\lambda = 3$ in Equation (1), it gives,

By Equating corresponding elements of both sides matrices

$-12x + 4y + 4z = 0;$

$-8x + 0y + 4z = 0;$

$-16x + 8y + 4z = 0$

→ Using Cramer's Rule

$$\frac{x}{\begin{vmatrix} 0 & 4 \\ 8 & 4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -8 & 4 \\ -16 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -8 & 0 \\ -16 & 8 \end{vmatrix}} = k$$

$\therefore \quad x = -32k, \quad y = -32k, \quad z = -64k$

$\therefore$ For Eigen value

$$\lambda_1 = 3, \quad \text{Eigen vector is, } X_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

For $\lambda_2 = \lambda_3 = -1$, Put $\lambda = -1$ in Equation (1), it gives,

For repeated Eigen value, find rank of matrix $[A - \lambda I]$ at $\lambda_2 = \lambda_3 = -1$

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \underset{R_2 - R_1 \,;\, R_3 - 2R_1}{\text{operate}} \sim \begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank = r = 1 and n = number of unknowns = 3

$\therefore \; n - r = 3 - 1 = 2$ i.e. Two linearly independent vectors are possible.

Here matrix A is unsymmetric.

$\therefore -8x + 4y + 4z = 0$ (From matrix)

Put $z = 0; \; -8x + 4y = 0 \Rightarrow 2x = y \qquad \therefore X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Put $y = 0; \; -8x + 4z = 0 \Rightarrow 2x = z \qquad \therefore X_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Hence Eigen vectors are,

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{\lambda_1 = 3} ; \quad X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}_{\lambda_2 = -1} ;$$

$$X_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}_{\lambda_3 = -1} \checkmark$$

...Ans.

Example 3.3.11

Find Eigen values and Eigen vectors for the following matrix :

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution : Given matrix is,

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

The characteristics equation is,

$$|A - \lambda I| = 0$$

Since,

$$[A - \lambda I] = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Using Scalar Multiplication

$$= \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

→ Using Subtraction of Matrices

$$[A - \lambda I] = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$$

$\therefore$ Characteristic equations is,

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - |A| = 0$$

Where S_1 and S_2 are sum of the minors of order 1 and 2 along principal diagonal respectively.

$$S_1 = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \leftarrow S_1 = \text{Sum of Diagonal elements}$$

$= (-2) + 1 = -1$ (Addition of diagonal Elements)

S_2 = Minor of $(-2)]$ + [Minor of 1] + [Minor of 0]

$$= \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

(Sum of Minors of diagonal elements)

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= [\underbrace{(1)(0)}_{0} - \underbrace{(-2)(-6)}_{12}] + [\underbrace{(-2)(0)}_{0} - \underbrace{(-1)(-3)}_{3}] + [\underbrace{(-2)(1)}_{-2} - \underbrace{(2)(2)}_{2}]$$

$= [0 - 12] + [0 - 3] + [-2 - 4]$

$= -12 - 3 - 6 = -21$

$|A| = -2(-12) - 2(-6) - 3(-3) = 45$

→ Using Synthetic Division

$$\therefore \; \lambda^3 + \lambda^2 - 21\lambda - 45 \qquad \begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & - & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$\lambda = -3, \; \lambda^2 - 2\lambda - 15$

$(\lambda - 5)(\lambda + 3) = 0$

$\therefore \quad \lambda = 5, -3$

Hence **Eigen value are $\lambda_1 = 5, \lambda_2 = -3, \lambda_3 = -3$** ✓ **...Ans.**

{**Verification :** $S_1 = \lambda_1 + \lambda_2 + \lambda_3$ and $|A| = \lambda_1 \lambda_2 \lambda_3$}

Eigen vectors

To find Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to Eigen values, $\lambda_1 = 5, \; \lambda_2 = \lambda_3 = -3$

Consider, $\quad [A - \lambda I] X = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

For $\lambda_1 = 5$, Put $\lambda = 5$ in Equation (1), it gives,

By Equating corresponding elements of both sides matrices

$-7x + 2y - 3z = 0;$

$2x - 4y - 6z = 0$

$; -x - 2y - 5z = 0$

→ Using Cramer's Rule

$$\frac{x}{\begin{vmatrix} -4 & -6 \\ -2 & -5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 2 & -6 \\ -1 & -5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & -4 \\ -1 & -2 \end{vmatrix}} = k$$

$\therefore \quad x = 8k, \quad y = 16k, \quad z = -8k$

$\therefore$ For Eigen value

$$\lambda_1 = 5, \qquad \text{Eigen vector is, } X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

For $\lambda_2 = \lambda_3 = -3$, Put $\lambda = -3$ in Equation (1), it gives,

Eigen value are repeated.

First find rank of $[A - \lambda I]$ at $\lambda = -3$

$$[A - \lambda I] = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

Operate $R_2 - 2R_1 \,;\, R_3 + R_1 \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ Rank = r = 1 and n = number of unknowns = 3

$\therefore \; n - r = 3 - 1 = 2$ i.e. Two linearly independent Eigen vectors are possible.

Here matrix A is unsymmetric

$x + 2y - 3z = 0$ (From matrix)

Put $z = 0; \; x + 2y = 0 \Rightarrow x = -2y;$

$$\therefore \; X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Put $y = 0; \; x - 3z = 0 \Rightarrow x = 3z$

$$\therefore \; X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Hence Eigen vectors are,

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}_{\lambda_1 = 5} ; \quad X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}_{\lambda_2 = -3} ;$$

$$X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}_{\lambda_3 = -3} \checkmark$$

...Ans.

Type IV : Examples on Repeated Eigen Values for Symmetric Matrix

Example 3.3.12

Find Eigen values and Eigen vectors for the following matrix : $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Solution : Given matrix is,

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation is,

$$|A - \lambda I| = 0$$

Since,

$$[A - \lambda I] = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Using Scalar Multiplication

$$= \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

→ Using Subtraction of matrices

$$[A - \lambda I] = \begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}$$

∴ Characteristic equations is,

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda^1 - |A| = 0$$

Where S_1 and S_2 are sum of the minors of order 1 and 2 along principal diagonal respectively.

$$S_1 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

S_1 = sum of diagonal elements

$$= 3 + 3 + 3 = 9$$

(Addition of diagonal Elements)

S_2 = [Minor of 3] + [Minor of 3] + [Minor of 3]

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$

(Sum of Minors of diagonal elements)

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= [\underbrace{(3)(3)}_{9} - \underbrace{(-1)(-1)}_{1}] + [\underbrace{(3)(3)}_{9} - \underbrace{(1)(1)}_{1}] - [\underbrace{(3)(3)}_{9} - \underbrace{(1)(1)}_{1}]$$

$$= [9-1] + [9-1] + [9-1] = 8 + 8 + 8 = 24$$

$$|A| = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$= 3 \times$ [Minor of 3] $- 1 \times$ [Minor of 1] $+ 1 \times$ [Minor of 1]

$$= 3 \times \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} - 1 \times \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix}$$

$$= 3 \times \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} - 1 \times \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix}$$

$$= 3 \times [\underbrace{(3)(3)}_{9} - \underbrace{(-1)(-1)}_{1}] - 1 \times [\underbrace{(1)(3)}_{3} - \underbrace{(1)(-1)}_{-1}] - 1[\underbrace{(1)(-1)}_{-1} - \underbrace{(1)(3)}_{3}]$$

$$= 3 \times [9-1] - 1 \times [3+1] + 1 \times [-1-3]$$

$$= 3(8) - 1(4) + 1(-4) = 24 - 4 - 4 = 16$$

→ Using Synthetic division

$$\therefore \lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$$

$$\lambda = 1, \quad \lambda^2 - 8\lambda + 16 = 0$$

$$(\lambda - 4)^2 = 0$$

1	1	−9	24	−16
	–	1	−8	16
	1	−8	16	0

$$\therefore \lambda = 1, 4, 4$$

Eigen value are $\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 4$ ✓ **...Ans.**

{**Verification :** $S_1 = \lambda_1 + \lambda_2 + \lambda_3$ and $|A| = \lambda_1 \lambda_2 \lambda_3$ }

Eigen vectors:

To find Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

corresponding to Eigen values, $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 4$

Consider, $[A - \lambda I] X = 0$

$$\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\because \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ...(1)$$

For $\lambda_1 = 1$, Put $\lambda = 1$ in Equation (1), it gives,

By equating corresponding elements of both sides matrices

$$2x + y + z = 0$$
$$x + 2y - z = 0$$
$$x - y + 2z = 0$$

→ Using Cramer's Rule

$$\frac{x}{\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}} = k$$

$$\therefore x = 3k, \quad y = -3k, \quad z = -3k$$

∴ For Eigen value $\lambda_1 = 1$,

Eigen vector is, $X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = \lambda_3 = 4$, Put $\lambda = 4$ in Equation (1), it gives,

Here Eigen value are repeated. First find rank of $[A - \lambda I]$ at $\lambda_2 = 4$

$$\therefore [A - \lambda I] = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

Operate $R_2 + R_1$; $R_3 + R_1$ ~ $\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

∴ Rank = r = 1 and n = number of unknowns = 3

∴ n − r = 3 − 1 = 2, i.e. Two Eigen vectors are possible.

Here, matrix A is symmetric, therefore Eigen vectors are orthogonal.

By above matrix, $\quad -x + y + z = 0$

Put $z = 0$; $\quad x = y$;

$$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and consider } X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

$$\therefore X_1 X_3^T = 0 \quad \text{and} \quad X_2 X_3^T = 0$$

Hence Eigen vectors are,

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} [l\ m\ n] = 0 \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} [l\ m\ n] = 0$$

$-l + m + n = 0$ and $l + m = 0 \Rightarrow l = -m$, $\quad$ Put m = k

$$\therefore l = -k \quad \text{and} \quad n = l - m = -k - k = -2k$$

$$\therefore X_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Hence Eigen vectors are,

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}_{\lambda_1 = 1} ; \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\lambda_2 = 4} ; \quad X_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}_{\lambda_3 = 4}$$ ✓

...Ans.

Example 3.3.13

Find Eigen values and independent Eigen for the following matrix :

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution : The given matrix is,

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic equation is,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

Where S_1 and S_2 are sum of the minors of order 1 an 2 along principal diagonal respectively.

$$S_1 = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

S_1 = sum of diagonal elements

$$S_1 = 6 + 3 + 3 = 12$$

(Addition of diagonal Elements)

$S_2 = $ [Minor of 6] + [Minor of 3] + [Minor of 3]

$$= \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

(Sum of Minors of diagonal elements)

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= [\underbrace{(3)(3)}_{9} - \underbrace{(-1)(-1)}_{1}] + [\underbrace{(6)(3)}_{18} - \underbrace{(2)(2)}_{4}] + [\underbrace{(6)(3)}_{18} - \underbrace{(-2)(-2)}_{4}]$$

$$= [9-1] + [18-4] + [18-4]$$

$$= 8 + 14 + 14 = 36$$

$$|A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6 \times [\text{Minor of } 6] - (-2) \times [\text{Minor of } (-2)] + 2 \times [\text{Minor of } 2]$$

$$= 6 \times \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} + 2 \times \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} + 2 \times \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6 \times \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + 2 \times \begin{vmatrix} -2 & -1 \\ 2 & 3 \end{vmatrix} + 2 \times \begin{vmatrix} -2 & 3 \\ 2 & -1 \end{vmatrix}$$

$$= 6 \times [(3)(3) - (-1)(-1)] + 2 \times [(-2)(3) - (2)(-1)] + 2 \times [(-2)(-1) - (2)(3)]$$

$$= 6 \times [9-1] + 2 \times [-6+2] + 2 \times [2-6]$$

$$= 6(8) + 2(-4) + 2(-4)$$

$$= 48 - 8 - 8 = 32$$

→ Using Synthetic Division

$\therefore \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

2	1	−12	36	−32
	−	2	−20	32
	1	−10	16	$\boxed{0}$

$\lambda = 2, \ (\lambda^2 - 10\lambda + 16) = 0$

$(\lambda - 8)(\lambda - 2) = 0$

$$\therefore \quad \lambda = 2, 2, 8$$

Eigen value are $\lambda_1 = 8, \lambda_2 = 2, \lambda_3 = 2$✓ ...**Ans.**

{**Verification :** $S_1 = \lambda_1 + \lambda_2 + \lambda_3$ and $|A| = \lambda_1 \lambda_2 \lambda_3$ }

Eigen vectors: To find Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to Eigen values, $\lambda_1 = 8, \ \lambda_2 = \lambda_3 = 2$

Consider, $\quad [A - \lambda I] X = \mathbf{0}$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\because \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ...(1)$$

For $\lambda_1 = 8$, Put $\lambda = 8$ in Equation (1), it gives,

By equating corresponding elements of both sides matrices

$$-2x - 2y + 2z = 0$$
$$-2x - 5y - z = 0$$
$$2x - y - 5z = 0$$

→ Using Cramer's Rule

$$\frac{x}{\begin{vmatrix} -5 & -1 \\ -1 & -5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & -1 \\ 2 & -5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & -5 \\ 2 & -1 \end{vmatrix}} = k$$

$\therefore \quad x = 24k, \qquad y = -12k, \qquad z = 12k$

$\therefore$ For Eigen value $\quad \lambda_1 = 8,$

Eigen vector is $\quad X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

For $\lambda_2 = \lambda_3 = 2$: Put $\lambda = 2$ in Equation (1), it gives,

Now, for repeated Eigen value $\lambda_2 = \lambda_3 = 2$

Find rank of $[A - \lambda I]$ at $\lambda = 2$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \text{Operate } R_{13} \sim \begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ 4 & -2 & 2 \end{bmatrix}$$

$$\text{Operate } R_2 + R_1 \, ; R_3 - 2R_1 \sim \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of $[A - \lambda I] = r = 1$ and n = number of unknowns = 3

$\therefore \quad (n - r) = 3 - 1 = 2$ i.e. Two Eigen vectors are possible.

The matrix A is symmetric, $\qquad$ [i.e. $A = A^T$]

$\therefore \quad$ Eigen vectors are orthogonal.

From above matrix, $\qquad 2x - y + z = 0$

Put $z = 0$; $\qquad 2x = y$;

$\therefore$ Eigen vector is, $\qquad X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Consider $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

Solve ,

$\therefore \quad X_1 X_3^T = 0 \quad$ and $\quad X_2 X_3^T = 0$

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} [l \ \ m \ \ n] = 0 \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} [l \ \ m \ \ n] = 0$$

$$2l - m + n = 0$$
$$l + 2m + on = 0$$

→ Using Cramer's Rule

$$\frac{l}{\begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix}} = \frac{-m}{\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}}$$

$$\Rightarrow \frac{l}{-2} = \frac{-m}{-1} = \frac{n}{5}$$

$$\therefore \quad X_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$$

Hence Eigen vectors are,

$$X_1 = \begin{bmatrix} \mathbf{2} \\ \mathbf{-1} \\ \mathbf{1} \end{bmatrix}_{\lambda_1 = 8} ; \quad X_2 = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{0} \end{bmatrix}_{\lambda_2 = 2} ; \quad X_3 = \begin{bmatrix} \mathbf{-2} \\ \mathbf{1} \\ \mathbf{5} \end{bmatrix}_{\lambda_3 = 2} ✓$$

...**Ans.**

Type V : Examples on Matrix Involving only Two Eigen Vectors

Example 3.3.14

Find the Eigen values and Eigen vectors of the following Matrix :

$$A \begin{bmatrix} \mathbf{4} & \mathbf{6} & \mathbf{6} \\ \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{-1} & \mathbf{-5} & \mathbf{-2} \end{bmatrix}$$

Solution : Given matrix is,

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$$

The characteristic equation is, $|A - \lambda I| = 0$

Since,

$$[A - \lambda I] = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Using scalar multiplication

$$[A - \lambda I] = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

→ Using subtraction of matrices

$$[A - \lambda I] = \begin{bmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{bmatrix}$$

$\therefore$ Characteristic equations is,

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - |A| = 0$$

Where S_1 and S_2 are sum of the minors of orders 1 and 2 along principal diagonal respectively.

$$S_1 = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$$

← S_1 = Sum of Diagonal elements

$= 4 + 3 - 2 = 5 \qquad$ (Addition of diagonal Elements)

$$S_2 = \begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}$$

$$= 4 - 2 + 6 = 8$$

$|A| = 4(4) - 6(0) + 6(-2) = 4$

$\therefore \quad \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

→ Using Synthetic Division

1	1	−5	8	−4
	−	1	−4	4
	1	−4	4	$\boxed{0}$

$\lambda = 1 \, ; \lambda^2 - 4\lambda + 4 = 0$

$\lambda = 1, (\lambda - 2)^2 = 0 \Rightarrow \lambda = 1, 2, 2$

$\therefore \quad$ **Eigen value are $\lambda_1 = 1, \lambda_2 = \lambda_3 = 2$** ✓ ...**Ans.**

{**Verification** : $S_1 = \lambda_1 + \lambda_2 + \lambda_3$ and $|A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$}

Eigen vectors

To find Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to Eigen values, $\lambda_1 = 1, \ \lambda_2 = \lambda_3 = 2$

Consider, $[A-\lambda I]X = 0$

$$\begin{bmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \because \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots(1)$$

For $\lambda_1 = 1$, Put $\lambda = 1$ in Equation (1), it gives,

By equating corresponding elements of both sides matrices

$$3x + 6y + 6z = 0$$
$$x + 2y + 2z = 0$$
$$-x - 5y - 3z = 0$$

➔ **Using Cramer's Rule**

$$\frac{x}{\begin{vmatrix} 2 & 2 \\ -5 & -3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ -1 & -5 \end{vmatrix}} = k$$

$$\therefore \quad \frac{x}{4} = \frac{-y}{-1} = \frac{z}{-3} = k$$

$$x = 4k, \quad y = k, \quad z = -3k$$

$\therefore$ For Eigen value $\lambda = 1$,

Eigen vector is $X_1 = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$

For $\lambda_2 = \lambda_3 = 2$: For repeated Eigen value $\lambda_2 = \lambda_3 = 2$, Put $\lambda = 2$ in Equation (1), it gives,

Here $A \neq A^T$ i.e. **A is non-symmetric matrix.**

First find rank of matrix $[A - \lambda I]$ for $\lambda = 2$

$$[A-\lambda I] = \begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}$$

Operate R_{12} $\sim \begin{bmatrix} 1 & 1 & 2 \\ 2 & 6 & 6 \\ -1 & -5 & -4 \end{bmatrix}$

Operate, $R_2 - 2R_1$; $R_3 + R_1$ $\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & 2 \\ 0 & -4 & -2 \end{bmatrix}$

Operate $R_3 + R_2$ $\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ $\rho(A - \lambda I) = 2$ i.e. $r = 2$ and

Number of unknowns = n = 3

$\therefore$ $n - r = 3 - 2 = 1$ i.e. only one Eigen vector is possible.

$$\left.\begin{matrix} x + y + 2z \\ 4y + 2z \end{matrix}\right\} \quad \text{From matrix}$$

➔ **Using Cramer's Rule**

$$\frac{x}{\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix}} = k$$

$$\frac{x}{-6} = \frac{-y}{2} = \frac{z}{4} = k \Rightarrow \quad x = -6k,$$

$$y = -2k, \qquad z = 4k$$

$\therefore$ For eigen value $\lambda = 2$

Eigen value is $X_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

Hence Eigen vectors are,

$$X_1 = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}_{\lambda=1} ; \quad X_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}_{\lambda=2} \checkmark \qquad \ldots\text{Ans.}$$

Type VI : Examples on Matrix Involving any One Eigen Vector

Example 3.3.15

Find the characteristic roots and the corresponding characteristic vectors for the following matrix.

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{-3} & \mathbf{3} \end{bmatrix}$$

Solution : The given matrix is,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

The characteristic equation is,

$$|A - \lambda I| = 0$$

➔ **Using scalar multiplication**

$$[A - \lambda I] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

➔ **Using subtraction of matrices**

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$[A - \lambda I] = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3-\lambda \end{bmatrix}$$

$\therefore$ Characteristic equations is,

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

Where S_1 and S_2 are sum of the minors of order 1 and 2 along principal diagonal respectively.

$$S_1 = 0 + 0 + 3 = 3$$

(Addition of diagonal Elements)

$$S_2 = \begin{vmatrix} 0 & 1 \\ -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= 3 + 0 + 0 = 3$$

$$|A| = 0 - (-1) = 1$$

$$\therefore \lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$\Rightarrow (\lambda - 1)^3 = 0$$

$$\Rightarrow \lambda = 1, 1, 1$$

Eigen values are $\lambda_1 = \lambda_2 = \lambda_3 = 1$✓ ...Ans.

Eigen vectors

To find Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

corresponding to Eigen values, $\lambda_1 = \lambda_2 = \lambda_3 = 1$

Consider, $[A - \lambda I] X = 0$

For repeated Eigen values $\lambda_1 = \lambda_2 = \lambda_3 = 1$

Here $A \neq A^T$ i.e. **A is non-symmetric matrix.**

First find the rank of matrix $[A - \lambda I]$ at $\lambda = 1$

$$[A - \lambda I] = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

Operate $R_3 + R_1$ $\sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}$

Operate $R_3 - 2R_2$ $\sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ $\rho[A - \lambda I] = 2$

i.e. $r = 2$ and Number of unknowns = n = 3

$n - r = 3 - 2 = 1$

$\therefore$ There is only one Eigen vector is possible.

From above matrix,

$$-x - y + 0z = 0$$
$$0x - y + z = 0$$

➔ **Using Cramer's Rule**

$$\frac{x}{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}} = k$$

$$\frac{x}{1} = \frac{-y}{-1} = \frac{z}{1} = k \quad \therefore \quad x = k,$$

$$y = k, \qquad z = k$$

Hence, Eigen vector,

$$\therefore \quad X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\lambda=1} \checkmark \qquad \ldots\text{Ans.}$$

Exercise 1

Find Eigen values and Eigen vectors

(A) Non-Symmetric and Non repeated

(i) $\begin{bmatrix} 0 & 2 & 0 \\ 3 & -2 & 3 \\ 0 & 3 & 0 \end{bmatrix}$

$\left\{\textbf{Ans. :} \lambda^3 + 2\lambda^2 - 15\lambda = 0; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_{\lambda=1}; \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}_{\lambda=3}; \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}_{\lambda=5}\right\}$

(ii) $\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$\left\{\textbf{Ans. :} \lambda^3 + 3\lambda^2 + 2\lambda = 0; \lambda = 0, 1, 2 \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}_{\lambda=0}; \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}_{\lambda=1}; \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}_{\lambda=2}\right\}$

(iii) $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$

$\left\{\textbf{Ans. :} \lambda = 1, -1, 3 \text{ and } \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}_{\lambda=1}; \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}_{\lambda=-1}; \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}_{\lambda=3}\right\}$

(iv) $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$\left\{\textbf{Ans. :} \lambda = 2, 1, -1 \text{ and } \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}_{\lambda=2}; \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}_{\lambda=1}; \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{\lambda=-1}\right\}$

(B) Non-symmetric and repeated

(i) $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

$\left\{\textbf{Ans. :} \lambda^3 + 2\lambda^2 - \lambda = 0; \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}_{\lambda=0}; \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}_{\lambda=1}; \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}_{\lambda=1}\right\}$

(ii) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$\left\{\textbf{Ans. :} \lambda^3 + 7\lambda^2 - 11\lambda - 5 = 0; \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\lambda=5}; \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}_{\lambda=1}; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_{\lambda=1}\right\}$

(iii) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$

$\left\{\textbf{Ans. :} \lambda^3 + \lambda^2 - \lambda + 1 = 0; \lambda = 1, 1, 1, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}_{\lambda=-1}; \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_{\lambda=1}\right\}$

(iv) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\left\{ \textbf{Ans. :} \lambda^3 + 5\lambda^2 - 7\lambda - 3 = 0; \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\lambda=3}; \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}_{\lambda=1}; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_{\lambda=1} \right\}$

(v) $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$\left\{ \textbf{Ans. :} \lambda^3 - 2\lambda^2 - \lambda + 2 = 0; \lambda = 1, -1, 2 \begin{bmatrix} \\ \\ \end{bmatrix}_{\lambda=1}; \begin{bmatrix} \\ \\ \end{bmatrix}_{\lambda=-1}; \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}_{\lambda=2} \right\}$

(C) Symmetric and Non repeated

(i) $\begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

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$\left\{ \textbf{Ans. :} \lambda^3 + 8\lambda^2 - 4\lambda + 48 = 0; \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\lambda=-2}; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_{\lambda=4}; \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{\lambda=6} \right\}$

(ii) $\begin{bmatrix} 2 & 4 & -6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{bmatrix}$

$\left\{ \textbf{Ans. :} \lambda^3 + 11\lambda^2 - 144\lambda + 324 = 0; \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}_{\lambda=-2}; \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}_{\lambda=9}; \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}_{\lambda=-18} \right\}$

(iii) $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

$\left\{ \textbf{Ans. :} \lambda^3 + 6\lambda^2 - 11\lambda - 6 = 0; \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{\lambda=1}; \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\lambda=2}; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_{\lambda=3} \right\}$

(D) Symmetric and repeated

(i) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(E) Special

(i) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

$\left\{ \textbf{Ans. :} \lambda^2 + 5\lambda^2 + 8\lambda - 4 = 0; \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}_{\lambda=1}; \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}_{\lambda=2} \right\}$

(ii) $\begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

$\left\{ \textbf{Ans. :} \lambda^2 - 3\lambda^2 + 3\lambda - 1 = 0; \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}_{\lambda=1} \right\}$

(F) Mixed

(i) $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\left\{ \textbf{Ans. :} \lambda^3 + 5\lambda^2 - 4\lambda - 3 = 0; \lambda = 0, 1, 4 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}_{\lambda=0}; \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\lambda=1}; \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\lambda=4} \right\}$

(ii) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

$\left\{ \textbf{Ans. :} \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0; \lambda = 3, 1, 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{\lambda=3}; \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}_{\lambda=1}; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_{\lambda=1} \right\}$

(iii) $\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & -4 \\ 1 & -1 & -2 \end{bmatrix}$

$\left\{ \textbf{Ans. :} \lambda^3 - 9\lambda = 0; \lambda = 0, 3, 3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}_{\lambda=0}; \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\lambda=3} \right\}$

(iv) $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

3.4 Cayley - Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

Example 3.4.1

Verify Cayley - Hamilton Theorem and hence find A^{-1} for the matrix.

$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ **Also find A^4**

Solution :

Cayley-Hamilton Theorem : States that "Every square matrix satisfies its own characteristic equation."

Given matrix is, $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Step I : The characteristic equation of matrix is,

$$|A - \lambda I| = 0$$

Since,

$$[A - \lambda I] = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

→ **Using scalar multiplication**

$$= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

→ **Using subtraction of matrices**

$$[A - \lambda I] = \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}$$

∴ Characteristic equations is,

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

Where S_1 and S_2 are sum of the minors of order one and two along the principal diagonal elements respectively.

$$S_2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

S_1 = sum of diagonal elements

$= 2 + 2 + 2 = 6$ (Addition of diagonal Elements)

S_2 = [Minor of 2] + [Minor of 2] + [Minor of 2]

$$= \begin{vmatrix} \cancel{2} & \cancel{-1} & \cancel{1} \\ \cancel{-1} & 2 & -1 \\ \cancel{1} & -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & \cancel{-1} & 1 \\ \cancel{-1} & \cancel{2} & \cancel{-1} \\ 1 & \cancel{-1} & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 & \cancel{1} \\ -1 & 2 & \cancel{-1} \\ \cancel{1} & \cancel{-1} & \cancel{2} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

(Sum of Minors of diagonal elements)

$= [(2)(2) - (-1)(-1)] + [(2)(2) - (1)(1)] + [(2)(2) - (-1)(-1)]$

$= [4 - 1] + [4 - 1] + [4 - 1]$

$= 3 + 3 + 3 = 9$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$= 2 \times$ [Minor of 2] $- (-1) \times$ [Minor of (-1)] $+ (1) \times$ [Minor of 1]

$$= 2 \times \begin{vmatrix} \cancel{2} & \cancel{-1} & \cancel{1} \\ \cancel{-1} & 2 & -1 \\ \cancel{1} & -1 & 2 \end{vmatrix} - 1 \times \begin{vmatrix} \cancel{2} & \cancel{-1} & \cancel{1} \\ -1 & \cancel{2} & -1 \\ 1 & \cancel{-1} & 2 \end{vmatrix} + 1 \times \begin{vmatrix} \cancel{2} & \cancel{-1} & \cancel{1} \\ -1 & 2 & \cancel{-1} \\ 1 & -1 & \cancel{2} \end{vmatrix}$$

$$= 2 \times \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}$$

$= 2 \times [(2)(2) - (-1)(-1)] + 1 \times [(-1)(2) - (1)(-1)] + 1\,[(-1)(-1) - (1)(2)]$

$= 2 \times [4 - 1] + 1 \times [-2 + 1] + 1\,[1 - 2]$

$= 2(3) + 1(-1) + 1(-1) = 4$

$\therefore \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

This is a characteristic equation.

Step II : Now, we have to verify Cayley-Hamilton Theorem

i.e. we have to show,

$$A^3 - 6A^2 + 9A - 4 = 0$$

Now, $A^2 = A \cdot A$

$$= \begin{array}{c} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix} \end{array} \begin{array}{c} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ C_1 & C_2 & C_3 \end{matrix} \end{array}$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix} \quad \text{...Standard form}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 - A^2 \cdot A = \begin{array}{c} \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix} \end{array} \begin{array}{c} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ C_1 & C_2 & C_3 \end{matrix} \end{array}$$

→ **Using scalar multiplication**

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix} \quad \text{...Standard form}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$\therefore\ A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore\ A^3 - 6A^2 + 9A - 4I = \mathbf{0} \qquad \ldots(1)$$

Hence Caley-Hamilton Theorem verified,

Step III : To find A^{-1}

Now, Equation (1) multiply by A^{-1}

$$\therefore\ A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\therefore\ 4A^{-1} = A^2 - 6A + 9I$$

$$4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Step IV : To find A^4

Since $A^3 - 6A^2 + 9A - 4I = 0$

Multiply by A $\quad A^4 - 6A^3 + 9A^2 - 4A = 0$

$$\therefore\ A^4 = 6A^3 - 9A^2 + 4A$$

$$A^4 = 6\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 9\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 4\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{bmatrix}$$

Example 3.4.2

Verify Cayley-Hamilton theorem for A and hence find A^{-1}.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution : Given matrix is,

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

Step I : The characteristics equation of matrix A is,

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

➔ **Using scalar multiplication**

$$= \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

➔ **Using subtraction of matrices**

$$[A - \lambda I] = \begin{bmatrix} 1-\lambda & 0 & -2 \\ 2 & 2-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$\therefore$ Characteristic equations is,

$$\begin{vmatrix} 1-\lambda & 0 & -2 \\ 2 & 2-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore\ \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

Where S_1 and S_2 are sum of minors of order 1 and 2 along principal diagonal elements respectively.

$$\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

S_1 = sum of diagonal elements

$= 1 + 2 + 2$

$= 5$ (Addition of diagonal Elements)

S_2=[minor of 1] + [Minor of 2] + [Minor of – 2]

$$= \begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{vmatrix}$$

(Sum of Minors of diagonal elements)

$$= \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}$$

$= [(2)(2) - (0)(4)] + [(1)(2) - (0)(-2)] + [(1)(2) - (2)(0)]$

$= [4 - 0] + [2 - 0] + [2 - 0]$

$= 4 + 2 + 2 = 8$

$$|A| = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 1 \times [\text{Minor of } 1] - \underbrace{(0) \times [\text{Minor of } 0]}_{0} + (-2) \times [\text{Minor of } (-2)]$$

$$= 1 \times [\text{Minor of } 1] + (-2) \times [\text{Minor of } (-2)]$$

$$= 1 \times \begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{vmatrix} + (-2) \times \begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 1 \times \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 2 \\ 0 & 0 \end{vmatrix}$$

$= 1 \times [(2)(2) - (0)(4)] - 2 \times [(2)(0) - (0)(2)]$

$= 1 \times [4 - 0] - 2 \times [0 - 0]$

$= 1(4) - 0 - 2(0) = 4$

$$\therefore\ \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

This is characteristics equation.

Step II : **We have to verify Cayley-Hamilton theorem**

i.e. we have to prove,

$$A^3 - 5A^2 + 8A - 4I = 0$$

Now, $A^2 = A \cdot A$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\quad\quad\quad\quad\quad\quad\quad\quad\quad C_1 \quad C_2 \quad C_3$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix} \ldots\text{Standard form}$$

$$= \begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{array} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\quad\quad\quad\quad\quad\quad\quad\quad\quad C_1 \quad C_2 \quad C_3$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix} \ldots\text{Standard form}$$

$$= \begin{bmatrix} 1 & 0 & -14 \\ 14 & 8 & 28 \\ 0 & 0 & 8 \end{bmatrix}$$

$$A^3 - 5A^2 + 8A - 4I = \begin{bmatrix} 1 & 0 & -14 \\ 14 & 8 & 28 \\ 0 & 0 & 8 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore\ A^3 - 5A^2 + 8A - 4I = 0 \qquad \ldots(1)$$

Hence Caylay Hamilton theorem verified.

Step III : **To find A^{-1} :**

Mutiply Equation (1) by A^{-1}

$$\therefore\ A^2 - 5A + 8I - 4A^{-1} = 0$$

$$4A^{-1} = A^2 - 5A + 8I \Rightarrow A^{-1}$$

$$= \frac{1}{4}[A^2 - 5A + 8I]$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 4 \\ -4 & 2 & -8 \\ 0 & 0 & 2 \end{bmatrix} ✓ \qquad \ldots\text{Ans.}$$

Example 3.4.3 **DR. BATU - May 2012**

Find the characteristic equation of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$ **show that the equation is satisfied by A and hence obtain the inverse of the given matrix A.**

Solution :

Step I : The characteristic equation of the matrix A is,

We know

$$|A - \lambda I| = 0$$

Since,

$$A-\lambda I = \begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix} - \lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix} - \begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}$$

$$[A-\lambda I] = \begin{bmatrix}1-\lambda&3&7\\4&2-\lambda&3\\1&2&1-\lambda\end{bmatrix}$$

$\therefore$ Characteristic equations is,

$$\therefore \begin{vmatrix}1-\lambda&3&7\\4&2-\lambda&3\\1&2&1-\lambda\end{vmatrix} = 0$$

→ Using Standard formula

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \qquad \ldots(1)$$

Where S_1 and S_2 are sum of the minors of order 1 and 2 along principal diagonal.

$$\therefore S_1 = \begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}$$

S_1 = sum of diagonal elements

$$= 1+2+1 = 4$$

S_2 = [Minor of 1] + [Minor of 2] + [Minor of 1]

$$= \begin{vmatrix}\cancel{1}&\cancel{3}&\cancel{7}\\\cancel{4}&2&3\\\cancel{1}&2&1\end{vmatrix} + \begin{vmatrix}1&\cancel{3}&7\\\cancel{4}&\cancel{2}&\cancel{3}\\1&\cancel{2}&1\end{vmatrix} + \begin{vmatrix}1&3&\cancel{7}\\4&2&\cancel{3}\\\cancel{1}&\cancel{2}&\cancel{1}\end{vmatrix}$$

(Sum of Minors of diagonal elements)

$$= \begin{vmatrix}2&3\\2&1\end{vmatrix} + \begin{vmatrix}1&7\\1&1\end{vmatrix} + \begin{vmatrix}1&3\\4&2\end{vmatrix}$$

$$= [(2\times1)-(2\times3)]+[(1\times1)-(1\times7)]+[(1\times2)-(4\times3)]$$

$$= (2-6)+(1-7)+(2-12) = (-4)+(-6)+(-10)$$

$$= (-20)$$

Now,

$$|A| = \begin{vmatrix}1&3&7\\4&2&3\\1&2&1\end{vmatrix}$$

$= 1 \times$ (Minor of 1) $- 3 \times$ (Minor of 3) $+ 7 \times$ (Minor of 7)

$$= 1\times\begin{vmatrix}\cancel{1}&\cancel{3}&\cancel{7}\\\cancel{4}&2&3\\\cancel{1}&2&1\end{vmatrix} - 3\times\begin{vmatrix}\cancel{1}&\cancel{3}&\cancel{7}\\4&\cancel{2}&3\\1&\cancel{2}&1\end{vmatrix} + 7\times\begin{vmatrix}\cancel{1}&\cancel{3}&\cancel{7}\\4&2&\cancel{3}\\1&2&\cancel{1}\end{vmatrix}$$

$$= 1\times\begin{vmatrix}2&3\\2&1\end{vmatrix} - 3\times\begin{vmatrix}4&3\\1&1\end{vmatrix} + 7\times\begin{vmatrix}4&2\\1&2\end{vmatrix}$$

$$= [(2\times1)-(2\times3)] - 3\times[(4\times1)-(1\times3)] + 7\times[(4\times2)-(1\times2)]$$

$$= [2-6] - 3\times[4-3] + 7\times[8-2]$$

$$= (-4) - 3\,(1) + 7(6) = -4-3+42 = 35$$

$\therefore$ Substitute these values (S_1, S_2 and $|A|$) in Equation (1)

$$\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0 \qquad \ldots(2)$$

Which is characteristic Equation

Step II : By Cayley Hamilton Theorem : Every square matrix satisfies its own characteristic equation.

$\therefore$ Matrix A satisfies its characteristic Equation (2)

$$\therefore A^3 - 4A^4 - 20A - 35I = 0$$

To find A^{-1} : Multiply throughout by A^{-1}

$$\therefore A^{-1}[A^3 - 4A^2 = 0 \qquad \ldots(\because A^{-1}A = I)$$

$$\therefore 35A^{-1} = A^2 - 4A - 20I$$

$$A^{-1} = \frac{1}{35}[A^2 - 4A - 20I] \qquad \ldots(3)$$

Find $A^2 = A \times A$

$$= \begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}\begin{array}{l}\to R_1\\\to R_2\\\to R_3\end{array}\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}$$

$$\begin{array}{ccc}\uparrow&\uparrow&\uparrow\\C_1&C_2&C_3\end{array}$$

→ Using Matrix Multiplication

$$= \begin{bmatrix}R_1C_1&R_1C_2&R_1C_3\\R_2C_1&R_2C_2&R_2C_3\\R_3C_1&R_3C_2&R_3C_3\end{bmatrix} \quad \ldots\text{(Standard form)}$$

$$\therefore R_1C_1 = (1\times1)+(3\times4)+(7\times1) = 1+12+7 = 20$$

$$R_1C_2 = (1\times3)+(3\times2)+(7\times2) = 3+6+14 = 23$$

$$R_1C_3 = (1\times7)+(3\times3)+(7\times1) = 7+9+7 = 23$$

$$R_2C_1 = (4\times1)+(2\times4)+(3\times1) = 4+8+3 = 15$$

$$R_2C_2 = (4\times3)+(2\times2)+(3\times2) = 12+4+6 = 22$$

$$R_2C_3 = (4\times7)+(2\times3)+(3\times1) = 28+6+3 = 37$$

$$R_3C_1 = (1\times1)+(2\times4)+(1\times1) = 1+8+1 = 10$$

$$R_3C_2 = (1\times3)+(2\times2)+(1\times2) = 3+4+2 = 9$$

$$R_3C_3 = (1\times7)+(2\times3)+(1\times1) = 7+6+1 = 14$$

$$\therefore A^2 = \begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix}$$

Substitute these values in Equation (3)

$$A^{-1} = \frac{1}{35}\left\{\begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix} - 4\times\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix} - 20\times\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right\}$$

$$= \frac{1}{35}\left\{\begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix} - \begin{bmatrix}4&12&28\\16&8&12\\4&8&4\end{bmatrix} - \begin{bmatrix}20&0&0\\0&20&0\\0&0&20\end{bmatrix}\right\}$$

$$= \frac{1}{35}\begin{bmatrix}20-4-20&23-12-0&23-28-0\\15-16-0&22-8-20&37-12-0\\10-4-0&9-8-0&14-4-20\end{bmatrix}$$

$$A^{-1} = \frac{1}{35}\begin{bmatrix}\mathbf{-4}&\mathbf{11}&\mathbf{-5}\\\mathbf{-1}&\mathbf{-6}&\mathbf{25}\\\mathbf{6}&\mathbf{1}&\mathbf{-10}\end{bmatrix} ✓ \qquad \ldots\textbf{Ans.}$$

Example 3.4.4

Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix}1&2&2\\0&2&1\\-1&2&2\end{bmatrix}$ and hence find A^{-1}.

Solution :

Given matrix is, $A = \begin{bmatrix}1&2&2\\0&2&1\\-1&2&2\end{bmatrix}$

Step I : The characteristic equation is,

$$|A-\lambda I| = 0$$

$$A-\lambda I = \begin{bmatrix}1&2&2\\0&2&1\\-1&2&2\end{bmatrix} - \lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

→ Using scalar multiplication

$$= \begin{bmatrix}1&2&2\\0&2&1\\-1&2&2\end{bmatrix} - \begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}$$

→ Using subtraction of matrices

$$[A-\lambda I] = \begin{bmatrix}1-\lambda&2&2\\0&2-\lambda&1\\-1&2&2-\lambda\end{bmatrix}$$

$\therefore$ Characteristic equations is,

$$\begin{vmatrix}1-\lambda&2&2\\0&2-\lambda&1\\-1&2&2-\lambda\end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

Where S_1, S_2 are sum of the minors of order 1 and 2 along the principal diagonal elements respectively.

$$\therefore S_1 = \begin{bmatrix}1&2&2\\0&2&1\\-1&2&2\end{bmatrix}$$

S_1 = sum of diagonal elements

$$S_1 = +2+2 = 5 \qquad \text{(Addition of diagonal Elements)}$$

$$S_2 = [\text{Minor of 1}] + [\text{Minor of 2}] + [\text{Minor of 2}]$$

$$= \begin{vmatrix}\cancel{1}&\cancel{2}&\cancel{2}\\\cancel{0}&2&1\\\cancel{-1}&2&2\end{vmatrix} + \begin{vmatrix}1&\cancel{2}&2\\\cancel{0}&\cancel{2}&\cancel{1}\\-1&\cancel{2}&2\end{vmatrix} + \begin{vmatrix}1&2&\cancel{2}\\0&2&\cancel{1}\\\cancel{-1}&\cancel{2}&\cancel{2}\end{vmatrix}$$

(Sum of Minors of diagonal elements)

$$= \begin{vmatrix}2&1\\2&2\end{vmatrix} + \begin{vmatrix}1&2\\-1&2\end{vmatrix} + \begin{vmatrix}1&2\\0&2\end{vmatrix}$$

$$= [(2)(2)-(2)(1)] + [(1)(2)-(1)(2)] + [(1)(2)-(0)(2)]$$

$= [4-2]+[2+2]+[2-0] = 2+4+2 = 8$

$$|A| = \begin{vmatrix} \boxed{1 \quad 2 \quad 2} \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{vmatrix}$$

$= 1 \times [\text{minor of } 1] - 2 \times [\text{minor of } 2] + 2 \times [\text{Minor of } 2]$

$$= 1 \times \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{2} \\ \cancel{0} & 2 & 1 \\ \cancel{-1} & 2 & 2 \end{vmatrix} - 2 \times \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{2} \\ 0 & \cancel{2} & 1 \\ -1 & \cancel{2} & 2 \end{vmatrix} + 2 \times \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{2} \\ 0 & 2 & \cancel{1} \\ -1 & 2 & \cancel{2} \end{vmatrix}$$

$$= 1 \times \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} - 2 \times \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} + 2 \times \begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix}$$

$= 1 \times [(2)(2)-(2)(1)] - 2 \times [(0)(2) - (-1)(1)]$

$+ 2 \times [(0)(2) - (-1)(2)]$

$= 1 \times [4-2] - 2 \times [0+1] + 2 \times [0+2]$

$= 1(2) - 2(1) + 2(2) = 4$

$\therefore \ \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

This is characteristic equation.

Step II : We have to verify Cayley Hamilton theorem

i.e. We have to prove,

Now, $A^2 = A \cdot A$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \quad \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$
$$\qquad\qquad\qquad\qquad\quad C_1 \ \ C_2 \ \ C_3$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix} \quad \text{...Standard form}$$

$$= \begin{bmatrix} -1 & 10 & 8 \\ -1 & 6 & 4 \\ -3 & 6 & 4 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 10 & 8 \\ -1 & 6 & 4 \\ -3 & 6 & 4 \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \quad \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$
$$\qquad\qquad\qquad\qquad\qquad\quad C_1 \ \ C_2 \ \ C_3$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix} \quad \text{...Standard form}$$

$$= \begin{bmatrix} -9 & 34 & 24 \\ -5 & 18 & 12 \\ -7 & 14 & 8 \end{bmatrix}$$

$$A^3 - 5A^2 + 8A - 4I = \begin{bmatrix} -9 & 34 & 24 \\ -5 & 18 & 12 \\ -7 & 14 & 8 \end{bmatrix}$$

$$-5\begin{bmatrix} -1 & 10 & 8 \\ -1 & 6 & 4 \\ -3 & 6 & 4 \end{bmatrix} + 8\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix} - 4$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e. $A^3 - 5A^2 + 8A - 4I = 0 \qquad \text{...(1)}$

Hence Cayley-Hamilton theorem verified.

Step III : To find A^{-1}

Multiply Equation (1) by A^{-1}

$\therefore \ A^2 - 5A + 8I - 4A^{-1} = 0$

$4A^{-1} = A^2 - 5A + 8I \Rightarrow A^{-1} = \frac{1}{4}[A^2 - 5A + 8I]$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{-2} \\ \mathbf{-1} & \mathbf{4} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-4} & \mathbf{2} \end{bmatrix} \checkmark \qquad \textbf{...Ans.}$$

Example 3.4.5

Verify Cayley Hamilton theorem for the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \textbf{ and final } \mathbf{A^{-1}} \textbf{ and } \mathbf{A^4 - 5A^3 + 8A}$$

Solution :

Given matrix is, $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Step I : The characteristic equation of matrix A is,

$|A - \lambda I| = 0$

$$[A - \lambda I] = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

➔ **Using scalar multiplication**

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

➔ **Using subtraction of matrices**

$$[A - \lambda I] = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix}$$

$\therefore$ Characteristic equations is,

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

Where S_1, S_2 are sum of the minors of order 1 and 2 along principal diagonal elements respectively.

$$\therefore \ S_1 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

S_1 = sum of diagonal elements

$= 2 + 1 + 2 = 5$ (Addition of diagonal Elements)

$S_2 = [\text{Minor of } 2] + [\text{Minor of } 1] + [\text{Minor of } 2]$

$$= \begin{vmatrix} \cancel{2} & \cancel{1} & \cancel{1} \\ \cancel{0} & 1 & 0 \\ \cancel{1} & 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & \cancel{1} & 1 \\ \cancel{0} & \cancel{1} & \cancel{0} \\ 1 & \cancel{1} & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 & \cancel{1} \\ 0 & 1 & \cancel{0} \\ \cancel{1} & \cancel{1} & \cancel{2} \end{vmatrix}$$

(Sum of Minors of diagonal elements)

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$= [(1)(2) - (1)(0)] + [(2)(2) - (1)(1)] + [(2)(1) - (0)(1)]$

$= (2-0) + (4-1) + (2-0)$

$= 2 + 3 + 2 = 7$

$$|A| = \begin{vmatrix} \boxed{2 \quad 1 \quad 1} \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$= 2 \times [\text{Minor of } 2] - 1 \times [\text{Minor of } 1] + 1 \times [\text{Minor of } 1]$

$$= 2 \times \begin{vmatrix} \cancel{2} & \cancel{1} & \cancel{1} \\ \cancel{0} & 1 & 0 \\ \cancel{1} & 1 & 2 \end{vmatrix} - 1 \times \begin{vmatrix} \cancel{2} & \cancel{1} & \cancel{1} \\ 0 & \cancel{1} & 0 \\ 1 & \cancel{1} & 2 \end{vmatrix} + 1 \times \begin{vmatrix} \cancel{2} & \cancel{1} & \cancel{1} \\ 0 & 1 & \cancel{0} \\ 1 & 1 & \cancel{2} \end{vmatrix}$$

$$= 2 \times \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} - 1 \times \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$$

$= 2 \times [(1)(a) - (1)(0)] - 1 \times [(0)(2) - (1)(0)]$

$+ 1 \times [(0)(1) - (1)(1)]$

$= 2 \times [2-0] - 1 \times [0-0] + 1\,[0-1]$

$= 2(2) - (0) + (-1) = 4 - 0 - 1 = 3$

$\therefore \ \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$ This is a characteristic equation.

Step II : We have to verify Cayley Hamilton theorem,

i.e. we have to prove,

$A^3 - 5A^2 + 7A - 3I = 0$

Now, $A^2 = A \cdot A$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \quad \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$
$$\qquad\qquad\qquad\qquad\quad C_1 \ \ C_2 \ \ C_3$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix}$$

...Standard form

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{array}{c} \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \end{array} \begin{array}{c} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \\ \uparrow \;\; \uparrow \;\; \uparrow \\ C_1 \;\; C_2 \;\; C_3 \end{array}$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix} \quad \text{...Standard form}$$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$$A^3 - 5A^2 + 7A - 3I = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $\mathbf{A^3 - 5A^2 + 7A - 3I = 0}$ ...(1)

Hence **Caylay Hamilton theorem verified.**

Step III : To find A^{-1}

Multiply Equation (1) by A^{-1}

$$\therefore A^2 - 5A + 7I - 3A^{-1} = 0$$

$$\therefore 3A^{-1} = A^2 - 5A + 7I$$

$$A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$$

$$\therefore A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Step IV : Now,

$$A^4 - 5A^3 + 8A^2 = (A^4 - 5A^3 + 7A^2 - 3A) + (A^2 + 3A)$$

$$= A(A^3 - 5A^2 + 7A - 3I) + (A^2 + 3A)$$

$$= A(0) + A^2 + 3A \quad \text{[From Equation (1)]}$$

$$= A^2 + 3A$$

$$\mathbf{A^4 - 5A^3 + 8A^2} = \begin{bmatrix} \mathbf{11} & \mathbf{7} & \mathbf{7} \\ \mathbf{0} & \mathbf{4} & \mathbf{0} \\ \mathbf{7} & \mathbf{7} & \mathbf{11} \end{bmatrix} \checkmark \quad \text{...Ans.}$$

Example 3.4.6

Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$**.**

Hence simplify the expression $A^5 - 3A^4 - 8A^3 - 7A^2 - 10A - 4I$ and obtain corresponding matrix.

Solution : Given matrix is,

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Step I : The characteristic equation of matrix A is,

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Using scalar multiplication

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

→ Using subtraction of matrices

$$[A - \lambda I] = \begin{bmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{bmatrix}$$

$\therefore$ Characteristic equations is,

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

Where S_1 and S_2 are sum of the minors of order 1 and 2 along the principle diagonal elements respectively.

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

S_1 = sum of diagonal elements

$$S_1 = 1 + 1 + 1 = 3$$

$$S_2 = [\text{Minor of } 1] + [\text{Minor of } 1] + [\text{Minor of } 1]$$

$$= \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{2} \\ \cancel{2} & 1 & 2 \\ \cancel{2} & 2 & 1 \end{vmatrix} + \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{2} \\ 2 & \cancel{1} & 2 \\ 2 & \cancel{2} & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & \cancel{2} \\ 2 & 1 & \cancel{2} \\ \cancel{2} & \cancel{2} & \cancel{1} \end{vmatrix}$$

(Sum of Minors of diagonal elements)

$$= \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= [(1)(1) - (2)(2)] + [(1)(1) - (2)(2)] + [(1)(1) - (2)(2)]$$

$$= [1 - 4] + [1 - 4] + [1 - 4]$$

$$= -3 - 3 - 3 = -9$$

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$

$$= 1 \times [\text{Minor of } 1] - (2) \times [\text{Minor of } 2] + 2 \times [\text{Minor of } 2]$$

$$= 1 \times \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{2} \\ \cancel{2} & 1 & 2 \\ \cancel{2} & 2 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{2} \\ 2 & \cancel{1} & 2 \\ 2 & \cancel{2} & 1 \end{vmatrix} + 2 \times \begin{vmatrix} 1 & 2 & \cancel{2} \\ 2 & 1 & \cancel{2} \\ \cancel{2} & \cancel{2} & \cancel{1} \end{vmatrix}$$

$$= 1 \times \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} + 2 \times \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= 1 \times [\underbrace{(1)(1)}_{1} - \underbrace{(2)(2)}_{4}] - 2 \times [\underbrace{(2)(1)}_{2} - \underbrace{(2)(2)}_{4}] + 2 \times [\underbrace{(2)(2)}_{4} - \underbrace{(2)(1)}_{2}]$$

$$= 1 \times [1 - 4] - 2 \times [2 - 4] + 2 \times [4 - 2]$$

$$= 1(-3) - 2(-2) + 2(2)$$

$$= -3 + 4 + 4 = 5$$

$\therefore$ $\mathbf{\lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0}$ This is a characteristic equation.

Step II : We have to verify Caylay Hamilton theorem,

i.e. we have to prove,

$$A^3 - 3A^2 - 9A - 5I = 0$$

$$\text{Now, } A^2 = A \cdot A$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \begin{array}{c} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \\ \uparrow \;\; \uparrow \;\; \uparrow \\ C_1 \;\; C_2 \;\; C_3 \end{array}$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix} \quad \text{...Standard form}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \begin{array}{c} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \\ \uparrow \;\; \uparrow \;\; \uparrow \\ C_1 \;\; C_2 \;\; C_3 \end{array}$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix} \quad \text{...Standard form}$$

$$= \begin{bmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix}$$

$$A^3 - 3A^2 - 9A - 5I = \begin{bmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix} - 3\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 9\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $A^3 - 3A^2 - 9A - 5I = 0$ ...(1)

Hence Caylay Hamilton theorem verified.

Step III :

$$A^5 - 3A^4 - 8A^3 - 7A^2 - 10A - 4I = (A^5 - 3A^4 - 9A^3 - 5A^2) + (A^3 - 2A^2 - 10A - 4I)$$

$= A^2(A^3 - 3A^2 - 9A - 5I) + (A^3 - 3A^2 - 9A - 5I) + (A^2 - A + I)$

$= A^2(0) + (0) + A^2 - A + I$ [$\because$ From Equation (1)]

$= A^2 - A + I$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^5 - 3A^4 - 8A^3 - 7A^2 - 10A - 4I = \begin{bmatrix} 9 & 6 & 6 \\ 6 & 9 & 6 \\ 6 & 6 & 9 \end{bmatrix} \checkmark \quad \text{...Ans.}$$

Exercise 2

Verify Caylay-Hamilton theorem

Ex. 1 : $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and find A^{-1}

Ans. : $A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ex. 2 : $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ and find A^{-1}

Ans. : $A^{-1} = \begin{bmatrix} \frac{-2}{11} & \frac{5}{11} & \frac{-1}{11} \\ \frac{-1}{11} & \frac{-3}{11} & \frac{5}{11} \\ \frac{7}{11} & \frac{-1}{11} & \frac{-2}{11} \end{bmatrix}$

Ex. 3 $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ and find A^{-1}

Ans. : $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{-3}{10} \\ \frac{1}{2} & \frac{-2}{5} & \frac{1}{10} \\ \frac{-1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

Ex. 4 $A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix}$ and find A^{-1}

Ans. : $A^{-1} = \begin{bmatrix} \frac{2}{3} & 1 & \frac{-1}{3} \\ \frac{2}{3} & 3 & \frac{-4}{3} \\ \frac{-1}{3} & -1 & \frac{2}{3} \end{bmatrix}$

Ex. 5 $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and find A^{-1} find A^{-2}

Ans. : $A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & 0 \\ \frac{2}{5} & \frac{-1}{5} & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Ex. 6 $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ and find A^{-1}

Ans. : $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

Ex. 7 $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ and find A^{-1}

Ans. : $A^{-1} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 0 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$

Ex. 8 $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$ and find A^{-1}

Ans. : $A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Ex. 9 $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ and find A^{-1}

Ans. : $A^{-1} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ \frac{-5}{4} & \frac{-1}{4} & \frac{-3}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix}$

Ex. 10 $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -2 & 0 \\ 3 & -1 & 2 \end{bmatrix}$ and find A^{-1}

Ans. : $A^{-1} = \begin{bmatrix} \frac{4}{7} & \frac{5}{7} & \frac{-2}{7} \\ \frac{2}{7} & \frac{-1}{7} & \frac{-1}{7} \\ \frac{-5}{7} & \frac{-8}{7} & \frac{6}{7} \end{bmatrix}$

Ex. 11 $A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$ and find A^{-1}

Ans. : $A^{-1} = \begin{bmatrix} \frac{-3}{7} & \frac{8}{7} & \frac{6}{7} \\ 1 & -2 & -1 \\ \frac{-1}{7} & \frac{5}{7} & \frac{2}{7} \end{bmatrix}$

Ex. 12 $A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & -1 & 1 \\ 3 & -1 & 2 \end{bmatrix}$ and find A^{-1}

Ans. : $A^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & \frac{1}{5} \\ \frac{-1}{5} & -2 & \frac{4}{5} \\ \frac{-2}{5} & -1 & \frac{3}{5} \end{bmatrix}$

Ex. 13 $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$ and find A^{-1}

Ans. : $A^{-1} = \frac{1}{81}\begin{bmatrix} 1 & 16 & -20 \\ 16 & 13 & 4 \\ -20 & 4 & -5 \end{bmatrix}$

Ex. 14 $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$ and find the inverse.

Ans. : $A^{-1} = \frac{1}{12}\begin{bmatrix} 5 & -6 & -2 \\ 1 & 6 & 2 \\ -11 & 18 & 2 \end{bmatrix}$

Ex. 15 $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, find A^{-1}

Ans. : $A^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

Ex. 16 $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ and A^{-1}

Ans. : $A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

3.5 University Questions and Answers

→ Oct 17

Q.1 Find the eigen values and the corresponding

$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

(Refer Section 3.3.4) **(6 Marks)**

→ Dec 17

Q.2 Find the eigen values and the corresponding eigen vectors for the matrix

$A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$

(Refer Ex. 3.3.8) **(6 Marks)**

→ May 18

Q. 3 Using Cayley-Hamilton theorem, find A^{-1} where the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ **(6 Marks)**

Ans. :

The characteristic equation of the matrix A is given by,

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0$

The cayley Hamilton theorem is verified if A satisfies the above characteristic equation i.e.

$A^3 - 11A^2 - 4A + I = 0$ ...(1)

We have,

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$$

$$\therefore A^3 - 11A^2 - 4A + I = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$$

$$-11\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

From equation (1), we have

$$A^{-1} = -A^2 + 11A + 4I$$

$$= -\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + 11\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

Chapter Ends...

□□□

Multiple Choice Questions (MCQ)

UNIT 1

Linear Algebra-Matrices

- Short Questions and Answers
- Fill in the Blanks
- Multiple Choice Questions

UNIT I

Linear Algebra - Matrices

Short Questions and Answers

Ex. 1 : Find the rank of a matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 7 & 8 & 9 \\ 11 & 12 & 13 \end{bmatrix}$

Soln. : Given matrix,

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 7 & 8 & 9 \\ 11 & 12 & 13 \end{bmatrix}$$

Operate $R_2 - 3R_1$ $\sim \begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & -3 \\ 11 & 12 & 13 \end{bmatrix}$

Operate $R_2 \leftrightarrow R_1$ (or R_{12}) $\sim \begin{bmatrix} 1 & -1 & -3 \\ 2 & 3 & 4 \\ 11 & 12 & 13 \end{bmatrix}$

Operate $R_2 - 2R_1$ $R_3 - 11\ R_1$ $\sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & 5 & 10 \\ 0 & 23 & 46 \end{bmatrix}$

Operate $\frac{R_2}{5}$ and $\frac{R_3}{23}$ $\sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

Operate $R_3 - R_2$ $\sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Which is echelon form Matrix A

$\therefore$ Rank of A $=\rho(A) = \begin{pmatrix} \text{Total} \\ \text{Number of} \\ \text{Rows} \end{pmatrix} - \begin{pmatrix} \text{Number of} \\ \text{Rows} \\ \text{conctaining} \\ \text{all zero} \end{pmatrix}$

$\therefore \rho(A) = 3 - 1 - 2$

$\therefore$ **$\rho(A) = 2$ ✓**

Ex. 2 : Find the rank of a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Soln. : Given Matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Operate $R_2 - 4\ R_1$ and $R_3 - 7R_1$ $\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$

Operate $\frac{R_2}{-3}$ and $\frac{R_3}{-6}$ $\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

Operate $R_3 - R_2$ $\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

which is echelon form of matrix A

$\therefore$ Rank of A $=\rho(A) = \begin{pmatrix} \text{Total} \\ \text{Number of} \\ \text{Rows} \end{pmatrix} - \begin{pmatrix} \text{Number of} \\ \text{Rows} \\ \text{conctaining} \\ \text{all zero} \end{pmatrix}$

$\therefore \rho(A) = 3 - 1 = 2$

$=$ **$\rho(A) = 2$ ✓**

Ex. 3 : Find the rank of $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

Soln. : Given Matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

By $R_2 - R_1$; $R_3 - R_1$ $\sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -1 & -4 \\ 0 & -3 & -2 \end{bmatrix}$

By $R_3 - 3R_2$ $\sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -1 & -4 \\ 0 & 0 & 10 \end{bmatrix}$

$\therefore$ **$\rho(A) = 3$ ✓**

Ex. 4 : How many solution of the linear system of equations $x + 2y = 0, 2x - y + z = 0, 4x + 3y + z = 0$.

Soln. : Given system $AX = Z$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 4 & 3 & 1 \end{bmatrix} x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|A| = 1(-4) - 2(-2) = -4 + 4 = 0$$

$|A| = 0 \Rightarrow \rho(A) < 3 \Rightarrow$ system has infinite many solutions.

Ex. 5 : Solve the linear system of equations $x + 2y + 3z = 0$; $3x + 6y + 9z = 0$; $4x + 8y + 12z = 0$.

Soln. : The system of equations,

$$AX = Z$$

$$[A \mid Z] = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 0 \\ 3 & 6 & 9 & 0 \\ 4 & 8 & 12 & 0 \end{array}\right]$$

Operate $R_2 - 3R_1$; $R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = \rho(A \mid Z) = 1$ i.e. $r = 1$

and $n = 3$ (Number of unknowns)

$\therefore$ **$r < n \Rightarrow$ system have infinite many solutions.** ✓

Ex. 6 : How many solutions of the linear system of equations $x + 3y + 2z = 0$; $4x + 6y + 5z = 0$; $3x + 5y + 4z = 0$.

Soln. : The system of equations, $AX = Z$

$$[A \mid Z] = \left[\begin{array}{ccc:c} 1 & 3 & 2 & 0 \\ 4 & 6 & 5 & 0 \\ 3 & 5 & 4 & 0 \end{array}\right]$$

Operate, $R_2 - 4R_1$; $R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 2 & 0 \\ 0 & -6 & -3 & 0 \\ 0 & -4 & -2 & 0 \end{array}\right]$$

Operate $\frac{R_2}{-3} \cdot \frac{R_3}{2}$

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

Operate $R_3 - R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \rho(A) = \rho(A \mid Z) = 2$

$r = 2$ and $n = 3$ i.e. $r < n$

$\Rightarrow$ system have infinite many solution. ✓

Ex. 7 : How many solutions of the system of linear equation $x + y + z = 0$; $2x + 2y + 2z = 0$; $3x + 3y + 3z = 0$.

Soln. : Given system in matrix from,

$$\text{Now } [A \mid B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right]$$

$$\rho(A) = \rho(A \mid B) = 1$$

Since 2^{nd} and 3^{rd} rows are multiples of I^{st} row.

$\therefore$ $r = 1$ and n = number of unknowns = 3

$r < 3$ $\therefore$ **system have infinite many solutions.** ✓

Ex. 8 : Solve the system of linear equations $x + 3y + 5z = 0$; $x + 2y + z = 0$; $x + 3z = 0$.

Soln. : Given system ;

$$\text{Now } [A \mid Z] = \left[\begin{array}{ccc:c} 1 & 3 & 5 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 3 & 0 \end{array}\right]$$

Operate $R_2 - R_1$; $R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 5 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -3 & -2 & 0 \end{array}\right]$$

$$\text{Operate } R_3 - 3R_2 \sim \left[\begin{array}{ccc:c} 1 & 3 & 5 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 10 & 0 \end{array}\right]$$

$\therefore \rho(A) = \rho(A \mid Z) = 3$ i.e. $r = 3$, $n = 3$

$r = n \Rightarrow$ **The system has unique solution which is trivial solution $x = y = z = 0$** ✓

Ex. 9 : How many solutions of the linear system of equations $x + 2y + 3z = 1$; $2x + 3y + 4z = 1$; $3x + 4y + 5z = 1$

Soln. : Given system of equations,

$$AX = B$$

$$[A \mid B] = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 5 & 1 \end{array}\right]$$

$R_2 - 2R_1$; $R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & -2 & -4 & -2 \end{array}\right]$$

$$R_3 - 2R_2 \sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ $\rho(A) = \rho(A \mid B) = 2$ and $n = 3$ (number of unknowns)

$r < n \Rightarrow$ **The system have infinite many solutions.** ✓

Ex. 10: How many solutions of the linear system of equations $x + y + 2z = 2$; $2x + 2y + 4z = 4$; $3x + 3y + 6z = 6$

Soln. : Given system as, $AX = B$

$$[A \mid B] = \left[\begin{array}{ccc:c} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 3 & 3 & 6 & 6 \end{array}\right]$$

$R_2 - 2R_1$; $R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ $\rho(A) = \rho(A \mid B) = 1$ i.e. $r = 1$

and $n = 3$ (number of unknowns)

$\therefore$ $r < n \Rightarrow$ system have infinite many solutions

Ex. 11: Solve the linear system of equations $x + 2y + 3z = 4$; $2x + 3y + 6z = 2$, $3x + 6y + 9z = 1$

Soln. : Given system, $AX = B$

$$[A \mid B] = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 2 \\ 3 & 6 & 9 & 1 \end{array}\right]$$

Operate $R_2 - 2R_1$; $R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -11 \end{array}\right]$$

$$\rho(A) \neq \rho(A \mid B)$$

$\therefore$ **System has no solution.** ✓

Ex. 12 : Find the characteristic equation for the matrix

$$A = \begin{bmatrix} 14 & -10 \\ 5 & 1 \end{bmatrix}$$

Soln. : Given matrix,

$$A = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}$$

The characteristic equation of matrix A is,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 14 - \lambda & -10 \\ 5 & -1 - \lambda \end{vmatrix} = 0$$

$\Rightarrow (14 - \lambda)(-1 - \lambda) + 50 = 0$

$(\lambda - 14)(\lambda + 1) + 50 = 0 \Rightarrow \lambda^2 - 13\lambda - 14 + 50 = 0$

$\therefore$ **$\lambda^2 - 13\lambda + 36 = 0$** ✓

Ex. 13 : Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Soln. : The characteristic equation is,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5 - \lambda & 0 & 1 \\ 0 & -2 - \lambda & 0 \\ 1 & 0 & 5 - \lambda \end{vmatrix} = 0$$

$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - |A| = 0$

$S_1 = 5 - 2 + 5 = 8$

$$S_2 = \begin{vmatrix} -2 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} + \begin{vmatrix} 5 & 0 \\ 0 & -2 \end{vmatrix}$$

$= -10 + 24 - 10 = 4$

$|A| = -48$

$\therefore$ **$\lambda^3 - 8\lambda^2 + 4\lambda + 48 = 0$** ✓

Ex. 14 : For a matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ the characteristic equation is $\lambda^3 - \lambda^2 - 5\lambda - 3 = 0$ then find the value of A^{-1} in terms of A.

Soln. : By Calay Hamilton theorem every square matrix satisfies it's own characteristic equation.

$\therefore A^3 - A^2 - 5A - 3I = 0$

$\therefore A^2 - A - 5I - 3A^{-1} = 0$ (...Multiply by A^{-1})

$\therefore 3A^{-1} = A^2 - A - 5I \Rightarrow A^{-1} = \frac{1}{3}[A^2 - A - 5I]$

Fill in the Blanks / Multiple Choice Questions

Rank of a Matrix

Q. 1 If A is a matrix of order m × n then rank of the matrix A is _______

(a) ≥ max, (m, n) (b) ≥ min (m, n)
(c) ≤ max (m, n) (d) ≤ min (m, n) **Ans. : (d)**

Q. 2 If A is a matrix and there is a non-vanishing (non-zero) minor of order r then rank of matrix A is_______

(a) ≥ r (b) = r (c) ≤ r (d) r + 1 **Ans. : (a)**

Q. 3 If A is a matrix and all minors of order ≥ r + 1 are zero then rank of matrix A is _______

(a) ≥ r (b) ≤ r (c) = r (d) r + 1 **Ans. : (b)**

Q. 4 If a square matrix of order n with |A| = 0 then rank of matrix A is _______

(a) > n (b) = n (c) 0 (d) < n **Ans. : (d)**

Q. 5 If a square matrix A of order n with | A |≠ 0 then rank of matrix A is _______

(a) > n (b) = n (c) 0 (d) < n **Ans. : (b)**

Q. 6 The rank of null matrix is _______

(a) Not defined (b) 0 (c) 1 d) Order of matrix **Ans. : (b)**

Q. 7 If A is a square matrix of order n with $\rho(A) = n$ then matrix A is _______

(a) Null matrix (b) Singular
(c) Non singular (d) orthogonal **Ans. : (c)**

Q. 8 If A is a square matrix of order n with $\rho(A) < n$ then matrix A is _____

(a) Null matrix (b) Singular
(c) Non singular (d) As above **Ans. : (b)**

Q. 9 Rank of the identify (unit) matrix is _______

(a) 1 (b) 0 (c) Order of matrix (d) None of these **Ans. : (c)**

Q. 10 If A is a matrix then by elementary transformation operating on A, rank of matrix is _______.

(a) Same (not alter) (b) Reduced by 1
(c) Reduced by 2 (d) None of these **Ans. : (a)**

Q. 11 Which of the following is not elementary transformation ?

(a) kR_i $k \neq 0$ (b) $R_i - R_j$
(c) $R_i + kR_j$ $k \neq 0$ (d) $R_i + k$ $k \neq 0$ **Ans. : (d)**

Q. 12 Which of the following is not elementary transformation ?

(a) $C_i - C_j$ $k \neq 0$ (b) $C_i - k$ $k \neq 0$
(c) kC_i $k \neq 0$ (d) $C_i + kC_j$ $k \neq 0$ **Ans. : (b)**

Q. 13 If A is a matrix with rank r then the rank of its transpose matrix A^T (OR A') is _______.

(a) r – 1 (b) r + 1 (c) r (d) None of above **Ans. : (c)**

Q. 14 The rank of the identity (unit) matrix is equal to _______.

(a) Zero (b) 1
(c) Order of a matrix (d) None of these **Ans. : (c)**

Q. 15 If A is a matrix and PAQ is in normal form then _______

(a) P = Q. (b) $P \neq Q^{-1}$
(c) P and Q are unique. (d) P and Q are not unique. **Ans. : (d)**

Q. 16 If A is a square matrix and PAQ is in normal form with two non-singular matrices P and Q then A^{-1} = _______

(a) PQ (b) $P^{-1}Q^{-1}$ (c) QP (d) $Q^{-1}P^{-1}$ **Ans. : (c)**

Q. 17 From the following matrices which is not normal form of a matrix _______.

(a) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ **Ans. : (a)**

Q. 18 Which of the following matrices is not a normal form of a matrix _______

(a) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (c) $[I_r\ 0]$ (d) $\begin{bmatrix} I_r & 0 \\ 0 & 1 \end{bmatrix}$ **Ans. : (d)**

Q. 19 Which of the following matrices is normal form of a matrix _______

(a) $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ (b) $[I_3\ 1]$

(c) $\begin{bmatrix} I_3 & 0 \\ 0 & 1 \end{bmatrix}$ (d) All above **Ans. : (a)**

Q. 20 The rank of the matrix A = $\begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \end{bmatrix}$ is _______

(a) 0 (b) 1 (c) 2 (d) 3 **Ans. : (b)**

Q. 21 Rank of a matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{bmatrix}$ is_______

(a) 0 (b) 1 (c) 2 (d) 3 **Ans. : (b)**

Q. 22 The rank of the matrix A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is _______

(a) 3 (b) 2 (c) 0 (d) 1 **Ans. : (a)**

Q. 23 The rank of the matrix A = $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is _______

(a) 0 (b) 1 (c) 2 (d) 3 **Ans. : (c)**

Q. 24 The normal for of matrix A is $[I_4]$ then rank of matrix A is _____

(a) 1 (b) 2 (c) 3 (d) 4 **Ans. : (d)**

Q. 25 If A is a matrix of order 3 × 4 reduces into normal form $[I_3\ 0]$ then rank of matrix A is _____

(a) 3 (b) 4 (c) 2 (d) 1 **Ans. : (a)**

Q. 26 If matrix A reduces by using elementary transformation into $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ then normal form of matrix A is _____

(a) $[I_2]$ (b) $[I_3]$ (c) $[I_2\ 0]$ (d) $\begin{bmatrix} I_2 \\ 0 \end{bmatrix}$ **Ans. : (c)**

Q. 27 If matrix A reduces by using elementary transformation into $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ then norm form of matrix A is ______

(a) $\begin{bmatrix} I_2 \\ 0 \end{bmatrix}$ (b) $[I_2\ 0]$ (c) $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ (d) $[I_2]$ **Ans. : (c)**

Q. 28 The rank of a matrix A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is _______

(a) 0 (b) 1 (c) 2 (d) 3 **Ans. : (c)**

Q. 29 Which of the following is a normal form a matrix _______

(a) $\begin{bmatrix} I_2 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **Ans. : (b)**

Q. 30 The rank of a matrix A = $\begin{bmatrix} 2 & 5 \\ 4 & 8 \end{bmatrix}$ is_______

(a) 0 (b) 1 (c) 2 (d) None of these **Ans. : (c**

Soln. : A = $\begin{bmatrix} 2 & 5 \\ 4 & 8 \end{bmatrix}$

Operate $R_2 - 2R_1$ $\sim \begin{bmatrix} 2 & 5 \\ 0 & -2 \end{bmatrix}$

$\therefore$ Rank of matrix is 2.

Q. 31 Which of the following is a echelon (cannonical form of matrix _______.

(a) $\begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 5 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ **Ans. : (**

Q. 32 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 8 & 9 \end{bmatrix}$ then rank of matrix A is______

(a) 0 (b) 2 (c) 1 (d) 3 **Ans. : (b)**

Soln. :

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1+4 & 2+4 & 3+4 \\ 1+6 & 2+6 & 3+6 \end{bmatrix}$$

$\therefore$ Rank of matrix A = $\rho(A) = 2$

Q. 33 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 10 & 15 \\ 3 & 6 & 9 \end{bmatrix}$ then rank of matrix A is______.

(a) 0 (b) 2 (c) 1 (d) 3 **Ans. : (c)**

Soln. : $A = \begin{bmatrix} 1 & 2 & 3 \\ 1\times 5 & 2\times 5 & 3\times 5 \\ 1\times 3 & 2\times 3 & 3\times 3 \end{bmatrix}$

$\therefore$ $\rho(A) = 1$

System of Linear Equations

Q. 34 A linear system of equation AX = B is said to be homogeneous system of equation if______.

(a) A = B (b) B = Z (null matrix)

(c) $B \neq Z$ (Null matrix) (d) $A \neq B$ **Ans. : (b)**

Q. 35 A linear system of equation AX = B is said to be non homogeneous system of equation if______.

(a) A = B (b) B = Z (null matrix)

(c) $B \neq Z$ (Null matrix) (d) $A \neq B$ **Ans. : (c)**

Q. 36 A system of linear equations AX = B is consistent if ______.

(a) $\rho(A) = \rho(B)$ (b) $\rho(A) \neq \rho(A|B)$

(c) $\rho(A) = \rho(A|B)$ (d) $\rho(A) < \rho(A|B)$ **Ans. : (c)**

Q. 37 A homogeneous system of linear equations AX = Z has only trivial solution if______.

(a) ρ (A) = Number of equations

(b) ρ (A) = Number of unknowns

(c) Number of equations = Number of unknowns

(d) ρ (A) < Number of unknowns **Ans. : (b)**

Q. 38 A homogeneous system of linear equations AX = Z has infinite many solution if______.

(a) ρ (A) = Number of equations

(b) ρ (A) = Number of unknowns

(c) Number of equations = Number of unknowns

(d) ρ (A) < Number of unknowns **Ans. : (d)**

Q. 39 A homogeneous system of linear equations AX = Z (Z – null matrix) is ______.

(a) Always inconsistent

(b) Always consistent

(c) May or may not be consistent

(d) None of above **Ans. : (b)**

Q. 40 A non homogeneous system of linear equations AX = B has a unique solution if ______.

(a) $\rho(A) = \rho(A|B)$ = Number of unknowns

(b) $\rho(A) = \rho(A|B)$ = Number of equations

(c) $\rho(A) \neq \rho(A|B)$

(d) Number of unknowns = Number of equations **Ans. : (a)**

Q. 41 A non homogeneous system of linear equations AX = B have infinite many solution if ______.

(a) $\rho(A) = \rho(A|B)$ = Number of unknowns

(b) $\rho(A) = \rho(A|B)$ = Number of equations

(c) $\rho(A) = \rho(A|B)$ = r and r < number of unknowns

(d) $\rho(A) \neq \rho(A|B)$ **Ans. : (c)**

Q. 42 A non homogeneous system of linear equation AX = B has no solution if ______.

(a) $\rho(A) = \rho(A|B)$ (b) $\rho(A) \neq \rho(A|B)$

(c) $\rho(A) < \rho(A|B)$ (d) $\rho(A) > \rho(A|B)$ **Ans. : (b)**

Q. 43 The system $2x + y + 2z = 0$, $x + y + 3z = 0$, $x + y + \lambda z = 0$ have a non-trivial solution if λ = ______

(a) 2 (b) 0 (c) 4 (d) 2 **Ans. : (b)**

Soln. : $|A| = 0 \Rightarrow$ These are infinite many solution i.e. non-trivial solution.

Q. 44 A system of linear equation AX = B with number of unknowns are equal to number of equations with $|A| \neq 0$ then X = ______.

(a) AB (b) $A^{-1}B$ (c) BA^{-1} (d) AB^{-1} **Ans. : (b)**

Q. 45 A homogeneous system of linear equations AX = Z with number of unknowns are equal to number of equations and $|A| \neq 0$ then system has ____

(a) Non trivial solution (b) Trivial solution

(c) Infinite many solution (d) No solution **Ans. : (b)**

Soln. : When system, AX = Z

$|A| \neq 0 \Rightarrow \rho(A)$ = order of A

i.e. $\rho(A)$ = number of unknown (r = n)

$\therefore$ system has only trivial solution

Q. 46 In a homogeneous linear system of equations AX = Z (Z = null matrix) with number of equations are equal to number of unknowns has a non trivial solution if ______.

(a) $|A| = 0$ (b) $|A| \neq 0$ (c) $|A| > 0$ (d) $|A| < 0$ **Ans. : (a)**

Soln. : Since system AX = Z has non irival solution.

i..e. $\rho(A)$ < number of unknowns

i.e. $\rho(A)$ < order of A $\Rightarrow |A| = 0$

Q. 47 A homogenous system of linear equations with 3 equations and 3 unknowns has a unique solution if ______.

(a) $|A| = 0$ (b) $|A| \neq 0$ (c) $|A| = 1$ (d) $|A| \neq 1$ **Ans. : (b)**

Q. 48 In the linear system of equations AX = B, by elementary transformation matrix

[A | B] reduces to

$$\left[\begin{array}{ccc:c} 1 & -6 & -17 & -19 \\ 0 & 15 & 39 & 47 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right]$$

then system has unique solution if ______

(a) $\lambda = 5$ $\mu = 9$ (b) $\lambda = 5, \mu \neq 9$

(c) $\mu \neq 9$, λ can have any value

(d) $\lambda \neq 5$ and μ can have any value **Ans. (d)**

Q. 49 In the linear system of equations AX = B, by elementary transformation matrix

[A | B] reduces to

$$\left[\begin{array}{ccc:c} 1 & -6 & -17 & -19 \\ 0 & 15 & 39 & 47 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right]$$

Then the system have infinite may solution if _____

(a) $\lambda = 5$ $\mu = 9$ (b) $\lambda = \lambda = 5, \mu \neq 9$

(c) $\mu \neq 9$, λ can have any value (d) As above

Ans. : (a)

Eigen values and Eigen Vectors / Cayley Hamilton Theorem

Q. 50 If A is a square matrix then the characteristic equation of a matrix A is ______.

(a) $|A| = 0$ (b) $|A - \lambda I| = 0$

(c) $|A| = |\lambda I|$ (d) $|A + \lambda I| = 0$ **Ans. : (b)**

Q. 51 If A is a square matrix of order 3 then the value of characteristic equation $|A - \lambda I|$ is ______. (S_1, S_2 are sum of minors of order 1 and 2 along principal diagonal respectively)

(a) $\lambda^3 + S_1\lambda^2 + S_2\lambda + |A| = 0$ (b) $\lambda^3 - S_1\lambda^2 - S_2\lambda - |A| = 0$

(c) $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$ (d) $\lambda^3 - S_1\lambda^2 - S_2\lambda + |A| = 0$

Ans. : (c)

Q. 52 If A is a square matrix then sum of the Eigen values of matrix A is equal to ______.

(a) Product of diagonal elements of A

(b) |A| i.e. determinant of A

(c) Sum of the diagonal elements of A

(d) None of the above **Ans. : (c)**

Q. 53 If A is a square matrix then the product of the Eigen values of matrix A is equal to______.

(a) Product of diagonal elements of A

(b) |A| i.e. determinant of A

(c) Sum of the diagonal elements of A

(d) None of above **Ans. : (b)**

Q. 54 If $\lambda_1, \lambda_2, \lambda_3$ are the Eigen values of a square matrix A with $|A| \neq 0$ then the Eigen values of A^{-1} are______.

(a) $\lambda_1, \lambda_2, \lambda_3$ (b) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$

(c) $\lambda_1 + \lambda_2, \lambda_2 + \lambda_3, \lambda_3 + \lambda_1$ (d) $-\lambda_1, -\lambda_2, -\lambda_3$

Ans. : (b)

Q. 55 A spectrum of a square matrix A is a set of______.

(a) Eigen vectors

(b) Diagonal elements

(c) Eigen values and Eigen vectors

(d) Eigen values **Ans. : (d)**

Q. 56 If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then trace of A is equal to______

(a) $\lambda_1 + \lambda_2 + \lambda_3$ Where $\lambda_1, \lambda_2, \lambda_3$ are Eigen values.

(b) $|A|$ i.e. determinant of A

(c) $a_{11} + a_{22} + a_{33}$

(d) Eigen vectors X_1, X_2, X_3 **Ans. : (c)**

Q. 57 If $\lambda_1, \lambda_2, \lambda_3$ are the Eigen values of a square matrix A then the Eigen values of a matrix A^m are______.

(a) $\lambda_1, \lambda_2, \lambda_3$ (b) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$

(c) $\lambda_1^m, \lambda_2^m, \lambda_3^m$ (d) $\frac{1}{\lambda_1^m}, \frac{1}{\lambda_2^m}, \frac{1}{\lambda_3^m}$ **Ans. : (c)**

Q. 58 If X_1, X_2, X_3 are the Eigen vectors of a square matrix A then X_1, X_2, X_3 are ______.

(a) Always equal (b) Linearly dependent

(c) Linearly independent (d) None of these

Ans. : (c)

Q. 59 If A is a square matrix and $A = A^T$ then the Eigen vectors of matrix A are______.

(a) Equals (b) Linearly dependent

(c) Zeros (d) Orthogonal **Ans. : (d)**

Q. 60 If A is a orthogonal matrix and $\lambda_1, \lambda_2, \lambda_3$ are Eigen values of matrix A then Eigen values of matrix A^T are______.

(a) $\lambda_1, \lambda_2, \lambda_3$ (b) $\lambda_1^2\ \lambda_2^2\ \lambda_3^2$ (c) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ (d) Zeros

Ans. : (c)

Soln. : For orthogonal matrix $AA^T = I$ i.e. $A^{-1} = A^T$

Q. 61 If A is a non-singular square matrix then ______.

(a) At least one Eigen value is zero

(b) All Eigen values are zero

(c) All Eigen values are non-zero

(d) All above **Ans. : (c)**

Q. 62 If A is a singular matrix then ______.

(a) At least one Eigen value is zero

(b) All Eigen values are non-zero

(c) All Eigen values always zero

(d) All above **Ans. : (a)**

Q. 63 If $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ are the Eigen values of a square matrix A then ______.

(a) $|A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \ldots \cdot \lambda_n$

(b) $\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n = a_{11} + a_{22} + a_{33} + \ldots + a_{nn}$

(c) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \ldots, \frac{1}{\lambda_n}$ are the Eigen values of A^{-1}

(d) All above **Ans. : (d)**

Q. 64 Every square matrix satisfies its own ______.

(a) Inverse (b) Transpose

(c) Characteristic Equation (d) None of above **Ans. : (c)**

Q. 65 If λ is a Eigen value of matrix A then the matrix $[A - \lambda I]$ is _____

(a) Singular (b) Non singular

(c) Orthogonal (d) Symmetrical **Ans. : (a)**

Q. 66 For an identity matrix I which of the following is true

(a) Rank < trace (b) Rank > trace

(c) Rank = trace (d) Rank + trace = 1 **Ans. : (c)**

Q. 67 The characteristic equation of a matrix A is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$ then _____

(a) A^{-1} exist (b) A^{-1} does not exist

(c) $A^{-1} = 18A^2 - 45A$ (d) $A^{-1} = A$ **Ans. : (b)**

Soln. : Since $\lambda^3 - 18\lambda^2 + 45\lambda = 0$ is characteristic equation of matrix A.

We have, characteristics equation as

$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$\therefore |A| = 0 \Rightarrow A^{-1}$ does not exist.

Q. 68 The degree of characteristic equation of the matrix A is equal to _____

(a) < 3 (b) = 3 (c) Order of matrix (d) Depend on $|A|$

Ans. : (c)

Q. 69 The trace of a matrix $A = \begin{bmatrix} 0 & 2 & 0 \\ 3 & -2 & 3 \\ 0 & 3 & 0 \end{bmatrix}$ is equal to______.

(a) 0 (b) 15 (c) – 2 (d) 2 **Ans. : (c)**

Soln. : Trace of matrix A = sum of the diagonal elements

$= 0 - 2 + 0 = -2$

Q. 70 For a matrix $A = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}$ the Eigen values are______.

(a) 10, 2 (b) 9, 4 (c) 7, 5 (d) 8, 4 **Ans. : (b)**

Soln. : Sum of the Eigen values is equal to sum of the diagonal elements.

$\therefore$ Here, $14 - 1 = 13$

In all above only in (b) addition is 13.

Q. 71 For a matrix $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ two Eigen values are 1, – 1. Then third Eigen value is,

(a) 4 (b) 3 (c) 0 (d) 2 **Ans. : (a)**

Soln. : Sum of the Eigen values is equal to the sum of the diagonal elements.

$\therefore \lambda_3 = 4$

Q. 72 For a matrix $A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ Eigen values are______.

(a) 4, 3, 1 (b) 1, 2, 5 (c) 2, 2, 4 (d) – 2, 5, 5 **Ans. : (b)**

Soln. :

(i) Sum of the Eigen values is equal to sum of the diagonal elements.

$a_{11} + a_{22} + a_{33} = 4 + 3 + 1 = 8$

(ii) Product of eigen value is equal to determinant of matrix A

$|A| = 10$

Here, (a) $4 + 3 + 1 = 8$, $4 \times 3 \times 1 = 12$

(b) $1 + 2 + 5 = 8$, $1 \times 2 \times 5 = 10$

(c) $2 + 2 + 4 = 8$, $2 \times 2 \times 4 = 16$

(d) $-2 + 5 + 5 = 8$, $-2 \times 5 \times 5 = -50$

Ans. : (b)

Q. 73 If matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ then Eigen values are ____.

(a) 0, 5, 1 (b) 1, 2, 3 (c) – 1, 2, 5 (d) 2, 2, 2

Ans. : (b)

Q. 74 The eigen values of the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 5 & 1 \end{bmatrix}$ ar

_____

(a) 4, 1, 0 (b) 2, 2, 1 (c) 5, 0, 0 (d) 1, 1, 3 **Ans. : (b)**

Q. 75 The eigen values of the matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ ar

_____

(a) 2, 2, 2 (b) 4, 2, 0 (c) 3, 3, 0 (d) 1, 2, 3 **Ans. : (d)**

Q. 76 If eigen values of matrix A are 1, 2, 5 then the value of $|A|$ is equal to

(a) 11 (b) 8 (c) 10 (d) 2 **Ans. : (c)**

Soln. : Since $|A|$ = product of the eigen values.

Q. 77 If eigen values of matrix A are 1, 2, 2 then trace o matrix A is ____

(a) 2 (b) 4 (c) 1 (d) 5 **Ans. : (d**

Soln. : Trace of A = sum of diagonal elemen = sum of eigen values

Q. 78 If eigen values of matrix A are 1, – 1, 2 then trace o matrix A is _____

(a) 1 (b) 2 (c) – 2 (d) – 1 **Ans. : (b**

Soln. : Trace of A = sum of diagonal element = sum o eigen values

Q. 79 X_1, X_2 are the Eigen vectors of the matrix A the X_1 and X_2 are orthogonalif ______.

(a) A is symmetric (b) A is non-symmetric

(c) A is singular (d) A is non-singular **Ans. : (a**

Q. 80 The sum and product of the Eigen values of a matri $A = \begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ is equal to ______.

(a) 5, 21 (b) 21, 5 (c) 5, 1 (d) 1, 5

Ans. : (

Soln. : We know, If $\lambda_1, \lambda_2, \lambda_3$ are Eigen values of matrix then.

$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33} = 2 + 1 + 2 = 5$

and $\lambda_1\ \lambda_2\ \lambda_3 = |A| = 2(2) - 3(-5) - 2(-1) = 21$

Note

UNIT 2
Partial Differentiation

>> Syllabus :

Partial derivatives of first and higher orders; Homogeneous functions – Euler's Theorem for functions containing two and three variables (with proofs); Total derivatives; Change of variables.

- **Chapter 4 : Partial Differentiation**

CHAPTER 4

UNIT II

Partial Differentiation

Syllabus

Partial derivatives of first and higher orders; Homogeneous functions – Euler's Theorem for functions containing two and three variables (with proofs); Total derivatives; Change of variables.

4.1 Introduction

As we know real world problems can be expressed in mathematical language by using parametric equations and functions. The differential calculus is very effective tool in engineering and technology. The problems in heat transfer, wave equations, multiple integrals, maxima, minima, dynamics, economics, electricity, statistics and probability, etc. deals with functions of two or more independent variables. In this chapter we will see the partial derivatives of functions of several variables, Euler's theorem for homogeneous function, total derivatives, composite functions.

4.2 Functions of Several Variables

In many applications, the values of the function depends upon more than one variable.

The area of an ellipse is πab ;

$\therefore A = f(a, b)$.

Area of a rectangle depends upon length and breadth, $A = f(l, b)$;

Volume of right circular cylinder depends upon radius and height ($V = \pi r^2 h$), $V = f(r, h)$.

If the rod is insulated then the temperature u of the rod depends on distance and time. $\therefore u = f(x, t)$

In these examples, the function value depends upon two variables.

The volume of cube depends on length, breadth and height; $V = f(l, b, h)$.

The surface area of a rectangular parallolepiped depends upon three variables x, y, z.

$$S_A = f(x, y, z).$$

Triple integration depends upon three variables. $I = f(r, \theta, \phi)$. The velocity of fluid particle in space depends upon x, y, z, t. i.e. $V = f(x, y, z, t)$ and so on.

So, the functions of two or more variables are very important in mathematics.

If $u = f(x, y)$ then the variables x and y are called **"independent"** variables or "arguments" and u is called **"dependent"** variable or "value" of the function. It is same for functions of several variables.

4.3 Domain of Dependent Variable

If the function $u = f(x, y)$ defined for the ordered pair (x, y) within a certain area in x-y plane, then that area is the domain of dependent variable u.

Here, for every x and y, the dependent variable u possess unique value.

4.4 Partial Derivatives

A partial derivative of a function of several variables is the ordinary derivative of the function with respect to one of the variables, keeping remaining all variables as constant.

Let $u = f(x, y)$ be a function of two independent variables x and y.

1. The partial derivative of $u = f(x, y)$ w.r.t x is the ordinary derivative of u w.r.t x, keeping variable y as

constant and it is denoted by $\frac{\partial u}{\partial x}$ or $\frac{\partial f}{\partial x}$ or u_x or f_x e

(read as, dabba u by dabba x or del u by del x),

It is defined as,

$$\frac{\partial u}{\partial x} = \lim_{\delta x \to 0}\left[\frac{f(x+\delta x, y) - f(x,y)}{\delta x}\right];$$

provided limit exists.

2. Similarly, we can define partial derivative of u w.r.t y, keeping x as constant;

it is denoted by $\frac{\partial u}{\partial y}$ or u_y or $\frac{\partial f}{\partial y}$ or f_y (read as, dabba u by dabba y or del u by del y)

$$\frac{\partial u}{\partial y} = \lim_{\delta y \to 0}\left[\frac{f(x, y+\delta y) - f(x,y)}{\delta y}\right];$$

provided limit exists

If $u = f(x_1, x_2, x_3, \ldots, x_n)$

i.e. if u is function of 'n' finite independent variables, then the partial derivative of function u w.r.t. x_1 (say) is obtained by differentiating u ordinarily w.r.t x_1, keeping all remaining variables constant, and it is denoted as $\frac{\partial u}{\partial x_1}$ or $\frac{\partial f}{\partial x_1}$.

Remarks

1. All the rules of ordinary differentiation are applicable in partial differentiation.
2. The partial derivatives, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial x_1}, \ldots$ are called first order partial derivatives.

4.5 Geometrical Interpretation

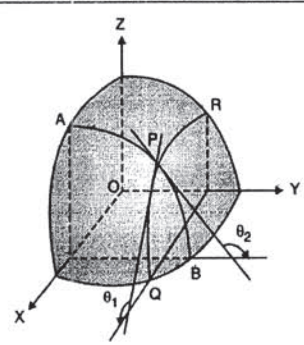


Fig. 4.5.1

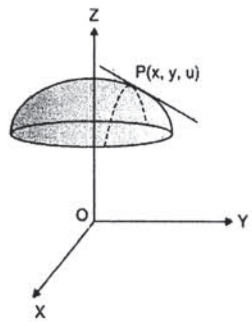


Fig. 4.5.2

The function u = f(x, y) represents a equation of surface in X, Y, Z - co-ordinate system. The point P [x, y, u (x, y)] on the surface u = f(x, y) corresponds to the values of independent variables (x, y). The intersection of the plane $y = y_1$ (parallel to ZOX – plane) and the surface u = f(x, y) represents a curve (dotted line in the Fig. 4.5.2). On this curve, x and u varies according the rule u = f(x, y).

The partial derivative of u = f(x, y) w.r.t x is $\left(\frac{\partial u}{\partial x}\right)_{(x,y)}$

Hence, $\frac{\partial u}{\partial x}$ represents the slope of the tangent to the curve of the intersection of the surface u = f(x, y) with the plane $y = y_1$ (parallel to ZOX- plane) at the point P[x, y, u(x, y)].

Similarly, $\frac{\partial u}{\partial y}$ represents the slope of the tangent to the curve of the intersection of the surface u = f(x, y) with the plane $x = x_1$ (parallel to ZOY-plane) at the point P[x, y, u(x, y)].

i.e. $\frac{\partial u}{\partial x} = \tan\theta_1$ and $\frac{\partial u}{\partial y} = \tan\theta_2$

4.6 Higher Order Partial Derivatives

Let, u = f(x, y) be a function of x and y.

The first order partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ may be the functions of x and y. So, we can differentiate partially again w.r.t. x as well as y. We get higher order partial derivatives such as,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \text{ or } \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \text{ or } u_{xx} \text{ or } f_{xx}$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \text{ or } \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \text{ or } u_{xy} \text{ or } f_{xy}$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \text{ or } \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \text{ or } u_{yx} \text{ or } f_{yx}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \text{ or } \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \text{ or } u_{yy} \text{ or } f_{yy}$$

are called second order partial derivatives.

If u = f(x, y) and the first order partial derivatives are continuous then,

$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$ is called **commutative property**. It is also called **cross or mixed partial derivatives**. Here, the order of derivative is immaterial.

Second order partial derivatives may again differentiate successively w.r.t. x and y we get the higher order derivative.

Differentiating $\frac{\partial^3 u}{\partial x^2}$, then

$$\frac{\partial^3 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}\right) \text{ or } \frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) \text{ or } u_{xxx} \text{ or } f_{xxx}$$

$$\frac{\partial^2 u}{\partial x\,\partial y\,\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\,\partial x}\right)$$

or $$\frac{\partial^3 f}{\partial x\,\partial y\,\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y\,\partial x}\right) \text{ or } u_{xyx} \text{ or } f_{xyx} \text{ and so on.}$$

Remark : If a function contains 'm' independent variables then we get m^n derivatives of order n.

Type I : Examples Based on Simple Partial Differentiation

Example 4.6.1

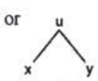
If $u = (1 - 2xy + y^2)^{-1/2}$ then prove that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = u^3 y^2$

Solution :

Step I : **Given :** $u\,(1 - 2xy + y^2)^{-1/2}$ ...(1)

Here, u is a function of x and y

$\because$ u→ x, y or



Step II : Differentiate Equation (1) partially w.r.t x, keeping y as a constant.

➔ **Using standard result**

$$\ldots\left[\frac{d}{dx} f(x)\,]^n = n\,f(x)^{n-1} \cdot \frac{d}{dx} f(x)\right]$$

$$\ldots f(x) = (1 - 2xy + y^2)^{-1/2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[1 - 2xy + y^2]^{-1/2}$$

$$= \frac{-1}{2}(1 - 2xy + y^2)^{-3/2} \cdot \underbrace{\frac{\partial}{\partial x}(1 - 2xy + y^2)}_{(0 - 2y + 0)}$$

$$= \frac{-1}{2}(1 - 2xy + y^2)^{-3/2} \cdot (0 - 2y + 0)$$

$$= \left(\frac{1}{\cancel{2}}\right)(-\cancel{2}\,y)\,(1 - 2xy + y^2)^{-3/2}$$

$$\frac{\partial u}{\partial x} = y\,(1 - 2xy + y^2)^{-3/2}$$

Multiplying both sides by x

$$\therefore\ x \cdot \frac{\partial u}{\partial x} = xy\,(1 - 2xy + y^2)^{-3/2} \qquad \ldots(2)$$

Again, differentiating Equation (1) w.r.t. y, partially, keeping x as a constant.

➔ **Using standard result :**

$$\ldots\left[\frac{d}{dx} f(x)\,]^n = n\,f(x)^{n-1} \cdot \frac{d}{dx} f(x)\right]$$

$$\ldots f(x) = (1 - 2xy + y^2)^{-1/2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[1 - 2xy + y^2]^{-1/2}$$

$$= \frac{-1}{2}(1 - 2xy + y^2)^{-3/2} \cdot \underbrace{\frac{\partial}{\partial y}(1 - 2xy + y^2)}_{(-2x + 2y)}$$

$$= \frac{-1}{2}(1 - 2xy + y^2)^{-3/2} \cdot (-2x + 2y)$$

$$(\because \text{ Using : } d\,[x]^n = n\,(x)^{n-1})$$

$$\frac{\partial u}{\partial y} = \left(\frac{-1}{\cancel{2}}\right)(-\cancel{2})\,(1 - 2xy + y^2)^{-3/2}\,(x - y)$$

Multiplying both sides by 'y'

$$\therefore\ y \cdot \frac{\partial u}{\partial y} = (1 - 2xy + y^2)^{-3/2}\,(xy - y^2) \qquad \ldots(3)$$

Step III : Subtracting Equation (3) from Equation (2) and take common term

$$x \cdot \frac{\partial u}{\partial x} - y \cdot \frac{\partial u}{\partial y} = [\,(1 - 2xy + y^2)^{-3/2}\,(xy)\,]$$
$$- [\,(1 - 2xy + y^2)^{-3/2} \cdot (xy - y^2)\,]$$

Taking $(1 - 2xy + y^2)^{-3/2}$ common, we get

$$x \cdot \frac{\partial u}{\partial x} - y \cdot \frac{\partial u}{\partial y} = (1 - 2xy + y^2)^{-3/2}\,(\cancel{xy} - \cancel{xy} + y^2)$$

$$= \underbrace{[\,(1 - 2xy + y^2)^{-1/2}]^3}_{u^3}\,(y^2)$$

$x \cdot \frac{\partial u}{\partial x} - y \cdot \frac{\partial u}{\partial y} = u^3 \cdot y^2$ ...from Equation (1) ✓ **...Ans.**

Example 4.6.2

If $u = \log(\tan x + \tan y + \tan z)$ then prove that $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$

Solution :

Step I :

Given : $u = \log(\tan x + \tan y + \tan z)$ ...(1)

Here, u is function of x, y, z

Differentiate Equation (1) partially w.r.t. x, keeping y and z as constants.

➔ **Using standard results :** $\ldots\left[\frac{d}{dx}[\log f(x)] = \frac{1}{f(x)} \cdot \frac{d}{dx} f(x)\right]$

$\ldots f(x) = \log(\tan x + \tan y + \tan z)$

$$\frac{\partial u}{\partial x} = \frac{1}{(\tan x + \tan y + \tan z)} \cdot \underbrace{\frac{d}{dx}(\tan x + \tan y + \tan z)}_{\sec^2 x}$$

$\left(\because \text{use } \frac{d}{dx}(\tan x) = \sec^2 x\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{(\tan x + \tan y + \tan z)}(\sec^2 x) \quad \ldots(2)$$

Similarly, we get

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{(\tan x + \tan y + \tan z)} \quad \ldots(3)$$

$\left(\because \text{use } \frac{d}{dy}(\tan y) = \sec^2 y\right)$

and $$\frac{\partial u}{\partial z} = \frac{\sec^2 z}{(\tan x + \tan y + \tan z)} \quad \ldots(4)$$

$\left(\because \text{use } \frac{d}{dz}(\tan z) = \sec^2 z\right)$

Step II :

$\sin 2x \times$ Equation (2) + $\sin 2y \times$ Equation (3) $\times \sin 2z \times$ Equation (4)

$$\sin 2x \cdot \frac{\partial u}{\partial x} + \sin 2y \cdot \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= \left(\frac{\sin 2x \cdot \sec^2 x}{\tan x + \tan y + \tan z}\right) + \left(\frac{\sin 2y \cdot \sec^2 y}{\tan x + \tan y + \tan z}\right) + \left(\frac{\sin 2z \cdot \sec^2 z}{\tan x + \tan y + \tan z}\right)$$

Hence,

$$\sin 2x \cdot \frac{\partial u}{\partial x} + \sin 2y \cdot \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= \frac{\sin 2x \cdot \sec^2 x + \sin 2y \cdot \sec^2 y + \sin 2z \cdot \sec^2 z}{(\tan x + \tan y + \tan z)}$$

...(Using $\sin 2x = 2 \sin x \cdot \cos x$)

$$= \frac{2 \sin x \cdot \cos x \cdot \sec^2 x + 2 \sin y \cdot \cos y \cdot \sec^2 y + 2 \sin z \cdot \cos z \cdot \sec^2 z}{(\tan x + \tan y + \tan z)}$$

$\left(\because \sec^2 x = \frac{1}{\cos x} \cdot \frac{1}{\cos x};\ \sec^2 y = \frac{1}{\cos y} \cdot \frac{1}{\cos y} \text{ and } \sec^2 z = \frac{1}{\cos z} \cdot \frac{1}{\cos z}\right)$

$$= 2\frac{\left[\sin x \cdot \cancel{\cos x} \cdot \frac{1}{\cancel{\cos x}} \cdot \frac{1}{\cos x} + \sin y \cdot \cancel{\cos y} \cdot \frac{1}{\cancel{\cos y}} \cdot \frac{1}{\cos y} + \sin z \cdot \cancel{\cos z} \cdot \frac{1}{\cancel{\cos z}} \cdot \frac{1}{\cos z}\right]}{(\tan x + \tan y + \tan z)}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)} = 2$$

Hence,

$$\sin 2x \cdot \frac{\partial u}{\partial x} + \sin 2y \cdot \frac{\partial u}{\partial y} + \sin 2z \cdot \frac{\partial u}{\partial z} = 2 \checkmark \quad \text{...Ans.}$$

Example 4.6.3

If $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$, then find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

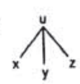
Solution :

Step I :

Given : $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}} = (x^2 + y^2 - z^2)^{-1/2}$ ...(1)

Here, u is a function of x, y, z

$\because$ u $\to$ x, y, z or



Step II : Differentiating Equation (1) partially w.r.t. x, keeping y, z as a constant

➔ **Using standard result :** $\ldots\left[\frac{d}{dx} f(x)\right]^n = n f(x)^{n-1} \cdot \frac{d}{dx} f(x)\Big]$

$\ldots f = (x^2 + y^2 + z^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = \left(\frac{-1}{2}\right)(x^2 + y^2 + z^2)^{-3/2} \cdot \underbrace{\frac{\partial}{\partial x}(x^2 + y^2 + z^2)}_{2x}$$

$$\frac{\partial u}{\partial x} = \frac{-1}{\cancel{2}}(x^2 + y^2 + z^2)^{-3/2}(\cancel{2}x)$$

$\left(\because \text{use } \frac{d}{dx}(x^2) = 2x\right)$

$$= -x(x^2 + y^2 + z^2)^{-3/2}$$

Again differentiate $\frac{\partial u}{\partial x}$ w.r.t. x, keeping y, z as constant.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\{-x(x^2 + y^2 + z^2)^{-3/2}\}$$

➔ **Using standard result**

$\ldots\left[\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx} g(x) + g(x) \cdot \frac{d}{dx} f(x)\right]$

Or

$\ldots\left[\frac{d}{dx}[I \cdot II] = I \cdot \frac{d}{dx}(II) + II \cdot \frac{d}{dx}(I)\right]$

$$= -\frac{\partial}{\partial x}\{\underbrace{x}_{I} \cdot \underbrace{(x^2 + y^2 + z^2)^{-3/2}}_{II}\}$$

$$= -\left[(x)\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-3/2} + (x^2 + y^2 + z^2)^{-3/2} \cdot \underbrace{\frac{\partial}{\partial x}(x)}_{1}\right]$$

$$= -\left[(x)\left(\frac{-3}{2}\right)(x^2 + y^2 + z^2)^{-5/2} \cdot \underbrace{\frac{\partial}{\partial x}(x^2 + y^2 + z^2)}_{2x} + (x^2 + y^2 + z^2)^{-3/2} \cdot 1\right]$$

$$= -\left[(x)\left(\frac{-3}{\cancel{2}}\right)(x^2 + y^2 + z^2)^{-5/2}(\cancel{2}x) + (x^2 + y^2 + z^2)^{-5/2}(1)\right]$$

$$= -\left[\frac{-3x^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{1}{(x^2 + y^2 + z^2)^{3/2}}\right]$$

Multiply and divide second term by $(x^2 + y^2 + z^2)$

$$= -\left[\frac{-3x^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{3/2}}\right]$$

$$= -\left[\frac{-3x^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}}\right]$$

$$= -\frac{1}{(x^2 + y^2 + z^2)^{5/2}}[-3x^2 + (x^2 + y^2 + z^2)]$$

$$\frac{\partial^2 u}{\partial x^2} = -\left[\frac{-2x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}}\right] \quad \ldots(2)$$

$$= \left[\frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}\right]$$

Similarly we get,

$$\frac{\partial^2 u}{\partial y^2} = \left[\frac{-x^2 + 2y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}\right] \quad \ldots(3)$$

and $$\frac{\partial^2 u}{\partial z^2} = \left[\frac{-x^2 - y^2 + 2z^2}{(x^2 + y^2 + z^2)^{5/2}}\right] \quad \ldots(4)$$

Step III : Now, adding Equations (2), (3) and (4), we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \left(\frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}\right) + \left(\frac{-x^2 + 2y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}\right) + \left(\frac{-x^2 - y^2 + 2z^2}{(x^2 + y^2 + z^2)^{5/2}}\right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{1}{(x^2 + y^2 + z^2)^{5/2}}[\cancel{2x^2} - \cancel{2x^2} - \cancel{2y^2} + \cancel{2y^2} - \cancel{2z^2} + \cancel{2z^2}]$$

$$= 0$$

Hence,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \checkmark \quad \text{...Ans.}$$

Example 4.6.4

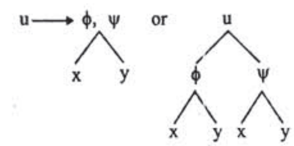
If $u = \phi(x + ay) + \psi(a - ay)$, then show that $\frac{\partial^2 u}{\partial y^2} = a^2 \cdot \frac{\partial^2 u}{\partial x^2}$

Solution :

Step I : **Given :** $u = \phi(x + ay) + \psi(a - ay)$ ...(1)

Here, u is a function of ϕ and ψ

Also, ϕ is a function of x, y and ψ is a function of x, y



Step II : Differentiating Equation (1) partially, w.r.t. y, keeping x as a constant

➔ **Using standard result :** $\ldots\left[\frac{d}{dx}[I + II] = \frac{d}{dx}(I) + \frac{d}{dx}(II)\right]$

$\ldots I = \phi(x + ay);\quad II = \psi(x - ay)$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial y}\phi(x + ay) + \frac{\partial u}{\partial y}\psi(x - ay)$$

$$= \phi'(x + ay) \cdot \underbrace{\frac{\partial}{\partial y}(x + ay)}_{a} + \psi'(x - ay) \cdot \underbrace{\frac{\partial}{\partial y}(x - ay)}_{-a}$$

$$\frac{\partial u}{\partial y} = \phi'(x + ay) \cdot a + \psi'(x - ay)(-a)$$

Again, differentiate $\frac{\partial u}{\partial y}$ w.r.t. y, keeping x as a constant

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial}{\partial y}\phi'(x + ay)\right)(a) + \left(\frac{\partial}{\partial y}\psi'(x - ay)\right)(-a)$$

$$= \phi''(x+ay)\cdot\underbrace{\frac{\partial}{\partial y}(x+ay)}_{a}(a)$$

$$+\psi''(x-ay)\underbrace{\frac{\partial}{\partial y}(x-ay)}_{(-a)}(-a)$$

$$\frac{\partial^2 u}{\partial y^2} = \phi''(x+ay)(a^2)+\psi''(x-ay)(-a^2)$$

$$= a^2\,|\,\phi''(x+ay)+\psi''(x-ay)\,| \quad ...(2)$$

$$\left[\because (-a)^2 = a^2\right]$$

Also, differentiating Equation (1) partially w.r.t. x, keeping y as a constant.

→ **Using standard result** : $...\left[\frac{d}{dx}[I+II] = \frac{d}{dx}(I)+\frac{d}{dx}(II)\right]$

$...I = \phi(x+ay)\,;\; II = \psi(x-ay)$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\phi(x+ay)+\frac{\partial}{\partial y}\psi(x-ay)$$

$$= \phi'(x+ay)\cdot\underbrace{\frac{\partial}{\partial x}(x+ay)}_{1}+\psi'(x-ay)\cdot\underbrace{\frac{\partial}{\partial x}(x-ay)}_{1}$$

$$\frac{\partial u}{\partial x} = \phi'(x+ay)\cdot 1+\psi'(x-ay)\cdot 1$$

Again, differentiate $\frac{\partial u}{\partial x}$ w.r.t. x, keeping y as a constant

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}[\phi'(x+ay)+\psi'(x-ay)]$$

$$= \frac{\partial}{\partial x}\phi'(x+ay)+\frac{\partial}{\partial x}\psi'(x-ay)$$

$$= \phi''(x+ay)\underbrace{\frac{\partial}{\partial x}(x+ay)}_{1}$$

$$+\psi''(x-ay)\cdot\underbrace{\frac{\partial}{\partial x}(x-ay)}_{1}$$

$$\frac{\partial^2 u}{\partial x^2} = \phi''(x+ay)\times 1+\psi''(x-ay)\times 1 \quad ...(3)$$

Multiplying both sides of Equation (3) by a^2,

$$a^2\frac{\partial^2 u}{\partial x^2} = a^2\,[\phi''(x+ay)+\psi''(x-ay)] \quad ...(4)$$

Step III : From equation (2) and (4), we get,

$$\frac{\partial^2 u}{\partial y^2} = a^2\frac{\partial^2 u}{\partial x^2}\ ✓ \qquad ...\textbf{Ans.}$$

Example 4.6.5

If $u = \dfrac{e^{x+y+z}}{e^x+e^y+e^z}$ **then find the value of** $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}$.

Solution :

Step I : Given : $u = \dfrac{e^{x+y+z}}{e^x+e^y+e^z}$ ...(1)

Here, u is a function of x, y, z

$\therefore u = x, y, z$ or

$$\begin{array}{ccccc} & & u & & \\ & \swarrow & \downarrow & \searrow & \\ x & & y & & z\end{array}$$

Step II : Differentiating Equation (1) w.r.t. x, keeping y and z as a constant

→ **Using standard result**

$$...\left[\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[g(x)\cdot\frac{d}{dx}f(x)-f(x)\cdot\frac{d}{dx}g(x)\right]}{g(x)^2}\right]$$

Or

$$...\left[\frac{d}{dx}\left[\frac{I}{II}\right] = \frac{\left[II\cdot\frac{d}{dx}(I)-(I)\cdot\frac{d}{dx}(II)\right]}{(II)^2}\right]$$

$...f(x) = e^{x+y+z}\,;\; g(x) = e^x+e^y+e^z$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\frac{e^{x+y+z}}{e^x+e^y+e^z}\right]$$

$$= \frac{(e^x+e^y+e^z)\frac{\partial}{\partial x}(e^{x+y+z})-(e^{x+y+z})\cdot\frac{\partial}{\partial x}(e^x+e^y+e^z)}{(e^x+e^y+e^z)^2}$$

$$\left[\because \frac{\partial}{\partial x}(e^{x+y+z}) = e^{x+y+z} \text{ and } \frac{\partial}{\partial x}(e^{x+y+z}) = e^x\right]$$

$$\frac{\partial u}{\partial x} = \frac{(e^x+e^y+e^z)(e^{x+y+z})-(e^{x+y+z})(e^x)}{(e^x+e^y+e^z)^2}$$

Take common e^{x+y+z} ; we get,

$$\frac{\partial u}{\partial x} = e^{x+y+z}\left[\frac{\cancel{e^x}+e^y+e^z-\cancel{e^x}}{(e^x+e^y+e^z)^2}\right] \quad ...(2)$$

Similarly, we get

$$\frac{\partial u}{\partial y} = e^{x+y+z}\left[\frac{e^x+\cancel{e^y}+e^z-\cancel{e^y}}{(e^x+e^y+e^z)^2}\right] \quad ...(3)$$

(... Using above procedure)

and $$\frac{\partial u}{\partial z} = e^{x+y+z}\left[\frac{e^x+e^y+\cancel{e^z}-\cancel{e^z}}{(e^x+e^y+e^z)^2}\right] \quad ...(4)$$

(... Using above procedure)

Step III : Adding Equations (2), (3) and (4)

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \left[e^{x+y+z}\left(\frac{e^y+e^z}{(e^x+e^y+e^z)^2}\right)\right]$$

$$+\left[e^{x+y+z}\left(\frac{e^x+e^z}{(e^x+e^y+e^z)^2}\right)\right]$$

$$+\left[e^{x+y+z}\left(\frac{(e^x+e^y)}{(e^x+e^y+e^z)^2}\right)\right]$$

$$= \frac{e^{x+y+z}}{(e^x+e^y+e^z)^2}[2e^x+2\cdot e^y+2e^z]$$

$$= \left[\frac{e^{x+y+z}\cdot 2\cancel{(e^x+e^y+e^z)}}{(e^x+e^y+e^z)^{\cancel{2}}}\right] = 2\frac{(e^{x+y+z})}{(e^x+e^y+e^z)}$$

Hence,

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = 2u\ ✓ \quad \textbf{... from Equation (1)} \quad \textbf{...Ans.}$$

Example 4.6.6

If $z = x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$ **then prove that** $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{x^2-y^2}{x^2+y^2}$

Solution :

Step I : Given : $z = x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$ ...(1)

Here, z is a function of x and y

$$\begin{array}{ccc} & z & \\ \frac{\partial z}{\partial x}\swarrow & & \searrow\frac{\partial z}{\partial y} \\ x & & y\end{array} \quad \text{i.e} \quad z \longrightarrow x, y$$

Step II : Differentiating Equation (1) partially w.r.t. y, keeping x as constant.

→ **Using standard result :**

$$...\left[\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\frac{d}{dx}g(x)+g(x)\cdot\frac{d}{dx}f(x)\right.$$

$$\left. I = x^2\,;\; II = \tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$\frac{\partial z}{\partial y} = \left[x^2\cdot\frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)+\tan^{-1}\left(\frac{y}{x}\right)\cdot\underbrace{\frac{\partial}{\partial y}(x^2)}_{0}\right]$$

$$-\left[y^2\cdot\frac{\partial}{\partial y}\tan^{-1}+\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{x}{y}\right)\cdot\underbrace{\frac{\partial}{\partial y}(y^2)}_{2y}\right]$$

→ **Using standard result**

$$...\left[\frac{d}{dx}\tan^{-1}(f(x)) = \frac{1}{[1+f(x)^2]}\cdot\frac{d}{dx}f(x)\,;\right.$$

$$\left.\frac{d}{dy}(x^2) = 0;\quad \frac{d}{dy}(y^2) = 2y\right]$$

$$\frac{\partial z}{\partial y} = x^2\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)-\left[\cancel{y^2}\cdot\frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{\cancel{y^2}}\right)+2y\tan^{-1}\left(\frac{x}{y}\right)\right]$$

$$= x^2\cdot\frac{\cancel{x^2}}{x^2+y^2}\left(\frac{1}{\cancel{x}}\right)+\frac{xy^2}{x^2+y^2}-2y\cdot\tan\left(\frac{x}{y}\right)$$

$$= \underbrace{\frac{x^3}{x^2+y^2}+\frac{xy^2}{x^2+y^2}}-2y\tan^{-1}\left(\frac{x}{y}\right)$$

Cross multiplication

$$= \frac{x\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}}-2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial z}{\partial y} = x-2y\tan^{-1}\left(\frac{x}{y}\right)$$

Step III : Now, differentiate $\frac{\partial z}{\partial y}$ w.r.t. x, keeping y as constant

$$\frac{\partial^2 z}{\partial x\cdot\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[x-2y\tan^{-1}\left(\frac{x}{y}\right)\right]$$

$$= \frac{\partial}{\partial x}(x)-2y\cdot\frac{\partial}{\partial x}\tan^{-1}\left(\frac{x}{y}\right)$$

→ **Using standard result :**

$$...\left[\frac{d}{dx}\tan^{-1}(f(x)) = \frac{1}{1+[f(x)]^2}\cdot\frac{d}{dx}f(x)\right]$$

$$= 1-2\cancel{y}\frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{\cancel{y}}\right)$$

$$= 1-\frac{2}{\frac{x^2+y^2}{y^2}}$$

$$= 1-\frac{2y^2}{x^2+y^2}$$

$$= \frac{x^2+y^2+2y^2}{x^2+y^2}$$

...(Using cross multiplication)

Hence, we get,

$$\therefore \frac{\partial^2 z}{\partial x\,\partial y} = \frac{x^2-y^2}{x^2+y^2}\ ✓ \qquad ...\textbf{Ans.}$$

Example 4.6.7

If $u = z\tan^{-1}\left(\frac{x}{y}\right)$, then prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

Solution :

Step I : Given : $u = z\tan^{-1}\left(\frac{x}{y}\right)$ ...(1)

Here, u is a function of x, y and z

$U \rightarrow x, y, z$ or

$$\begin{array}{ccc} & u & \\ x & y & z \end{array}$$

Step II : Differentiating Equation (1), partially w.r.t. x, keeping y and z as constants.

$$\frac{\partial u}{\partial x} = z\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{x}{y}\right)\right]$$

→ Using standard result

$$\ldots\left[\frac{d}{dx}\tan^{-1}(f(x)) = \frac{1}{1+[f(x)]^2}\cdot\frac{d}{dx}f(x)\right]$$

$$\therefore \frac{\partial u}{\partial x} = z\cdot\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\underbrace{\frac{\partial}{\partial x}\left(\frac{x}{y}\right)}_{\left(\frac{1}{y}\right)}$$

$$= z\cdot\left(\frac{y^2}{x^2+y^2}\right)\cdot\left(\frac{1}{y}\right)$$

$$\frac{\partial y}{\partial x} = \frac{zy}{x^2+y^2}$$

Again, differentiate $\frac{\partial u}{\partial x}$ w.r.t. x, keeping y and z as constant

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{zy}{x^2+y^2}\right]$$

→ Using standard result : $\left[\frac{d}{dx}\frac{1}{f(x)} = -\frac{1}{[f(x)]^2}\cdot\frac{d}{dx}f(x)\right]$

$$= zy\left(\frac{-1}{(x^2+y^2)^2}\right)\cdot\underbrace{\frac{\partial}{\partial x}(x^2)}_{x^2}$$

$$= \frac{-zy}{(x^2+y^2)^2}\cdot(2x)$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{2xyz}{(x^2+y^2)^2} \quad \ldots(2)$$

Differentiating Equation (1), partially w.r.t. y keeping x and z as constant.

→ Using standard result :

$$\ldots\left[\frac{d}{dx}\tan^{-1}(f(x)) = \frac{1}{1+[f(x)]^2}f'(x)\ ;\ f(x) = \left(\frac{x}{y}\right)\right]$$

$$\frac{\partial u}{\partial y} = z\cdot\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\underbrace{\frac{\partial}{\partial y}\left(\frac{x}{y}\right)}_{-\left(\frac{x}{y^2}\right)}$$

$$= z\left(\frac{y^2}{x^2+y^2}\right)\left(-\frac{x}{y^2}\right)$$

$$\therefore \frac{\partial u}{\partial y} = z\cdot\frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right) = -\frac{zx}{x^2+y^2}$$

Again, differentiating $\frac{\partial u}{\partial y}$ w.r.t. y, keeping x and z as constant

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{zx}{x^2+y^2}\right)$$

→ Using standard rule : $\left[\frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -\frac{1}{y^2}\right]$

$$= -(zx)\cdot\left[\frac{-1}{(x^2+y^2)^2}\cdot\underbrace{\frac{\partial}{\partial y}(x^2+y^2)}_{2y}\right]$$

$$\frac{\partial^2 u}{\partial y^2} = \left[\frac{zx}{(x^2+y^2)^2}(2y)\right] = \frac{2zxy}{(x^2+y^2)^2} \quad \ldots(3)$$

And differentiating equation (1) partially w.r.t. z, keeping x and y as constants.

$$\frac{\partial u}{\partial z} = \tan^{-1}\left(\frac{x}{y}\right)\underbrace{\frac{\partial}{\partial z}(z)}_{1}$$

$$\frac{\partial u}{\partial z} = \tan^{-1}\left(\frac{x}{y}\right) \text{ and } \frac{\partial^2 u}{\partial z^2} = 0$$

$$\text{and } \frac{\partial^2 u}{\partial z^2} = \underbrace{\frac{\partial}{\partial z}\left[\tan^{-1}\left(\frac{x}{y}\right)\right]}_{0} = 0 \quad \ldots(4)$$

Step III : By adding equations (2), (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \left[-\frac{\cancel{2xyz}}{(x^2+y^2)^2}+\frac{\cancel{2xyz}}{(x^2+y^2)^2}+0\right]$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 0\ ✓ \quad \text{...Ans.}$$

Example 4.6.8

If $u = \log\left(\sqrt{x^2+y^2+z^2}\right)$, then prove that :

$$(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=1$$

Solution :

Step I : Given, $u = \log\left(\sqrt{x^2+y^2+z^2}\right)$ ...(1)

Here, u is a function of x, y, z

i.e. $u \longrightarrow x, y, z$

Step II : Differentiate equation (i) w.r.t. x partially, keeping y and z as constants.

→ Using standard result :

$$\ldots\left[\frac{d}{dx}\log\left(\sqrt{f(x)}\right) = \frac{1}{\sqrt{f(x)}}\cdot\frac{d}{dx}\sqrt{f(x)}\cdot\frac{d}{dx}f(x)\right]$$

$$\ldots\left[\frac{d}{dx}[f(x)]^n = n\,f(x)^{n-1}\cdot\frac{d}{dx}f(x)\right];\ f(x) = (x^2+y^2+z^2)^{1/2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2+z^2}}\cdot\underbrace{\frac{\partial}{\partial x}(x^2+y^2+z^2)\frac{\partial}{\partial x}x^2}_{\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot(2x)}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(\frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}\right)\cdot(\cancel{2}x)$$

$\left[\ldots\text{Here, Using standard result } d\left(\left(\sqrt{f(x)}\right)\right) = \frac{1}{2\sqrt{f(x)}}d(f(x))\right]$

$$= \frac{x}{x^2+y^2+z^2}$$

Differentiate $\frac{\partial u}{\partial x}$, w.r.t. x, keeping y and z as constants.

→ Using standard result :

$$\ldots\left[\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \left[\frac{g(x)\cdot\frac{d}{dx}f(x) - f(x)\cdot\frac{d}{dx}g(x)}{g(x)^2}\right]\right]$$

Or

$$\ldots\left[\frac{d}{dx}\left[\frac{I}{II}\right] = \left[\frac{II\cdot\frac{d}{dx}(I) - (I)\cdot\frac{d}{dx}(II)}{(II)^2}\right]\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)$$

$$\frac{(x^2+y^2+z^2)\cdot\underbrace{\frac{\partial}{\partial x}\cdot(x)}_{1} - (x)\cdot\underbrace{\frac{\partial}{\partial x}(x^2+y^2+z^2)}_{2x}}{(x^2+y^2+z^2)^2}$$

$$\text{and } \frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2+z^2)\cdot 1 - x\cdot 2x}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2+z^2+x^2}{(x^2+y^2+z^2)^2} \quad \ldots(2)$$

Similarly, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2+z^2-y^2}{(x^2+y^2+z^2)^2} \quad \ldots(3)$$

$$\text{and } \frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2} \quad \ldots(4)$$

Step III :

Adding Equations (2), (3) and (4) we get,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2}$$

Multiplying both sides by $(x^2+y^2+z^2)$, we get,

$$\therefore (x^2+y^2+z^2)\cdot\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

$$= (x^2+\cancel{y^2}+z^2)\cdot\frac{1}{(x^2+\cancel{y^2}+z^2)}$$

Hence,

$$(x^2+y^2+z^2)\cdot\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) = 1\ ✓ \quad \text{...Ans.}$$

Type II : Examples on verification of $\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}$

Example 4.6.9

If $u = x^3y + e^{xy^2}$, then prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$

Solution :

Step I : Given : $u = x^3y + e^{xy^2}$ ...(i)

Here, u us a function of x, y

$\therefore u \rightarrow x, y$ or

$$\begin{array}{cc} \multicolumn{2}{c}{u} \\ x & y \end{array}$$

Step II : Differentiating Equation (1) partially w.r.t. x, keeping y as constant

→ Using standard result : $\ldots\left[\frac{d}{dx}[I+II] = \frac{d}{dx}(I)+\frac{d}{dx}(II)\right]$

$$\ldots\left[\frac{d}{dx}(e^{f(x)}) = e^{f(x)}\cdot\frac{d}{dx}f(x)\right]$$

$$\left[\frac{d}{dx}e^{ax} = e^{ax}\cdot a\right]$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3y)+\frac{\partial}{\partial x}(e^{xy^2})$$

$$= y\Big(\underbrace{\frac{\partial}{\partial x}x^3}_{3x^2}\Big)+e^{xy^2}\cdot\underbrace{\frac{\partial}{\partial x}(xy^2)}_{y^2}$$

Again differentiating $\frac{\partial u}{\partial x}$ w.r.t. y, keeping x as constant

$$\frac{\partial^2 u}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial y}[3x^2y+y^2e^{xy^2}]$$

$$= \frac{\partial}{\partial y}(3x^2y)+\frac{\partial}{\partial y}(y^2\cdot e^{xy^2})$$

$$= 3x^2\cdot\underbrace{\frac{\partial}{\partial y}(y)}_{1}+\left[y^2\frac{\partial}{\partial y}e^{xy^2}+e^{xy^2}\underbrace{\frac{\partial}{\partial y}y^2}_{2y}\right]$$

$$= 3x^2+y^2.e^{xy^2}.\underbrace{\frac{\partial}{\partial y}(xy^2)}_{2xy}+e^{xy^2}.(2y)$$

$$= 3x^2+y^2e^{xy^2}(2xy)+2y\,e^{xy^2}$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = 3x^2+2xy^3e^{xy^2}+2y\,e^{xy^2} \quad ...(2)$$

Now, differentiating Equation (1) partilly w.r.t. y, keeping x as constant

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^3y+e^{xy^2})=\frac{\partial}{\partial y}(x^3y)+\frac{\partial}{\partial y}(e^{xy^2})$$

$$= x^3\underbrace{\frac{\partial}{\partial y}(y)}_{1}+e^{xy^2}.\underbrace{\frac{\partial}{\partial y}(xy^2)}_{2xy}$$

$$\frac{\partial u}{\partial y} = x^3+e^{xy^2}(2xy)$$

...(Using standard result $d(x^n)=n\,x^{n-1}$ and $d(e^{f(x)}\,df(x))$)

Again differentiating $\frac{\partial u}{\partial y}$ w.r.t. x, keeping y as constant.

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}[x^3+2xy\,.\,e^{xy^2}]$$

$$= \frac{\partial}{\partial x}(x^3)+2y\left[\frac{\partial}{\partial x}(x\,.\,e^{xy^2})\right]$$

$$= 3x^2+2y\Big[x\,.\underbrace{\frac{\partial}{\partial x}e^{xy^2}}_{e^{xy^2}.(y^2)}+e^{xy^2}.\underbrace{\frac{\partial}{\partial x}(x)}_{1}\Big]$$

$$= 3x^2+2y\,[x\,e^{xy^2}(y^2)+e^{xy^2}(1)]$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = 3x^2+2xy^3e^{xy^2}+2y\,e^{xy^2} \quad ...(3)$$

Step III : From Equations (2) and (3), it is clear that

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}\checkmark \quad \textbf{...Ans.}$$

Example 4.6.10

If $u=\log(x^2+y^2)$, show that $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$

Solution :

Step I : Given : $u=\log(x^2+y^2)$ ...(1)

$$u \longrightarrow x, y \quad \text{or} \quad \begin{array}{c} u \\ \swarrow\ \searrow \\ x \qquad y \end{array}$$

Step II : Differentiate Equation (1) w.r.t. x partially, keeping y as constants

→ **Using standard result :** $...\left[\frac{d}{dx}[\log f(x)]=\frac{1}{f(x)}\cdot\frac{d}{dx}f(x)\right]$

...$f(x)=x^2+y^2$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}\cdot\underbrace{\frac{\partial}{\partial x}(x^2+y^2)}_{2x}$$

$$\frac{\partial u}{\partial x} = \frac{1(2x)}{x^2+y^2}$$

Differentiate $\frac{\partial u}{\partial x}$ w.r.t. y, keeping x as constant.

→ **Using standard result :** $...\left[\frac{d}{dx}\frac{1}{f(x)}=-\frac{1}{[f(x)]^2}\cdot\frac{d}{dx}f(x)\right]$

...$f(x)=(x^2+y^2)$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left[\frac{2x}{x^2+y^2}\right]$$

$$= 2x\Big(\frac{-1}{(x^2+y^2)^2}\cdot\underbrace{\frac{\partial}{\partial y}(x^2+y^2)}_{2y}\Big)$$

$$= 2x\left[\frac{-1(2y)}{(x^2+y^2)^2}\right]$$

$$\frac{\partial^2 u}{\partial y\partial x} = \frac{-4xy}{(x^2+y^2)^2} \quad ...(2)$$

Step III : Now, differentiate Equation (1) partially w.r.t. y keeping x as constant

→ **Using standard result** $...\left[\frac{d}{dx}[\log f(x)]=\frac{1}{f(x)}\cdot\frac{d}{dx}f(x)\right]$

...$f(x)=x^2+y^2$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2}\cdot\underbrace{\frac{\partial}{\partial y}(x^2+y^2)}_{2y}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$$

Again differentiating $\frac{\partial u}{\partial y}$ w.r.t. x, keeping y as constant

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left[\frac{2y}{x^2+y^2}\right]$$

→ **Using standard result :** $...\left[\frac{d}{dx}\frac{1}{f(x)}=-\frac{1}{[f(x)]^2}\cdot\frac{d}{dx}f(x)\right]$

...$f(x)=(x^2+y^2)$

$$\therefore \frac{\partial^2 u}{\partial x\partial y} = 2y\Big[\frac{-1}{(x^2+y^2)^2}\cdot\underbrace{\frac{\partial}{\partial x}(x^2+y^2)}_{2x}\Big]$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = 2y\left(\frac{-1(2x)}{(x^2+y^2)^2}\right)=\frac{-4xy}{(x^2+y^2)^2} \quad ...(3)$$

Step IV : From Equations (2) and (3), we conclude that,

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}\checkmark \quad \textbf{...Ans.}$$

Example 4.6.11

If $u=\tan^{-1}\left(\frac{x}{y}\right)$ then prove that : $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$

Solution :

Given : $u = \tan^{-1}\left(\frac{x}{y}\right)$ ...(1)

Here, u is a function of x and y

$\Rightarrow u \to x.\ y$

Differentiate Equation (1) partially , w.r.t. x, keeping y as constant

→ **Using standard result :**

$...\left[\frac{d}{dx}\tan^{-1}(f(x))=\frac{1}{1+[f(x)]^2}f'(x)\right]$

... $f(x)=\left(\frac{x}{y}\right)$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\underbrace{\frac{\partial}{\partial x}=\left(\frac{x}{y}\right)}_{1/y}$$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(\frac{1}{y}\right)=\frac{y}{x^2+y^2}$$

Again, differentiate $\frac{\partial u}{\partial x}$ w.r.t. y, keeping x as constants.

→ **Using standard result**

$...\left[\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{\left[g(x)\cdot\frac{d}{dx}f(x)-f(x)\cdot\frac{d}{dx}g(x)\right]}{(g(x))^2}\right]$

Or

$...\left[\frac{d}{dx}\left[\frac{I}{II}\right]=\frac{\left[II\cdot\frac{d}{dx}(I)-(I)\cdot\frac{d}{dx}(II)\right]}{(II)^2}\right]$

...$I=y\ ;\ II=x^2+y^2$

$$\frac{\partial^2 u}{\partial y\partial x} = \frac{\partial}{\partial y}\left[\frac{y}{x^2+y^2}\right]$$

$$(x^2+y^2)\cdot\underbrace{\frac{\partial}{\partial y}(y)}_{1}-(y)\cdot\underbrace{\frac{\partial}{\partial y}(x^2+y^2)}_{2y}$$

$$= \overline{\qquad (x^2+y^2)^2 \qquad}$$

$$= \frac{(x^2+y^2)\cdot 1-y\cdot 2y}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y\partial x} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad ...(2)$$

Step II : Now, differentiate Equation (1) partially, w.r.t. y keeping x as constant

→ **Using standard result :**

$...\left[\frac{d}{dx}\tan^{-1}(f(x))=\frac{1}{1+[f(x)]^2}f'(x)\right]$

...$f(x)=\left(\frac{x}{y}\right)$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\underbrace{\frac{\partial}{\partial y}\left(\frac{x}{y}\right)}_{\left(\frac{-x}{y^2}\right)}$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(-\frac{x}{y^2}\right)$$

$$= \frac{\cancel{y^2}}{(x^2+y^2)}\left(-\frac{x}{\cancel{y^2}}\right)=\frac{-x}{(x^2+y^2)}$$

Again, differentiate $\frac{\partial u}{\partial y}$ w.r.t. x partially, keeping y as constant.

$$\therefore \frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left[-\frac{x}{(x^2+y^2)}\right]$$

→ **Using standard result :**

$...\left[\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{\left[g(x)\cdot\frac{d}{dx}f(x)-f(x)\cdot\frac{d}{dx}g(x)\right]}{(g(x))^2}\right]$

Or

$$\ldots\left[\frac{d}{dx}\left[\frac{I}{II}\right]=\frac{\left[II\cdot\frac{d}{dx}(I)-(I)\cdot\frac{d}{dx}(II)\right]}{(II)^2}\right]$$

$\ldots I = x\ ;\ II = (x^2+y^2)$

$$\frac{\partial u}{\partial y} = -\frac{(x^2+y^2)\cdot\overbrace{\frac{\partial}{\partial x}(x)}^{1}-(x)\cdot\overbrace{\frac{\partial}{\partial x}(x^2+y^2)}^{2x}}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x\partial y} = \left[-\frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}\right]$$

$$= -\left[-\frac{(x^2+y^2)}{(x^2+y^2)^2}\right]=\frac{x^2-y^2}{(x^2+y^2)^2} \qquad \ldots(3)$$

Step III : From Equations (2) and (3), we get

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}\checkmark \qquad \ldots\textbf{Ans.}$$

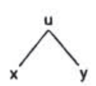
Example 4.6.12

If $u = x^y$, then show that $\frac{\partial^3 u}{\partial x^2\partial y}=\frac{\partial^3 u}{\partial x\,\partial y\,\partial x}$

Solution :

Step I : Given : $u = x^y$ ...(1)

Here, u is a function of x, y.

$u \longrightarrow x, y$ or 

Step II : Differentiate Equation (1) partially w.r.t. y, keeping x as constant.

➔ **Using standard result :** $\ldots\left[\frac{d}{dx}(a^x)=a^x\log a\right]$

Here, $a = x$ and $x = y$

$$\frac{\partial u}{\partial y} = x^y\log x$$

Again differentiate $\frac{\partial u}{\partial x}$ partially w.r.t. x, keeping y as constant.

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right]=\frac{\partial}{\partial x}[x^y\log x]$$

➔ **Using standard result :**

$$\ldots\left[\frac{d}{dx}[f(x)\cdot g(x)]=f(x)\cdot\frac{d}{dx}g(x)+g(x)\cdot\frac{d}{dx}f(x)\right]$$

Or

$$\ldots\left[\frac{d}{dx}[I\cdot II]=I\cdot\frac{d}{dx}(II)+II\cdot\frac{d}{dx}(I)\right]$$

$\ldots I = x^y\ ;\ II = \log x$

$$= (x^y)\cdot\underbrace{\frac{\partial}{\partial x}\log x}_{1/x}+\log x\cdot\underbrace{\frac{\partial}{\partial x}(x^y)}_{yx^{y-1}}$$

$$= x^y\cdot\frac{1}{x}+\log x\cdot yx^{y-1}$$

$(\because d(x^n)=n\cdot x^{n-1})$

$$= x^{y-1}+x^{y-1}\cdot y\log x \ldots(\because x^y x^{-1}=x^{y-1})$$

$$\frac{\partial^2 u}{\partial x\partial y} = x^{y-1}[1+y\log x]$$

Again, differentiate $\frac{\partial^2 u}{\partial x\partial y}$ w.r.t. x, keeping y as constant

$$\frac{\partial^3 u}{\partial x^2\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial^2 u}{\partial x\partial y}\right]=\frac{\partial}{\partial x}[x^{y-1}(1+y\log x)]$$

➔ **Using standard result :**

$$\ldots\left[\frac{d}{dx}[f(x)\cdot g(x)]=f(x)\cdot\frac{d}{dx}g(x)+g(x)\cdot\frac{d}{dx}f(x)\right]$$

Or

$$\ldots\left[\frac{d}{dx}[I\cdot II]=I\cdot\frac{d}{dx}(II)+II\cdot\frac{d}{dx}(I)\right]$$

$\ldots I = x^{y-1}$ and $II = y\log x$

Also, Using standard result : $d(x)^n = nx^{n-1}$ and $\frac{d}{dx}\log x=\frac{1}{x}$

$$= \left(x^{y-1}\cdot\frac{\partial}{\partial x}(1+y\log x)+(1+y\log x)\cdot\underbrace{\frac{\partial}{\partial x}(x^{y-1})}_{(y-1)\,x^{y-2}}\right)$$

$$= (x^{y-1})\left(0+y+\frac{1}{x}\right)+(1+y\log x)\cdot(y-1)\cdot x^{y-2}$$

$$= x^{y-2}\cdot y+(1-y)\cdot(1+y\log x)\cdot x^{y-2}$$

$$\frac{\partial^3 u}{\partial x^2\partial y} = x^{y-2}[y+(1-y)\cdot(1+y\log x)] \qquad \ldots(A)$$

Step III : Differentiate Equation (A) partially w.r.t. x, keeping y as constant

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^y) \qquad \therefore\ \frac{\partial u}{\partial x} = yx^{y-1}$$

Again differentiate $\frac{\partial u}{\partial x}$ w.r.t. y, keeping x as constant

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]=\frac{\partial}{\partial y}[yx^{y-1}]$$

➔ **Using standard result :**

$$\ldots\left[\frac{d}{dx}[f(x)\cdot g(x)]=f(x)\cdot\frac{d}{dx}g(x)+g(x)\cdot\frac{d}{dx}f(x)\right]$$

Or

$$\ldots\left[\frac{d}{dx}[I\cdot II]=I\cdot\frac{d}{dx}(II)+II\cdot\frac{d}{dx}(I)\right]$$

$$\ldots\left[\frac{d}{dx}(a^x)=a^x\cdot\log x\right]$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = y\cdot\underbrace{\frac{\partial}{\partial y}(x^{y-1})}_{x^{y-1}\log x}+(x^{y-1})\cdot\underbrace{\frac{\partial}{\partial y}(y)}_{1}$$

$$= y\cdot x^{y-1}\cdot\log x+x^{y-1}$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = x^{y-1}[1+y\log x]$$

Differentiate $\frac{\partial^2 u}{\partial y\,\partial x}$ w.r.t. x, keeping y as constant

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial x} = \frac{\partial}{\partial x}\left[\frac{\partial^2 u}{\partial y\,\partial x}\right]=\frac{\partial}{\partial x}[x^{y-1}(1+y\log x)]$$

➔ **Using standard result :**

$$\ldots\left[\frac{d}{dx}[f(x)\cdot g(x)]=f(x)\cdot\frac{d}{dx}g(x)+g(x)\cdot\frac{d}{dx}f(x)\right]$$

Or

$$\ldots\left[\frac{d}{dx}[I\cdot II]=I\cdot\frac{d}{dx}(II)+II\cdot\frac{d}{dx}(I)\right]$$

$$\ldots[d[x]^n = n\,(x)^{n-1}]$$

$$= x^{y-1}\cdot\underbrace{\frac{\partial}{\partial x}(1+y\log x)}_{y\cdot\frac{1}{x}}+(1+y\log x)\cdot\underbrace{\frac{\partial}{\partial x}x^{y-1}}_{(y-1)\,x^{y-2}}$$

$$= x^{y-1}\left[y\cdot\frac{1}{x}\right]+(y-1)\,x^{y-2}(1+y\log x)$$

$$= y\,x^{y-2}+(y-1)(1+y\log x)\,x^{y-2}$$

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial x} = x^{y-2}[y+(y-1)(1+y\log x)] \qquad \ldots(B)$$

Step IV : From Equations (A) and (B) we get,

$$\frac{\partial^3 u}{\partial x^2\,\partial y} = \frac{\partial^3 u}{\partial x\,\partial y\,\partial x}\checkmark \qquad \ldots\textbf{Ans.}$$

Type III : Some Particular Examples

Example 4.6.13

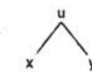
If $u=\frac{x^2+y^2}{x+y}$, then show that

$$\left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)^2=4\left[1-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right]$$

Solution :

Step I : Given : $u=\frac{x^2+y^2}{x+y}$ ...(1)

Here, u is a function of x and y

$u \longrightarrow x, y$ or 

Step II : Differentiate Equaion (1) w.r.t. x, partially keeping y constants

➔ **Using standard result of derivatives :**

$$\ldots\left[\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{\left[g(x)\cdot\frac{d}{dx}f(x)-f(x)\cdot\frac{d}{dx}g(x)\right]}{(g(x))^2}\right]$$

Or

$$\ldots\left[\frac{d}{dx}\left[\frac{I}{II}\right]=\frac{\left[II\cdot\frac{d}{dx}(I)-(I)\cdot\frac{d}{dx}(II)\right]}{(II)^2}\right]$$

$$\frac{\partial u}{\partial x}=\frac{(x+y)\,\underbrace{\frac{\partial}{\partial x}(x^2+y^2)}_{(2x)}-(x^2+y^2)\,\underbrace{\frac{\partial}{\partial x}(x+y)}_{(1)}}{(x+y)^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x+y)\cdot(2x)-(x^2+y^2)(1)}{(x+y)^2}$$

$$= \frac{2x^2+2xy-x^2-y^2}{(x+y)^2}$$

$$= \frac{x^2-y^2+2xy}{(x+y)^2} \qquad \ldots(A)$$

Similarly, we get,

$$\frac{\partial u}{\partial y}=\frac{y^2-x^2+2xy}{(x+y)^2} \qquad \ldots(B)$$

Step III : Substracting Equation (B) from Equation (A) we get,

$$\therefore\ \left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right) = \left(\frac{x^2-y^2+2xy}{(x+y)^2}\right)-\left(\frac{y^2-x^2+2xy}{(x+y)^2}\right)$$

$$= 2\,\frac{x^2-y^2}{(x+y)^2}=\frac{2(x-y)\cancel{(x+y)}}{(x+y)\cancel{(x+y)}}$$

$(\because$ Using $(a^2-b^2)=(a-b)(a+b))$

$$= 2\,\frac{x-y}{(x+y)}$$

Squaring both sides, we get,

$$\left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)^2 = 4\left(\frac{x-y}{x+y}\right)^2$$

$$\therefore\ \text{L.H.S} = \left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)^2 = 4\left(\frac{x-y}{x+y}\right)^2 \qquad \ldots(2)$$

Step IV : Also

$$\text{R.H.S} = 4\left[1-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right]$$

Putting the value of $\frac{\partial u}{\partial x}$ from equation (A) and $\frac{\partial u}{\partial y}$ from equation (B)

$$= 4\left[1-\frac{(x^2-y^2+2xy)}{(x+y)^2}-\frac{(y^2-x^2+2xy)}{(x+y)^2}\right]$$

$$= 4\left[1-\left[\frac{\cancel{x^2}-\cancel{y^2}+2xy+\cancel{y^2}-\cancel{x^2}+2xy)}{(x+y)^2}\right]\right]$$

$$= 4\left[\underbrace{1-\frac{4xy}{(x+y)^2}}_{\textit{cross multiplication}}\right] = 4\left[\frac{(x+y)^2-4xy}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2+y^2+2xy-4xy}{(x+y)^2}\right] = 4\left[\frac{x^2+y^2-2xy}{(x+y)^2}\right]$$

$(\because (a+b)^2 = a^2+b^2+2ab;$ put $a = x, b = y)$

$$\therefore 4\left[1-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right] = 4\left(\frac{x-y}{x+y}\right)^2 \quad \text{...(3)}$$

$(\because$ Use $x^2+y^2-2xy = (x-y^2))$

Step V : Hence from Equations (2) and (3) it is clear that

$$\left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)^2 = 4\left[1-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right] ✓ \quad \text{...Ans.}$$

Example 4.6.14

If $u = \log(x^3+y^3-x^2y-xy^2)$ then prove that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u = \frac{-4}{(x+y)^2}$

Solution :

Step I : Since, L.H.S. $= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)u$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right) = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\cdot w$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u = \frac{\partial w}{\partial x}+\frac{\partial w}{\partial y} \quad \text{...(1)}$$

$$\text{where,} \quad w = \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y} \quad \text{...(2)}$$

Step II : Now, given $u = \log(x^3+y^3-x^2y-xy^2)$

$$= \log[x^2(x-y)+y^2(y-x)]$$

$$= \log[x^2(x-y)-y^2(x-y)]$$

Take common,

$$= \log[(x-y)(x^2-y^2)]$$

$\ldots(\because x^2-y^2 = (x-y)(x+y))$

$$= \log[(x-y)(x-y)(x+y)]$$

$$= \log[(x-y)^2(x+y)]$$

$\ldots(\because$ Using : $\log(AB) = \log A + \log B;$

Here, $A = (x+y)^2; B = (x+y)$

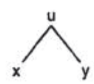
$$= \log(x-y)^2 + \log(x+y)$$

$\ldots(\because$ Using : $\log(a^b) = b \log a$; Here $a = (x-y); b = 2)$

$$u = 2\log(x-y)+\log(x+y) \quad \text{...(3)}$$

Here, u is a function of x, y.

u ⟶ x, y or



Step III : Differentiating Equation (3) partially, w.r.t x, keeping y as constant,

$$\frac{\partial u}{\partial x} = 2\cdot\frac{1}{(x-y)}\cdot\underbrace{\frac{\partial}{\partial x}(x-y)}_{1}+\frac{1}{(x+y)}\cdot\underbrace{\frac{\partial}{\partial y}(x+y)}_{1}$$

$$\therefore \frac{\partial u}{\partial x} = 2\cdot\frac{1}{(x-y)}+\frac{1}{(x+y)}$$

Differentiate (3) partially, w.r.t. y, keeping x as constant

➔ **Using standard result :** $\ldots\left[\frac{d}{dx}[\log f(x)] = \frac{1}{f(x)}\cdot\frac{d}{dx}f(x)\right]$

$$\frac{\partial u}{\partial y} = 2\cdot\frac{1}{(x-y)}\cdot\underbrace{\frac{\partial}{\partial y}(x-y)}_{(-1)}+\frac{1}{(x+y)}\cdot\underbrace{\frac{\partial}{\partial y}(x+y)}_{(1)}$$

$$= \left[2\cdot\frac{1}{(x-y)}(-1)+\frac{1}{(x+y)}(1)\right]$$

$$\text{and} \quad \frac{\partial u}{\partial y} = \frac{-2}{(x-y)}+\frac{1}{(x+y)}$$

$$\text{Therefore,} \quad w = \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}$$

$$= \left[\frac{\cancel{2}}{\cancel{x-y}}+\frac{1}{x+y}\right]+\left[\frac{\cancel{-2}}{\cancel{x-y}}+\frac{1}{x+y}\right]$$

$$= \frac{(1+1)}{(x+y)}$$

$$w = \left(\frac{2}{x+y}\right)$$

Step IV : Equation (2) becomes,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u = \frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}$$

$$= \underbrace{\frac{\partial}{\partial x}\left[\frac{2}{x+y}\right]}_{} + \underbrace{\frac{\partial}{\partial y}\left[\frac{2}{x+y}\right]}_{}$$

Differentiating w.r.t. x, keeping y as contant   *Differentiate w.r.t. y, keeping x as constant*

$$(-1)\cdot\underbrace{\frac{\partial}{\partial x}(x+y)}_{} \qquad (-1)\cdot\underbrace{\frac{\partial}{\partial y}(x+y)}_{}$$

$$= 2\cdot\frac{(1)}{(x+y)^2}+2\frac{(1)}{(x+y)^2}$$

$$= \frac{-2}{(x+y)^2}+\frac{-2}{(x+y)^2} = \frac{-4}{(x+y)^2} = \text{R.H.S.}$$

$$\text{Hence,} \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u = \frac{-4}{(x+y)^2} ✓ \quad \text{...Ans.}$$

Example 4.6.15 Dr. BATU - May 18

If $u = \log(x^3+y^3+z^3-3xyz)$

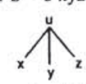
prove that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$

Solution :

Step I :

Given : $u = \log(x^3+y^3+z^3-3xyz)$

u ⟶ x, y, z or



$$\text{L.H.S.} = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$$

$$= \left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]u$$

$$= \left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right]$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]w = \frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}+\frac{\partial w}{\partial z}$$

$$\text{where,} \quad w = \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} \quad \text{...(1)}$$

Step II : Now given, $u = \log(x^3+y^3+z^3-3xyz)$ ...(2)

Here, u is a function of x, y, z.

Diffrentiating Equation (2) w.r.t. x, keeping y and z as constant

➔ **Using standard result of derivative :**

$$\ldots\left[\frac{d}{dx}[\log f(x)] = \frac{1}{f(x)}\cdot\frac{d}{dx}f(x)\right]$$

$\ldots f(x) = (x^3+y^3+z^3-3xyz)$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\log(x^3+y^3+z^3-3xyz)$$

$$= \frac{1}{x^3+y^3+z^3}\cdot\underbrace{\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)}_{(3x^2-3yz)}$$

$$\left(\because \frac{\partial}{\partial x}(x^3) = 3x^2\right)$$

$$\frac{\partial u}{\partial x} = \frac{1\cdot 3\cdot(x^2-yz)}{x^3+y^3+z^3-3xyz}$$

Similarly, we get,

$$\frac{\partial u}{\partial y} = \frac{3(y^2-xz)}{x^3+y^3+z^3-3xyz} \quad \text{and}$$

$$\frac{\partial u}{\partial z} = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}$$

Step III : Adding, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$, we get,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \left(\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}\right)+\left(\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}\right)+\left(\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}\right)$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-yz-xz-xy)}{x^3+y^3+z^3-3xyz}$$

[since, $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-ac-bc)$]

$$\therefore \left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) = \frac{3\cancel{(x^2+y^2+z^2-xy-yz-zx)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = w = \frac{3}{(x+y+z)}$$

Step IV : Therefore, Equation (1) becomes,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}+\frac{\partial w}{\partial z}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

➔ **Using standard result of derivative**

$$\ldots\left[\frac{d}{dx}\frac{1}{f(x)} = -\frac{1}{[f(x)]^2}\cdot\frac{d}{dx}f(x)\right]$$

...Here, $f(x) = (x+y+z)$

$$= -\frac{3}{(x+y+z)^2}\cdot\underbrace{\frac{\partial}{\partial x}(x+y+z)}_{1}+\frac{-3}{(x+y+z)^2}\cdot\underbrace{\frac{\partial}{\partial y}(x+y+z)}_{1}$$

$$+\frac{-3}{(x+y+z)^2}\cdot\underbrace{\frac{\partial}{\partial z}(x+y+z)}_{1}$$

$$= -\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}$$

Hence,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2} \checkmark \quad \text{...Ans.}$$

4.7 To find The Values of Constants or Unknowns

Example 4.7.1

Find the value 'n' for which $u = A e^{-gx} \sin(nt - gx)$ satisfies the partial differential equation $\frac{\partial u}{\partial t} = m\frac{\partial^2 u}{\partial x^2}$ where, A, g, m are constant.

Solution :

Step I : Given $= u\ Ae^{-gx}\sin(nt-gx)$ ...(1)

Here, u is a function of x, t

$u \to x, t$ or

Step II : Diffferentiate Equation (1) partially w.r.t. t, keeping x as constant.

➔ **Using standard result :** $\frac{d}{dx}(\sin(ax+b))$

$$= \cos(ax+b)\cdot\frac{\partial}{\partial x}(ax+b)$$

$$\frac{\partial u}{\partial t} = Ae^{-gx}\frac{\partial}{\partial t}\sin(nt-gx)\cdot\underbrace{\frac{\partial}{\partial t}(nt-gx)}_{(n)}$$

$$= Ae^{-gx}\cdot\cos(nt-gx)\cdot(n)$$

$$\frac{\partial u}{\partial t} = n\,Ae^{-gx}\cos(nt-gx) \quad ...(2)$$

Now, Differentiate Equation (1) partially w.r.t. x, keeping t as constant.

➔ **Using standard rule :**

$$...\left[\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\frac{d}{dx}g(x)+g(x)\cdot\frac{d}{dx}f(x)\right]$$

Or

$$...\left[\frac{d}{dx}[I\cdot II] = I\cdot\frac{d}{dx}(II)+II\cdot\frac{d}{dx}(I)\right]$$

$...I = Ae^{-gx}$ and $II = \sin(nt-gx)$

$$\frac{\partial u}{\partial x} = A\left[e^{-gx}\cdot\frac{\partial}{\partial x}\sin(nt-gx)\cdot\frac{\partial}{\partial x}(nt-gx) + \sin(nt-gx)\frac{\partial}{\partial x}e^{-gx}\right]$$

$$= A\,[e^{-gx}\cos(nt-gx)(-g)+\sin(nt-gx)\cdot e^{-gx}(-g)]$$

$$\frac{\partial u}{\partial x} = Ae^{-gx}\cdot\cos(nt-gx)\cdot(-g)(-g)+Ae^{-gx}\sin(nt-gx)$$

Again, Differentiate $\frac{\partial u}{\partial x}$ w.r.t. x, keeping t as constant

And $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

➔ **Using standard rule :**

$$...\left[\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\frac{d}{dx}g(x)+g(x)\cdot\frac{d}{dx}f(x)\right]$$

Or

$$...\left[\frac{d}{dx}[I\cdot II] = I\cdot\frac{d}{dx}(II)+II\cdot\frac{d}{dx}(I)\right]$$

$$\frac{\partial^2 u}{\partial x^2} = -gA\left[e^{-gx}\cdot\frac{\partial}{\partial x}\cos(nt-gx)\cdot\frac{\partial}{\partial x}(nt-gx) + \cos(nt-gx)\cdot\frac{\partial}{\partial x}e^{-gx}\right]$$

$$-gA\left[e^{-gx}\cdot\frac{\partial}{\partial x}\sin(nt-gx)\cdot\frac{\partial}{\partial x}(nt-gx) + \sin(nt-gx)\cdot\frac{\partial}{\partial x}e^{-gx}\right]$$

$$= -gA\,[e^{-gx}\cdot(-\sin(nt-gx))(-g)+\cos(nt-gx)\ (-g)\,e^{-gx}]$$

$$-gA\,[e^{-gx}\ \cos(nt-gx)(-g)+\sin(nt-gx)\cdot e^{-gx}(-g)]$$

$$= -g^2Ae^{-gx}\cancel{\sin}\ (nt-gx)+g^2Ae^{-gx}\cos(nt-gx)$$

$$+g^2Ae^{-gx}\cos(nt-gx)+g^2Ae^{-gx}\cancel{\sin}\,(\cancel{nt}-gx)$$

$$\frac{\partial^2 u}{\partial x^2} = 2g^2\,Ae^{-gx}\cos(nt-gx) \quad ...(3)$$

Step III : Given :

$$\frac{\partial u}{\partial t} = m\frac{\partial^2 u}{\partial x^2}$$

From equation (2) and from equation (3)

$$A\cancel{n}e^{-\cancel{gx}}\cos(nt\cancel{-gx}) = 2g^2m\ \cancel{A}e^{-\cancel{gx}}\cos(nt\cancel{-gx})$$

$$\Rightarrow \quad n = 2mg^2\checkmark \quad \text{...Ans.}$$

Example 4.7.2

If $u = t^n e^{-\frac{r^2}{4kt}}$, then find the value of n for which $\frac{\partial u}{\partial t} = k\left[\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}\right]$; k is any constant.

Solution : Step I : Given : $u = t^n e^{-\frac{r^2}{4kt}}$ ...(1)

Since, u is a function of t, r

$u \to t, r$ or

Step II : Differentiate Equation (1) partially w.r.t. t keeping r as constant.

➔ **Using standard results :**

(i) $$...\left[\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\frac{d}{dx}g(x)+g(x)\cdot\frac{d}{dx}f(x)\right]$$

Or

$$...\left[\frac{d}{dx}[I\cdot II] = I\cdot\frac{d}{dx}(II)+II\cdot\frac{d}{dx}(I)\right]$$

(ii) $d(x^n) = n\,x^{n-1}$

(iii) $d(e^{f(x)}) = e^{f(x)}d\cdot f(x)$

$$\frac{\partial u}{\partial t} = t^n\cdot\frac{\partial}{\partial t}e^{-\frac{r^2}{4at}}+e^{-\frac{r^2}{4at}}\cdot\underbrace{\frac{\partial}{\partial t}\cdot t^n}_{nt^{n-1}}$$

$$\frac{\partial u}{\partial t} = nt^{n-1}\ e^{-\frac{r^2}{4kt}}+t^n e^{-\frac{r^2}{4kt}}\left(\frac{r^2}{4kt^2}\right)$$

$$\therefore\ \frac{\partial u}{\partial t} = e^{-\frac{r^2}{4at}}\left[nt^{n-1}+\frac{1}{4k}r^2t^{n-2}\right] \quad ...(A)$$

Again differentiate Equation (1) w.r.t. r, keeping t as constant.

➔ **Using standard results :**

$$...\left[\frac{d}{dx}(e^{f(x)}) = e^{f(x)}\cdot\frac{d}{dx}f(x)\right]$$

$$...\left[\frac{d}{dx} = e^{a(x)}\cdot a\right]$$

$$...f(x) = \left(\frac{-r^2}{4kt}\right)$$

$$\frac{\partial u}{\partial r} = t^n\underbrace{\frac{\partial}{\partial r}\ e^{-\frac{r^2}{4kt}}}_{e^{-\frac{r^2}{4kt}}\left(\frac{-2r}{4kt}\right)}$$

$$\therefore\ \frac{\partial u}{\partial r} = \cdot t^n e^{-\frac{r^2}{4kt}}\cdot\left(\frac{-\cancel{2}t}{\cancel{4}_2kt}\right)$$

$$= \frac{-1}{2k}\,r\,t^{n-1}\ e^{-\frac{r^2}{4kt}} \quad ...(B)$$

Again differentiate $\frac{\partial u}{\partial r}$ w.r.t r, keeping t as constant

and $$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r}\right]$$

$$= -\frac{1}{2k}\ t^{n-1}\left[e^{-\frac{r^2}{4kt}}\underbrace{\frac{\partial}{\partial r}(r)}_{1}+(r)\underbrace{\frac{\partial}{\partial t}\cdot e^{-\frac{r^2}{4kt}}}_{e^{-\frac{r^2}{4kt}}\left(-\frac{2r}{4kt}\right)}\right]$$

$$= -\frac{1}{2k}\ t^{n-1}\left[e^{-\frac{r^2}{4kt}}+r.\,e^{-\frac{r^2}{4kt}}\left(\frac{-2r}{4kt}\right)\right]$$

$$= -\frac{1}{2k}\ t^{n-1}\cdot e^{-\frac{r^2}{4kt}}\cdot\left[1-\frac{t}{2kt}\right] \quad ...(C)$$

Adding $\frac{\partial^2 u}{\partial r^2}$ and $\frac{2}{r}\cdot\frac{\partial u}{\partial r}$

From equation (B) and (C)

Step III : Now, $k\left[\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\ \frac{\partial u}{\partial r}\right]$

$$= k\left\{-\frac{1}{2k}t^{n-1}e^{-\frac{r^2}{4kt}}\left(1-\frac{2r^2}{4kt}\right)+\frac{2}{r}\left(-\frac{r}{2k}t^{n-1}e^{-\frac{r^2}{4kt}}\right)\right\}$$

$$\therefore k\left[\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\ \frac{\partial u}{\partial r}\right] = -\frac{1}{2}\ t^{n-1}\ e^{-\frac{r^2}{4kt}}\left(1-\frac{2r^2}{4kt^2}\right)-t^{n-1}\ e^{-\frac{r^2}{4kt}} \quad ...(D)$$

Given, $\frac{\partial u}{\partial t} = k\left[\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\ \frac{\partial u}{\partial r}\right]$

Now, from Equations (A), (B) and (C), we get

$$\therefore\ \cancel{e^{-\frac{r^2}{4kt}}}\left[n\,t^{n-1}+\frac{1}{4k}r^2t^{n-2}\right]$$

$$= -\frac{1}{2}\ t^{n-1}\ \cancel{e^{-\frac{r^2}{4kt}}}\left(1-\frac{2r^2}{4kt^2}\right)-t^{n-1}\ \cancel{e^{-\frac{r^2}{4kt}}}$$

$$\Rightarrow nt^{n-1}+\frac{1}{4k}\cancel{r^2}t^{n-2} = \frac{t^{n-1}}{2}+\frac{t^{n-2}}{\cancel{4k}}r^2-t^{n-1} = \left(-\frac{1}{2}-1\right)t^{n-1}$$

$$nt^{n-1} = -\frac{3}{2}\,t^{n-1}$$

$$n = -\frac{3}{2}\checkmark \quad \text{...Ans.}$$

Example 4.7.3

Find the value of n for which $u = k\,t^{-\frac{1}{2}}\,e^{\left(-\frac{x^2}{na^2t}\right)}$ satisfies the partial differential equation $4\cdot\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$.

Solution :

Step I : Given : $u = k\,t^{-\frac{1}{2}}\ e^{-\frac{x^2}{na^2t}}$ ...(1)

Here, u is a function of x, t

$u \to x, t$ or

Step II : Differentiate equation (1) partially w.r.t. t, keeping x as constant

→ Using standard Rules :

(i) $\ldots\left[\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\frac{d}{dx}g(x) + g(x)\cdot\frac{d}{dx}f(x)\right]$

Or

$\ldots\left[\frac{d}{dx}[I\cdot II] = I\cdot\frac{d}{dx}(II) + II\cdot\frac{d}{dx}(I)\right]$

(ii) $\ldots\left[\frac{d}{dx}(e^{f(x)}) = e^{f(x)}\cdot\frac{d}{dx}f(x)\right]$

$\left[\frac{d}{dx} = e^{a(x)}\cdot a\right]$

(iii) $\ldots[\,d[x]^n = n\,(x)^{n-1}]$

$$\frac{\partial u}{\partial t} = k\cdot t^{-\frac{1}{2}}\frac{\partial}{\partial t}e^{-\frac{x^2}{na^2t}} + k\cdot e^{-\frac{x^2}{na^2t}}\underbrace{\frac{\partial}{\partial t}t^{-\frac{1}{2}}}_{\left(-\frac{1}{2}t^{-\frac{3}{2}}\right)}$$

$$\frac{\partial u}{\partial t} = k\,t^{-\frac{1}{2}}\cdot e^{-\frac{x^2}{na^2t}}\left(\frac{x^2}{na^2t^2}\right) - k\frac{1}{2}t^{-\frac{3}{2}}e^{-\frac{x^2}{na^2t}} \qquad \ldots(A)$$

Multiplying both sides by 4 and take common $e^{-\frac{x^2}{na^2t}}$

$$4\frac{\partial u}{\partial t} = e^{-\frac{x^2}{na^2t}}\left[\frac{4kx^2t^{-5/2}}{na^2} - 2kt^{-3/2}\right]$$

Again, differentiate Equation (1) partially w.r.t. x, keeping t as constant

→ Using standard results :

$\ldots\left[\frac{d}{dx}e^{f(x)} = e^{f(x)}\frac{d}{dx}f(x)\right]$

$$\frac{\partial u}{\partial x} = k\,t^{-\frac{1}{2}}\underbrace{\frac{\partial}{\partial x}e^{-\frac{x^2}{na^2t}}}_{e^{-\frac{x^2}{na^2t}}\left(\frac{-2x}{na^2t}\right)}$$

$$= \frac{2k}{na^2}\;t^{-\frac{3}{2}}\left(x\,e^{-\frac{x^2}{na^2t}}\right)$$

Again differentiate $\frac{\partial u}{\partial x}$ w.r.t x, keeping t as constant

→ Using standard Results

$\ldots\left[\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\frac{d}{dx}g(x) + g(x)\cdot\frac{d}{dx}f(x)\right]$

Or

$\ldots\left[\frac{d}{dx}[I\cdot II] = I\cdot\frac{d}{dx}(II) + II\cdot\frac{d}{dx}(I)\right]$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{-2k}{na^2}t^{-\frac{3}{2}}\cdot\left(x\,e^{-\frac{x^2}{na^2t}}\right)\right]$$

$$= \frac{-2k}{na^2}t^{-\frac{3}{2}}\left\{(x)\cdot\frac{\partial}{\partial x}e^{-\frac{x^2}{na^2t}} + e^{-\frac{x^2}{na^2t}}\underbrace{\frac{\partial}{\partial x}x}_{1}\right\}$$

$$\text{and } \frac{\partial^2 u}{\partial x^2} = \frac{-2k}{na^2}t^{-\frac{3}{2}}\left[xe^{-\frac{x^2}{na^2t}}\left(-\frac{2x}{na^2t}\right) + e^{-\frac{x^2}{na^2t}}\right]$$

Multiplying both sides by a^2

$$a^2\frac{\partial^2 u}{\partial x^2} = \left[\frac{4x^2t^{-\frac{5}{2}}}{n^2a^2} - \frac{2kt^{-\frac{3}{2}}}{n}\right]e^{-\frac{x^2}{na^2t}} \qquad \ldots(B)$$

Step III : From equation (A) and (B) we get

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$$

$$\therefore\left[kt^{-\frac{5}{2}}\frac{4x^2}{n^2a^2} - 2kt^{-\frac{3}{2}}\right]\cancel{e^{-\frac{x^2}{na^2t}}} = \left[\frac{4x^2t^{-\frac{5}{2}}}{n^2a^2} - \frac{2kt^{-\frac{3}{2}}}{n}\right]\cancel{e^{-\frac{x^2}{na^2t}}}$$

$$\Rightarrow kt^{-\frac{5}{2}}\cancel{\frac{4x^2}{n^2a^2}} - 2kt^{-\frac{3}{2}} = \cancel{\frac{4x^2t^{-\frac{5}{2}}}{n^2a^2}} - \frac{2kt^{-\frac{3}{2}}}{n}$$

$$\Rightarrow -2kt^{-\frac{3}{2}} = -\frac{2k}{n}t^{-\frac{3}{2}}$$

$$\Rightarrow \quad 1 = \frac{2}{n} \Rightarrow n = 2✓ \qquad \textbf{...Ans.}$$

Example 4.7.4

Find the value of n so that $u = r^n(3\cos^2\theta - 1)$ satisfies the partial differential equation

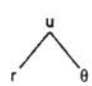
$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0$$

Solution :

Step I : **Given :** $u = r^n(3\cos^2\theta - 1)$ ...(1)

Here, u is a function of r, θ.

$u \to r, \theta$ 'or'



Step II : Differentiate Equation (1) partially w.r.t. r, keeping θ as constant.

→ Using standard result : $\ldots d(x^n) = n\,x^{n-1}$

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}[\,r^n(3\cos^2\theta - 1)]$$

$$= \underbrace{\frac{\partial r^n}{\partial r}}_{n\,r^{n-1}}\cdot(3\cos^2\theta - 1)$$

$$\frac{\partial u}{\partial r} = n\,r^{n-1}(3\cos^2\theta - 1)$$

Multiplying both sides by r^2

$$r^2\frac{\partial u}{\partial r} = n\,r^{n+1}(3\cos^2\theta - 1)$$

$\ldots(\because r^2\cdot r^{n-1} = r^{2+n-1} = r^{n+1})$

Differentiate w.r.t. r, keeping θ as constant

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = n\cdot\underbrace{\left(\frac{\partial}{\partial r}r^{n+1}\right)}_{(n+1)\cdot r^n}(3\cos^2\theta - 1)$$

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial u}{\partial r}\right] = n\,(n+1)\,r^n(3\cos^2\theta - 1) \qquad \ldots(A)$$

Again differentiate Equation (1) partially w.r.t. θ, keeping r as constant

$$\frac{\partial u}{\partial\theta} = r^n\underbrace{\frac{\partial}{\partial\theta}[\,3\cos^2\theta - 1\,]}_{6\cos\theta\cdot(-\sin\theta)}$$

$$\therefore\quad \frac{\partial u}{\partial\theta} = r^n\,6\cos\theta\,(-\sin\theta)$$

$$= -r^n\cdot 6\cdot\sin\theta\cos\theta$$

$$= -3\,r^n\cdot(2\sin\theta\cos\theta) = -3\,r^n\sin 2\theta$$

$\ldots(\because 2\sin\theta\cos\theta = \sin 2\theta)$

Multiplying both sides by sin θ

$$\sin\theta\frac{\partial u}{\partial\theta} = -3\,r^n\sin\theta\sin 2\theta$$

Differentiating both sides w.r.t. θ, keeping r as constant

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = \frac{\partial}{\partial\theta}[-3\,r^n\sin\theta\sin 2\theta]$$

$$= -3r^n\left[\sin 2\theta\cdot\underbrace{\frac{\partial}{\partial\theta}\cdot\sin\theta}_{\cos\theta} + \sin\theta\cdot\underbrace{\frac{\partial}{\partial\theta}\cdot\sin\theta}_{2\cos 2\theta}\right]$$

$$= -3r^n(\cos\theta\sin 2\theta + 2\sin\theta\cos 2\theta)$$

$\ldots\ldots\left(\text{use } \cos^2\theta = \frac{1+\cos 2\theta}{2}\right.$

i.e. $2\cos^2\theta - 1 = \cos 2\theta$

$$= -3r^n[\cos\theta\cdot 2\sin\theta\cos\theta + 2\sin\theta\,(2\cos^2\theta - 1)\,]$$

$\ldots$ (Using $\sin 2\theta = 2\sin\theta\cos\theta$)

Multiplying both sides by $\frac{1}{\sin\theta}$

$$\therefore\ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial u}{\partial\theta}\right] = -3r^n(2\cos^2\theta + 4\cos^2\theta - 2)$$

$$= -3r^n[6\cos^2\theta - 2) = (-3\times 2)\,r^n\cdot[3\cos^2\theta - 1]$$

$$= -6\,r^n[3\cos^2\theta - 1\,] \qquad \ldots(B)$$

Step III : ∴ Adding Equations (A) and (B) and equating with zero.

$$\therefore\ \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0$$

$$\Rightarrow n\,(n+1)\,\underbrace{r^n(3\cos^2\theta - 1)} - 6\underbrace{r^n(3\cos^2\theta - 1)} = 0$$

$$[\,n\,(n+1) - 6\,]\,[\,r^n(3\cos\theta - 1\,)\,] = 0$$

$$[\,n\,(n+1) - 6\,] = \frac{0}{(3\cos\theta - 1)\,r^n} = 0$$

$$\Rightarrow n\,(n+1) - 6 = 0 \quad\Rightarrow\quad n^2 + n - 6 = 0$$

$$n^2 + 3n - 2n - 6 = 0$$

$$\Rightarrow n\,(n+3) - 2\,(n+3) = 0$$

$$\Rightarrow \quad (n+3)\,(n-2) = 0$$

$$\Rightarrow \quad \mathbf{n = 2, -3}\ ✓ \qquad \textbf{...Ans.}$$

Exercise 1

Ex. 1 Find the first order partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ if

(i) $u = e^x\cos y$ (ii) $u = \log\sqrt{x^2 + y^2}$

(iii) $u = \tan^{-1}\left(\frac{x}{y}\right)$ (iv) $x^2e^{2y+3x}\cos 4y$

(v) $u = e^{ax}\cos(\log(x^2 + y^2))$

(vi) $u = \frac{x-y}{x+y}$

Ex. 2 If $u = \log(2x + 2y) + \tan(2x - 2y)$ then prove that $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$

Ex. 3 Verify : $\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}$

(i) $u = \log(2x + 3y)$ (ii) $u = \log(y\sin x + x\sin y)$

Ex. 4 If $u = \frac{e^{x+y}}{e^x + e^y}$ then find the value of $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$

Ex. 5 Find the value of $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$ if $e^{\frac{-z}{x^2 - y^2}} = (x - y)$.

Ex. 6 If $w = (x^2 + y^2 + z^2)^{-1/2}$ then prove that $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 = w^4$

Ex. 7 If $v = x^y$ then show that $\frac{\partial^3 v}{\partial x^2 \partial y} = \frac{\partial^3 v}{\partial x \partial y \partial x}$

Ex. 8 If $u(x, y, z) = \cos 3x \cos 4y \sinh 5z$ then find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

Ex. 9 If $u = [1 - 2xy + y^2]^{-1/2}$, then prove that $\frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2 \frac{\partial u}{\partial y}\right\} = 0$

Ex. 10 If $x = e^{r\cos\theta} \cdot \cos(r \sin\theta)$ and $y = e^{r\cos\theta} \cdot \sin(r\sin\theta)$, then prove that

$$\frac{\partial x}{\partial r} = \frac{1}{r} \cdot \frac{\partial y}{\partial \theta} \cdot \frac{\partial y}{\partial r} = -\frac{1}{r}\frac{\partial x}{\partial \theta}$$

Ex. 11 If $u = \sqrt{x^2 + y^2 + z^2}$ then show that

(i) $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 1$

(ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$

Ex. 12 If $u = (x^2 + y^2 + z^2)^{n/2}$, find the value of n which satisfies the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

4.8 Variables to be Treated as Constants

Let us consider the implicit functions,

$$f_1(u, v, r, \theta) = 0 \quad \text{and} \quad f_2(u, v, r, \theta) = 0$$

$$\text{Let, } x = r\cos\theta; \quad y = r\sin\theta \quad \ldots(4.8.1)$$

$$r^2 = x^2 + y^2 \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

and we have to find $\frac{\partial r}{\partial \theta}$. We need a function which has a relation between r and θ. But in above equations, one more variable x or y is extra i.e. there are four variables x, y, r, θ. So, we have to eliminate only one variable out of four variables from Equation (4.8.1). We get two possible relations :

$$r = x \sec\theta \quad \ldots(4.8.2)$$

$$r = y \operatorname{cosec}\theta \quad \ldots(4.8.3)$$

Now, we can find very easily $\frac{\partial r}{\partial \theta}$ either from Equation (4.8.2) or from Equation (4.8.3). But, we get two different values of $\frac{\partial r}{\partial \theta}$. To avoid this, as to which variable is consider as constant, we use the following method

$\Rightarrow \left(\frac{\partial u}{\partial \theta}\right)_y$ = Means, first write u as a function of θ and y and then differentiate u partially w.r.t. θ, keeping y as constant.

$\Rightarrow \left(\frac{\partial u}{\partial r}\right)_x$ = Means, first write u as a function of r and x and then differentiate u partially, w.r.t r, keeping x as constant.

$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_r$ = Means, first write x as a function of y and r and then differentiate x w.r.t. y, keeping r as constant.

Example 4.8.1

If $x = \frac{\cos\theta}{u}$; $y = \frac{\sin\theta}{u}$;

evaluate $\left(\frac{\partial x}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial y}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial y}\right)_x$

OR If $x = \frac{\cos\theta}{r}$; $y = \frac{\sin\theta}{r}$ then find the value of

$$\left(\frac{\partial x}{\partial r}\right)_\theta \left(\frac{\partial r}{\partial x}\right)_y + \left(\frac{\partial y}{\partial r}\right)_\theta \left(\frac{\partial r}{\partial y}\right)_x$$

Solution :

Step I : Given : $x = \frac{\cos\theta}{u}$; $y = \frac{\sin\theta}{u}$

Differentiate x w.r.t. u, keeping θ as constant and y w.r.t. u, keeping θ as constant.

➔ **Using standard result :** $\ldots\left[d\left(\frac{1}{x}\right) = -\frac{1}{x^2}\right]$

$$\therefore \left(\frac{\partial x}{\partial u}\right)_\theta = -\frac{\cos\theta}{u^2} \quad \ldots(1)$$

$$\left(\frac{\partial y}{\partial u}\right)_\theta = \frac{-\sin\theta}{u^2} \quad \ldots(2)$$

Step II : Now to find u as a function of x and y

Squaring x and y and then add, we get,

$$\therefore x^2 + y^2 = \frac{\cos^2\theta}{u^2} + \frac{\sin^2\theta}{u^2} = \frac{\sin^2\theta + \cos^2\theta}{u^2} = \frac{1}{u^2}$$

$$(\because \sin^2\theta + \cos^2\theta = 1)$$

$$\Rightarrow u^2 = \frac{1}{x^2 + y^2}$$

(Here, we get u is a function of x and y)

$$u \rightarrow f(x, y)$$

Step III : Differentiate u w.r.t. x, keeping y as constant

➔ **Using standard result :**

$$\ldots\left[\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{1}{(f(x))^2}\frac{d}{dx}f(x)\right]$$

$$2u\left(\frac{\partial u}{\partial x}\right)_y = -\frac{1}{(x^2+y^2)^2} \cdot \underbrace{\frac{\partial}{\partial x}(x^2+y^2)}_{(2x)}$$

$$\cancel{2}u\left(\frac{\partial u}{\partial x}\right)_y = -\frac{1}{(x^2+y^2)^2} \cdot 2x = \frac{-\cancel{2}x}{(x^2+y^2)^2}$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y = -\frac{x}{u(x^2+y^2)^2} \quad \ldots(3)$$

Now, differentiate u^2 w.r.t. y, keeping x as constant

$$2u\left(\frac{\partial u}{\partial y}\right)_x = -\frac{1}{(x^2+y^2)^2} \cdot \underbrace{\frac{\partial}{\partial x}(x^2+y^2)}_{(2y)}$$

$$\text{and } 2u\left(\frac{\partial u}{\partial y}\right)_x = -\frac{1}{(x^2+y^2)^2} \cdot 2y = \frac{-\cancel{2}y}{(x^2+y^2)^2}$$

$$\therefore \left(\frac{\partial u}{\partial y}\right)_x = -\frac{y}{u(x^2+y^2)^2} \quad \ldots(4)$$

Step IV : By using Equations (1), (2), (3), (4) we get,

$$\left(\frac{\partial x}{\partial u}\right)_\theta \cdot \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial y}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial y}\right)_x$$

$$= \left(\frac{-\cos\theta}{u^2}\right)\left(\frac{-x}{u(x^2+y^2)^2}\right) + \left(\frac{-\sin\theta}{u^2}\right)\left(\frac{-y}{u(x^2+y^2)^2}\right)$$

$$= \frac{x\cos\theta}{u^3(x^2+y^2)^2} + \frac{y\sin\theta}{u^3(x^2+y^2)^2}$$

$$= \frac{1}{u^3(x^2+y^2)^2} \cdot [x\cos\theta + y\sin\theta]$$

$$= \frac{1}{u^3 \cdot \left(\frac{1}{u^2}\right)^2} \cdot \left[\frac{\cos\theta}{u} \cdot \cos\theta + \frac{\sin\theta}{u} \cdot \sin\theta\right] \ldots\text{Given}$$

$$= \frac{u^{\cancel{4}}}{\cancel{u^3}}\left[\frac{\cos^2\theta + \sin^2\theta}{u}\right]$$

$$= \cancel{u}\left[\frac{\cos^2\theta + \sin^2\theta}{\cancel{u}}\right] = 1(1) = 1$$

Hence,

$$\Rightarrow \left(\frac{\partial x}{\partial u}\right)_\theta \cdot \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial y}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial y}\right)_x = 1 ✓ \quad \ldots\text{Ans.}$$

Example 4.8.2

If $x = \frac{\cos\theta}{u}$; $y = \frac{\sin\theta}{u}$**, then prove that**

$$\left(\frac{\partial x}{\partial u}\right)_\theta \cdot \left(\frac{\partial u}{\partial x}\right)_y = \cos^2\theta$$

Solution :

Step I : Given : $x = \frac{\cos\theta}{u}$; $y = \frac{\sin\theta}{u}$ and

$$x^2 + y^2 = \frac{\cos^2\theta}{u^2} + \frac{\sin^2\theta}{u^2} = \frac{1}{u^2} \ldots(\because \sin^2\theta + \cos^2\theta = 1)$$

$$\Rightarrow u^2 = \frac{1}{x^2+y^2}$$

Here, u is function of x and y.

Step II : Now, differentiate x w.r.t. u keeping θ as constant.

➔ **Using standard formula :** $d\left(\frac{1}{u}\right) = -\frac{1}{u^2}$

$$\left(\frac{\partial x}{\partial u}\right)_\theta = \frac{\cos\theta}{-u^2};$$

and differentiate u w.r.t x keeping y as constants

$$2u\left(\frac{\partial u}{\partial x}\right)_y = -\frac{1}{(x^2+y^2)^2} \cdot \underbrace{\frac{\partial u}{\partial x}(x^2+y^2)}_{(2x)}$$

$$\cancel{2}u\left(\frac{\partial u}{\partial x}\right)_y = -\frac{1}{(x^2+y^2)^2} \cdot 2x = \frac{-\cancel{2}x}{(x^2+y^2)^2}$$

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{-x}{u(x^2+y^2)^2}$$

Step III : Multipling ,

$$\left(\frac{\partial x}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial x}\right)_y = \left(\frac{\cos\theta}{-u^2}\right)\left(\frac{-x}{u(x^2+y^2)^2}\right)$$

$$= \frac{x\cos\theta}{u^3(x^2+y^2)^2} = \frac{x\cos\theta \cancel{u^4}}{\cancel{u^3}}$$

$$\ldots\left(\because u^2 = \frac{1}{(x^2+y^2)}\right)$$

$$= x\,u\,\cos\theta = \frac{\cos\theta}{u} \cdot u\cos\theta = \cos^2\theta$$

...(From given)

Hence,

$$\therefore \left(\frac{\partial x}{\partial u}\right)_\theta \cdot \left(\frac{\partial u}{\partial x}\right)_y = \cos^2\theta ✓ \quad \ldots\text{Ans.}$$

Example 4.8.3

If $x = \frac{r}{2}(e^\theta + e^{-\theta})$; $y = \frac{r}{2}(e^\theta - e^{-\theta})$ **then prove that**

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y$$

Solution :

Step I : Given : $x = \frac{r}{2}(e^\theta + e^{-\theta}) = r\cosh\theta$ and

$$y = \frac{r}{2}(e^\theta - e^{-\theta}) = r\sinh\theta$$

Squaring and subtracting

$$\text{and } x^2 - y^2 = r^2\cosh^2\theta - r^2\sinh^2\theta$$

$$= r^2 \underbrace{[\cosh^2\theta - \sinh^2\theta]}_{1}$$

$= r^2 \qquad \ldots(\because \cosh^2\theta - \sinh^2\theta = 1)$

$\Rightarrow r^2 = x^2 - y^2$

Differentiate x w.r.t. r, keeping θ as constant.

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \cosh\theta \underbrace{\left(\frac{\partial r}{\partial r}\right)}_{1}$$

Step II : $\left(\frac{\partial x}{\partial r}\right)_\theta = \cosh\theta$

Differentiating r^2 w.r.t x, keeping y as constant

and $\cancel{2} r \left(\frac{\partial r}{\partial x}\right)_y = \cancel{2} x$

$$\Rightarrow \left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r} = \frac{\cancel{r}\cosh\theta}{\cancel{r}} = \cosh\theta$$

Hence, $\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y$ ✓ **...Ans.**

Example 4.8.4

If $ux + vy = 0 ; \frac{u}{x} + \frac{v}{y} = 1$ then prove that :

$$\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{x^2+y^2}{y^2-x^2}$$

Solution :

Step I : Given : $ux + vy = 0$ ...(1)

and $\frac{u}{x} + \frac{v}{y} = 1$ ...(2)

Here, We have to find u as function of x, y and find v as a function of x, y

$u = f(x, y)$ and $v = f(x, y)$

Step II : ∴ From Equation (1),

$u = -\frac{vy}{x}$, put this value of u in Equation (2)

$$\frac{-vy}{x^2} + \frac{v}{y} = 1$$

$$\Rightarrow v\left[\frac{1}{y} - \frac{y}{x^2}\right] = 1 \qquad \Rightarrow v\left[\frac{x^2-y^2}{yx^2}\right] = 1$$

$$\Rightarrow v = \frac{yx^2}{x^2-y^2} = -\left[\frac{yx^2}{y^2-x^2}\right] \qquad \ldots(A)$$

Hence, $v = f(x, y)$

Step III : Also, from Equation (1) ; $v = -\frac{ux}{y}$ put in Equation (2),

$$\frac{u}{x} - \frac{ux}{y^2} = 1 \Rightarrow u\left(\frac{1}{x} - \frac{x}{y^2}\right) = 1$$

$$\Rightarrow u\left[\frac{y^2-x^2}{xy^2}\right] = 1$$

$$u = \frac{xy^2}{y^2-x^2} \qquad \ldots(B)$$

Here, u as function of x and y

Step IV : Now differentiate Equation (B) w.r.t. x, keeping y as constant

➔ **Using standard result**

$$\ldots\left[\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[g(x)\cdot\frac{d}{dx}f(x) - f(x)\cdot\frac{d}{dx}g(x)\right]}{(g(x))^2}\right]$$

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{(y^2-x^2)\underbrace{\frac{\partial}{\partial x}(xy^2)}_{(y^2)} - (xy^2)\underbrace{\frac{\partial}{\partial x}(y^2-x^2)}_{(-2x)}}{(y^2-x^2)^2}$$

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{(y^2-x^2)\cdot y^2 - xy^2(-2x)}{(y^2-x^2)^2}$$

$$= \frac{y^4 - x^2y^2 + 2x^2y^2}{(y^2-x^2)^2} = \frac{y^4 + x^2y^2}{(y^2-x^2)^2}$$

$$\left(\frac{\partial u}{\partial x}\right)_y = y^2\frac{(y^2+x^2)}{(y^2-x^2)^2} \qquad \ldots(3)$$

Similarly, differentiate Equation (A) w.r.t. y, keeping x as constant.

$$\left(\frac{\partial v}{\partial y}\right)_x = \frac{(y^2-x^2)\cdot\underbrace{\frac{\partial}{\partial y}(yx^2)}_{(x^2)} - (yx^2)\cdot\underbrace{\frac{\partial}{\partial y}(y^2-x^2)}_{(2y)}}{(y^2-x^2)^2}$$

$$\left(\frac{\partial v}{\partial y}\right)_x = -\left[\frac{(y^2-x^2)\cdot x^2 - yx^2(2y)}{(y^2-x^2)^2}\right]$$

$$= -\left[\frac{y^2x^2 - x^4 - 2x^2y^2}{(y^2-x^2)^2}\right]$$

$$= -\left[\frac{-x^2y^2 - x^4}{(y^2-x^2)^2}\right] = x^2\frac{(x^2+y^2)}{(y^2-x^2)^2} \ldots(4)$$

Step V : Subtracting Equations (4) from (3), we get

$$\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \left(y^2\frac{(y^2+x^2)}{(y^2-x^2)^2} - x^2\frac{(x^2+y^2)}{(y^2-x^2)^2}\right)$$

$$= \frac{(x^2+y^2)\cancel{(y^2-x^2)}}{(y^2-x^2)^{\cancel{2}}} = \left(\frac{x^2+y^2}{(y^2-x^2)}\right)$$

Hence,

$$\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{x^2+y^2}{y^2-x^2} \checkmark \qquad \textbf{...Ans.}$$

Example 4.8.5

If $ux + vy = 0; \frac{u}{x} + \frac{v}{y} = 1$; prove that

$$\left(\frac{u}{x}\right)\left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{v}{y}\right)\left(\frac{\partial y}{\partial v}\right)_u = 0$$

Solution :

Step I : Given : $ux + vy = 0$ ...(1)

$\frac{u}{x} + \frac{v}{y} = 1$ ...(2)

We have to find x and y are functions of u and v.

i.e. Here, we have to find $x = f(u, v)$ and $y = f(u, v)$

Step II : ∴ From Equation (1), $x = -\frac{vy}{u}$,

Put this value of x in Equation (2)

$$\frac{-u}{\left(\frac{vy}{u}\right)} + \frac{v}{y} = 1 \qquad \Rightarrow -\frac{u^2}{vy} + \frac{v}{y} = 1$$

$$\Rightarrow \frac{1}{y}\left[\frac{v}{1} - \frac{u^2}{v}\right] = 1 \qquad \Rightarrow y = \left[\frac{v^2-u^2}{v}\right] \qquad \ldots(A)$$

$\Rightarrow y = f(u,v)$

Step III : From Equation (1), $y = -\frac{ux}{v}$, put this value of y in Equation (2),

$$\frac{u}{x} - \frac{v}{\left(\frac{ux}{v}\right)} = 1 \qquad \Rightarrow \frac{u}{x} - \frac{v^2}{ux} = 1$$

$$\Rightarrow \frac{1}{x}\left[\frac{u}{1} - \frac{v^2}{u}\right] = 1$$

$$x = \left[\frac{u^2-v^2}{u}\right] \qquad \ldots(B)$$

$\Rightarrow x = f(u,v)$

Step IV : Differentiate Equation (B) w.r.t. u, keeping v as constant

➔ **Using formula**

$$\ldots\left[\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[g(x)\cdot\frac{d}{dx}f(x) - f(x)\cdot\frac{d}{dx}g(x)\right]}{(g(x))^2}\right]$$

Or

$$\ldots\left[\frac{d}{dx}\left[\frac{I}{II}\right] = \frac{\left[II\cdot\frac{d}{dx}(I) - (I)\cdot\frac{d}{dx}(II)\right]}{(II)^2}\right]$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{u\cdot\underbrace{\frac{\partial}{\partial u}(u^2-v^2)}_{(2u)} - (u^2-v^2)\cdot\underbrace{\frac{\partial}{\partial u}(u)}_{(1)}}{u^2}$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{u(2u) - (u^2-v^2)\cdot 1}{u^2} = \frac{u^2+v^2}{u^2}$$

Multiplying both sides by $\left(\frac{u}{x}\right)$

$$\left(\frac{u}{x}\right)\cdot\left(\frac{\partial x}{\partial u}\right)_v = \left(\frac{\cancel{u}}{x}\right)\cdot\left(\frac{u^2+v^2}{\cancel{u^2}}\right)$$

$$\left(\frac{u}{x}\right)\left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{x}\left(\frac{u^2+v^2}{u}\right) \qquad \ldots(3)$$

Now, differentiate Equation (A) w.r.t. v, keeping u as constant

$$\left(\frac{\partial u}{\partial v}\right)_u = \frac{v\cdot\underbrace{\frac{\partial}{\partial v}(v^2-u^2)}_{(2v)} - (v^2-u^2)\cdot\underbrace{\frac{\partial}{\partial v}\cdot v}_{(1)}}{v^2}$$

$$\left(\frac{\partial y}{\partial v}\right)_u = \frac{v(2v) - (v^2-u^2)\cdot 1}{v^2} = \frac{v^2+u^2}{v^2}$$

Multiplying both sides by $\frac{v}{y}$

$$\left(\frac{v}{y}\right)\cdot\left(\frac{\partial y}{\partial v}\right)_u = \left(\frac{\cancel{v}}{y}\right)\left(\frac{v^2+u^2}{\cancel{v^2}}\right)$$

$$\left(\frac{v}{y}\right)\left(\frac{\partial y}{\partial v}\right)_u = \frac{1}{y}\left(\frac{u^2+v^2}{v}\right) \qquad \ldots(4)$$

Step V : Adding Equations (3) and (4) we get,

$$\left(\frac{u}{x}\right)\cdot\left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{v}{y}\right)\left(\frac{\partial y}{\partial v}\right)_u = \left[\frac{1}{x}\cdot\left(\frac{u^2+v^2}{u}\right)\right] + \left[\frac{1}{y}\cdot\left(\frac{u^2+v^2}{v}\right)\right]$$

$$\left(\frac{u}{x}\right)\cdot\left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{v}{y}\right)\left(\frac{\partial y}{\partial v}\right)_u = (v^2+u^2)\underbrace{\left[\frac{1}{xu} + \frac{1}{vy}\right]}_{\textit{Cross multiplication}}$$

$$= (v^2+u^2)\left[\frac{vy+xu}{xy\,uv}\right]$$

$$= (v^2+u^2)\left(\frac{ux+vy}{xy\,uv}\right)$$

But given, $ux + vy = 0$

Hence,

$$\therefore \left(\frac{u}{x}\right)\cdot\left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{v}{y}\right)\left(\frac{\partial y}{\partial v}\right)_u = (v^2+u^2)\cdot\left(\frac{0}{xyuv}\right)$$

$$= 0 \checkmark \qquad \textbf{...Ans.}$$

Example 4.8.6

If $x^2 = a\sqrt{u} + b\sqrt{v}$ and $y^2 = a\sqrt{u} - b\sqrt{v}$, where a, b are constants, prove that

$$\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x\left(\frac{\partial y}{\partial v}\right)_u$$

OR If $x^2 = a\sqrt{u} + b\sqrt{v}$; $y^2 = a\sqrt{u} - b\sqrt{v}$, where a, b are constants, find $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v$.

Solution :

Step I : Given : $x^2 = a\sqrt{u} + b\sqrt{v}$;

$$y^2 = a\sqrt{u} - b\sqrt{v}$$

We have to find u as function of x, y

i.e. We have to find $u = f(x, y)$ and $x = f(u, v)$ (given)

$$\text{Since, } x^2 = a\sqrt{u} + b\sqrt{v} \qquad \ldots(1)$$

Step II : Differentiate Equation (1) partially, w.r.t. u, keeping v as constant..

→ **Using standard result :**

$$\ldots\left[d(\sqrt{x}) = \frac{1}{2\sqrt{x}}\right]$$

$$2x\left(\frac{\partial x}{\partial u}\right)_v = a\underbrace{\frac{\partial}{\partial u}(\sqrt{u})}_{(1/2\sqrt{u})} + 0$$

$$2x\left(\frac{\partial x}{\partial u}\right)_v = a\frac{1}{2\sqrt{u}} \Rightarrow \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{4x\sqrt{u}} \qquad \ldots(A)$$

Step III : To find $u = f(x, y)$

i.e. to find u as function of x, y

Adding x^2 and y^2

$$\therefore \quad x^2 + y^2 = 2a\sqrt{u}$$

$$\sqrt{u} = \frac{(x^2 + y^2)}{2a} \qquad \ldots(2)$$

Step IV : Differentiate Equation (2), w.r.t. x, keeping y as constant.

$$\frac{1}{2\sqrt{u}}\left(\frac{\partial u}{\partial x}\right)_y = \frac{1}{2a}\cdot\underbrace{\frac{\partial}{\partial x}(x^2 + y^2)}_{(2x)}$$

$$\frac{1}{2\sqrt{u}}\left(\frac{\partial u}{\partial x}\right)_y = \frac{1}{2a}\cdot 2x = \frac{x}{a}$$

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{2x\sqrt{u}}{a} \qquad \ldots(B)$$

Step V : ∴ From Equation (A) and (B)

$$\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \left(\frac{2x\sqrt{u}}{a}\right)\left(\frac{a}{4x\sqrt{u}}\right) = \frac{1}{2} \qquad \ldots(3)$$

Step VI : Also we have to find $v = f(x, y)$ and $y = f(u, v)$

$$\text{Since,} \quad y^2 = a\sqrt{u} - b\sqrt{v}$$

$$2y\left(\frac{\partial y}{\partial v}\right)_u = 0 - b\underbrace{\frac{\partial}{\partial v}\sqrt{v}}_{1.2\sqrt{v}}$$

$$2y\left(\frac{\partial y}{\partial v}\right)_u = -b\frac{1}{2\sqrt{v}} = \frac{-b}{2\sqrt{v}}$$

$$\left(\frac{\partial y}{\partial v}\right)_u = \frac{-b}{4y\sqrt{v}} \qquad \ldots(C)$$

and for $v = f(x, y)$

$$\therefore \quad x^2 - y^2 = (a\sqrt{u} + b\sqrt{v}) - (a\sqrt{u} - b\sqrt{v})$$

$$= 2b\sqrt{v}$$

$$\sqrt{v} = \frac{1}{2b}\cdot(x^2 - y^2)$$

Step VII : Differentiate v w.r.t. y, keeping x constant.

$$\frac{1}{2\sqrt{v}}\left(\frac{\partial v}{\partial y}\right)_x = \frac{1}{2b}(-2y) = -\frac{y}{b}$$

$$\left(\frac{\partial v}{\partial y}\right)_x = -\frac{2y\sqrt{v}}{b} \qquad \ldots(D)$$

Step VIII : From Equation (C) and (D).

$$\therefore \left(\frac{\partial v}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial v}\right)_u = \left(\frac{-2y\sqrt{v}}{b}\right)\left(\frac{-b}{4y\sqrt{v}}\right) = \frac{1}{2} \qquad \ldots(4)$$

Hence,from Equation (3) and (4), we get

$$\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u \checkmark \qquad \textbf{...Ans.}$$

Example 4.8.7

If $x^2 = au + bv$; $y^2 = au - bv$; then find the value of $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial v}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial v}\right)_u$

Solution :

Step I : Given : $x^2 = au + bv$; ...(1)

i.e. $\quad y^2 = au - bv$ ...(2)

We have to find u and v are functions of x, y

i.e. we have to find $u = f(x, y)$; $v = f(x, y)$

$$x = f(u, v) \;;\; y = f(u, v)$$

Given, x and y are functions of u, v

From Equation (1), differentiate (1) w.r.t. u, keeping v as constant

$$2x\left(\frac{\partial x}{\partial u}\right)_v = \underbrace{\frac{\partial}{\partial u}(au + bv)}_{a}$$

$$2x\left(\frac{\partial x}{\partial u}\right)_v = a \Rightarrow \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2x}$$

From Equation (2), differentiate (2) w.r.t. v, keeping u as constant,

$$2y\left(\frac{\partial y}{\partial v}\right)_u = \underbrace{\frac{\partial}{\partial v}(au - bv)}_{(-b)}$$

$$2y\left(\frac{\partial y}{\partial v}\right)_u = -b \Rightarrow \left(\frac{\partial y}{\partial v}\right)_u = -\frac{b}{2y}$$

Step II : Adding Equations (1) and (2), we get

$$x^2 + y^2 = (au + \cancel{bv}) + (au - \cancel{bv})$$

$$= 2au \Rightarrow u = \frac{1}{2a}(x^2 + y^2)$$

and subtracting (1) and (2), $x^2 - y^2 = (au + bv) - (au - bv)$

$$= (\cancel{au} + bv - \cancel{au} + bv)$$

$$= 2bv \Rightarrow v = \frac{1}{2b}(x^2 - y^2)$$

Differentiate u w.r.t. x, keeping y as constant

$$\left(\frac{\partial u}{\partial x}\right)_y = \underbrace{\frac{\partial}{\partial x}\left(\frac{1}{2a}(x^2 + y^2)\right)}_{\frac{1}{2a}(2x)}$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y = \frac{1}{\cancel{2}a}(\cancel{2}x) = \frac{x}{a}$$

And differentiate v w.r.t. y, keeping x as constant

$$\left(\frac{\partial v}{\partial y}\right)_x = \frac{1}{2b}\underbrace{\frac{\partial}{\partial x}(x^2 - y^2)}_{(0 - 2u)}$$

$$\left(\frac{\partial v}{\partial y}\right)_x = \frac{1}{2b}(0 - 2y) = \frac{-y}{b}$$

Hence,

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial v}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial v}\right)_u = \left(\frac{\cancel{x}}{\cancel{a}}\right)\left(\frac{\cancel{a}}{2\cancel{x}}\right) + \left(\frac{-\cancel{y}}{\cancel{b}}\right)\left(-\frac{\cancel{b}}{2\cancel{y}}\right)$$

$$= \frac{1}{2} + \frac{1}{2} = 1 \checkmark \qquad \textbf{...Ans.}$$

Example 4.8.8

If $x = u\tan v$; $y = u\sec v$, then prove that

Solution :

Step I : Given : $x = u\tan v$; $y = u\sec v$

We have to find u, v are functions of x, y

i.e. we have to find $u = f(x, y)$; $v = f(x, v)$

Step II : Squaring x, y and subtracting,

$$y^2 - x^2 = u^2\tan^2 v - u^2\sec^2 v$$

$$= u^2\underbrace{[\sec^2 v - \tan^2 v]}_{1} = u^2$$

$$y^2 - x^2 = u^2 \qquad \ldots(\because \sec^2 x - \tan^2 x = 1)$$

Differentiate u^2 w.r.t. x, keeping y as constant

$$u^2 = y^2 - x^2$$

$$2u\frac{\partial u}{\partial x} = \underbrace{\frac{\partial}{\partial x}(y^2 - x^2)}_{(-2x)}$$

$$2u\left(\frac{\partial u}{\partial x}\right)_y = -2x \Rightarrow \left(\frac{\partial u}{\partial x}\right)_y = \frac{-x}{u}$$

$$= -\left[\frac{\cancel{u}\tan v}{\cancel{u}}\right]$$

$$\left(\frac{\partial u}{\partial x}\right)_y = -\tan v \qquad \ldots(1)$$

Differentiate u^2 w.r.t. y, keeping x as constant

$$2u\left(\frac{\partial u}{\partial y}\right)_x = \underbrace{\frac{\partial}{\partial y}(y^2 - x^2)}_{(2y)}$$

$$\text{and} \quad 2u\left(\frac{\partial u}{\partial y}\right)_x = 2y \Rightarrow \left(\frac{\partial u}{\partial y}\right)_x = \frac{y}{u} = \frac{\cancel{u}\sec v}{\cancel{u}}$$

$$\left(\frac{\partial u}{\partial y}\right)_x = \sec v \qquad \ldots(2)$$

Step III : Now, to eliminate u, from x and y,

$$\therefore \quad \frac{x}{y} = \frac{\cancel{u}\tan v}{\cancel{u}\sec v} = \frac{\sin v}{\cancel{\cos v}\left(\frac{1}{\cancel{\cos v}}\right)} = \sin v$$

$$\therefore \quad \frac{x}{y} = \sin v \qquad \ldots(A)$$

Step IV : Differentiate Equation (A) w.r.t. x, keeping y as constant.

$$\frac{1}{y}\underbrace{\frac{\partial}{\partial x}(x)}_{1} = \underbrace{\frac{\partial}{\partial x}(\sin v)}_{\cos v\left(\frac{\partial v}{\partial x}\right)_y}$$

$$\frac{1}{y} = \cos v\left(\frac{\partial v}{\partial x}\right)_y$$

$$\Rightarrow \quad \left(\frac{\partial v}{\partial x}\right)_y = \frac{1}{y\cos v} \qquad \ldots(3)$$

Differentiate Equation (A) w.r.t. y, keeping x as constant.

$$x \cdot \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = \underbrace{\frac{\partial}{\partial y}(\sin v)}_{\cos v \cdot \left(\frac{\partial v}{\partial y}\right)_u}$$

$$-\frac{x}{y^2} = \cos v \left(\frac{\partial v}{\partial y}\right)_x$$

$$\Rightarrow \quad \left(\frac{\partial v}{\partial y}\right)_x = -\frac{x}{y^2 \cos v} \qquad \ldots(4)$$

Step V : From Equations (1), (2), (3), (4),

$$\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial v}{\partial x}\right)_y = (-\tan v)\left(\frac{1}{y \cos v}\right) = -\frac{\sin v}{y \cos^2 v} \qquad \left(\because \tan v = \frac{\sin v}{\cos v}\right)$$

and
$$\left(\frac{\partial u}{\partial y}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_x = (\sec v)\left(-\frac{x}{y^2 \cos v}\right) = -\frac{x}{y^2 \cos^2 v} \qquad \left(\because \sec v = \frac{1}{\cos v}\right)$$

$$= \frac{-u \tan v}{y (u \sec v) \cos^2 v} = -\frac{\sin v}{y \cos^2 v}$$

This shows that,

$$\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_x \checkmark \qquad \ldots\textbf{Ans.}$$

Example 4.8.9

If $x = r\cos\theta$; $y = r\sin\theta$ then prove that

(a) $\left(\frac{\partial r}{\partial x}\right)_y = \left(\frac{\partial x}{\partial r}\right)_\theta$

(b) $\frac{1}{r}\left[\frac{\partial x}{\partial \theta}\right]_r = r\left(\frac{\partial \theta}{\partial x}\right)_y$

(c) $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$

Solution :

Step I : Given : $x = r\cos\theta$; $y = r\sin\theta$

Squaring and adding,

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2(\cos^2\theta + \sin^2\theta)$$
$$= r^2(1) = r^2 \quad \text{and}$$
$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta;$$
$$\Rightarrow \frac{y}{x} = \tan\theta;$$

Hence,

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore x = r\cos\theta \qquad \ldots(1)$$
$$y = r\sin\theta \qquad \ldots(2)$$
$$r^2 = x^2 + y^2 \qquad \ldots(3)$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \ldots(4)$$

Step II :

(a) Differentiate Equation (3) w.r.t. x partially, keeping y as constant

$$\cancel{2}r\left(\frac{\partial r}{\partial x}\right)_y = \cancel{2}x$$

$$\Rightarrow \quad \left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r} = \frac{\cancel{r}\cos\theta}{\cancel{r}} = \cos\theta$$

Differentiate Equation (1) w.r.t. r partially, keeping θ as constant.

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$$

Hence,

$$\left(\frac{\partial r}{\partial x}\right)_y = \left(\frac{\partial x}{\partial r}\right)_\theta$$

Step III :

(b) Now differentiate Equation (1) w.r.t. θ, keeping r as constant,

$$\left(\frac{\partial x}{\partial \theta}\right)_r = r \cdot \underbrace{\frac{\partial}{\partial \theta}(\cos\theta)}_{(-\sin\theta)} = -r\sin\theta$$

Multiplying bothsides by $\frac{1}{r}$

$$\Rightarrow \quad \frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r = \frac{1}{\cancel{r}}(-\cancel{r}\sin\theta) = -\sin\theta \qquad \ldots(5)$$

Differentiate Equation (4) partially w.r.t. x, keeping y as constant.

$$\frac{\partial \theta}{\partial x} = \underbrace{\frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x}\right)}_{\frac{1}{1+\left(\frac{y}{x}\right)^2}} \cdot \underbrace{\frac{\partial}{\partial x}\left(\frac{y}{x}\right)}_{\left(-\frac{y}{x^2}\right)}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$$
$$= \left(\frac{\cancel{x^2}}{x^2+y^2}\right)\left(-\frac{y}{\cancel{x^2}}\right) = -\frac{y}{x^2+y^2}$$
$$= -\frac{\cancel{r}\sin\theta}{r^{\cancel{2}}} = -\frac{\sin\theta}{r}$$

$$\therefore \quad \left(\frac{\partial \theta}{\partial x}\right)_y = -\frac{\sin\theta}{r};$$

$$r\left(\frac{\partial \theta}{\partial x}\right)_y = -\sin\theta \qquad \ldots(6)$$

From Equations (5) and (6), we get,

$$\therefore \quad \frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r = r\left(\frac{\partial \theta}{\partial x}\right)_y$$

Step IV : since, $\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} = -\frac{y}{r}\cdot\frac{1}{r} = \frac{y}{r^2} = \frac{-y}{x^2+y^2}$;

$$\frac{\partial r}{\partial x} = \cos\theta$$

Differentiate $\frac{\partial \theta}{\partial x}$ w.r.t. x, keeping y as constant,

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial \theta}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{-y}{x^2+y^2}\right]$$

$$= -\left[\frac{-y\underbrace{\frac{\partial}{\partial x}(x^2+y^2)}_{(2x)}}{(x^2+y^2)^2}\right]$$

$$= -\left[\frac{-y \cdot 2x}{(x^2+y^2)^2}\right]$$

$$\frac{\partial^2 \theta}{\partial x^2} = \left[\frac{2xy}{(x^2+y^2)^2}\right] \qquad \ldots(A)$$

and $\because \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Differenate θ, w.r.t. y, keeping x as constant

$$\frac{\partial \theta}{\partial y} = \underbrace{\frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)}_{\frac{1}{1+\left(\frac{y}{x}\right)^2}} \cdot \underbrace{\frac{\partial}{\partial y}\left(\frac{y}{x}\right)}_{\frac{1}{x}}$$

$$\frac{\partial \theta}{\partial y} = \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} = \left(\frac{x^{\cancel{2}}}{x^2+y^2}\right)\cdot\frac{1}{\cancel{x}}$$
$$= \frac{x}{x^2+y^2}$$

Again differentate $\frac{\partial \theta}{\partial y}$ w.r.t. y, keeping x as constant,

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial \theta}{\partial y}\right] = \frac{\partial}{\partial y}\left[\frac{x}{x^2+y^2}\right]$$

$$= \frac{-x \cdot \underbrace{\frac{\partial}{\partial y}(x^2+y^2)}_{(2y)}}{(x^2+y^2)^2} = \frac{-x \cdot 2y}{(x^2+y^2)^2}$$

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2} \qquad \ldots(B)$$

Step V : $\therefore$ Adding Equations (A) and (B), we get

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \left[\frac{\cancel{2xy}}{\cancel{(x^2+y^2)^2}}\right] + \left[\frac{\cancel{-2xy}}{\cancel{(x^2+y^2)^2}}\right] = 0 \checkmark \qquad \ldots\textbf{Ans.}$$

Example 4.8.10

If $u = mx + ny$; $v = nx - my$; then find the value of

$$\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial y}{\partial v}\right)_x \cdot \left(\frac{\partial x}{\partial u}\right)_v \cdot \left(\frac{\partial v}{\partial y}\right)_u$$

Solution :

Step I : Given : $u = mx + ny \qquad \ldots(1)$

$v = nx - my \qquad \ldots(2)$

We have to find

$y \to f(x, v)$; $v \to f(y, u)$

$u \to f(x, y)$; $x \to f(u, v)$

Step II : From Equation (1), differentiate Equation (1) w.r.t. x, keeping y as constant.

$$\left(\frac{\partial u}{\partial x}\right)_y = \underbrace{\frac{\partial}{\partial x}(mx + ny)}_{m}$$

$$\left(\frac{\partial u}{\partial x}\right)_y = m \qquad \ldots(A)$$

Step III : To find $y \to f(x, v)$; from Equation (2)

$$y = \frac{nx - v}{m}$$

Differentiate y w.r.t. v, keeping x as constant

$$\left(\frac{\partial y}{\partial v}\right)_x = \underbrace{\frac{\partial}{\partial v}\left(\frac{nx-v}{m}\right)}_{\frac{-1}{m}}$$

$$\Rightarrow \left(\frac{\partial y}{\partial v}\right)_x = \left(-\frac{1}{m}\right) \quad \text{...(B)}$$

Step IV : To find $v \rightarrow f(y, u)$;

$\therefore$ From Equation (1) ;

$$x = \frac{u-ny}{m} \quad \text{put in Equation (2)}$$

$$v = n\left[\frac{u-ny}{m}\right] - my$$

Differentiate v w.r.t. y, keeping u as constant

$$\left(\frac{\partial v}{\partial y}\right)_u = \frac{n}{m}\cdot\underbrace{\frac{\partial}{\partial y}(u-ny)}_{(-n)} - m\cdot\underbrace{\frac{\partial}{\partial y}(y)}_{1}$$

$$\therefore \left(\frac{\partial v}{\partial y}\right)_u = \frac{n}{m}[-n] - m = -\frac{n^2}{m} - m$$

$$\left(\frac{\partial v}{\partial y}\right)_u = -\left[\frac{(m^2+n^2)}{m}\right] \quad \text{...(C)}$$

Step V : To find $x \rightarrow f(u, v)$

Eliminate y from Equations (1) and (2)

Therefore,

$m \times$ Equation (1) + $n \times$ Equation (2)

$$\begin{aligned} mu &= m^2x + mny \\ nv &= n^2x - mny \\ \Rightarrow mu + nv &= m^2 x + \cancel{mny} + n^2 x - \cancel{mny} \\ mu + nv &= (m^2+n^2)\cdot x \\ x &= \frac{mu+nv}{m^2+n^2} \end{aligned}$$

Differentiate x w.r.t. y, keeping v as constant

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{m^2+n^2}\cdot\underbrace{\frac{\partial}{\partial u}(mu+nv)}_{m}$$

$$\therefore \left(\frac{\partial x}{\partial u}\right)_v = \frac{m}{m^2+n^2} \quad \text{...(D)}$$

Step VI : $\therefore$ From Equations (A), (B), (C) and (D), we get

$$\text{Hence, } \left(\frac{\partial u}{\partial x}\right)_y\cdot\left(\frac{\partial y}{\partial v}\right)_x\cdot\left(\frac{\partial x}{\partial u}\right)_v\cdot\left(\frac{\partial v}{\partial y}\right)_u$$

$$= (\cancel{m})\cdot\left(\frac{-1}{\cancel{m}}\right)\cdot\left(\frac{\cancel{m}}{m^2+\cancel{n^2}}\right)\cdot\left(\frac{-(m^2+\cancel{n^2})}{\cancel{m}}\right) = 1 \checkmark \quad \text{...Ans.}$$

Exercise 2

Ex. 1 If $u = x^2 + 2y^2$; $x = r\cos\theta$; $y = r\sin\theta$, then find $\left(\frac{\partial u}{\partial x}\right)_y ; \left(\frac{\partial u}{\partial y}\right)_\theta ; \left(\frac{\partial u}{\partial \theta}\right)_r$

Ex. 2 If $u = lx + my$; $v = mx - ly$ then show that

(i) $\left(\frac{\partial u}{\partial x}\right)_y\cdot\left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2+m^2}$

(i) $\left(\frac{\partial y}{\partial v}\right)_x\cdot\left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2+m^2}{l^2}$

4.9 Composite Function (Function of Function)

If $u = f(x, y)$ and $x = \phi_1(r, \theta)$; $y = \phi_2(r, \theta)$ then u is called composite function of independent variable r and θ.

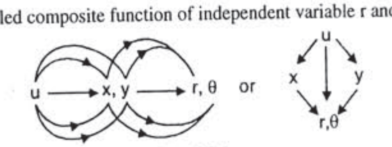


Fig. 4.9.1

Chain rule for partial derivative

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} ;$$

$$\text{and } \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} ;$$

Similarly, If $u = f(x, y, z)$ and $x = \phi_1(s,t)$, $y = \phi_2(s,t)$, $z = \phi_3(s,t)$ then u is a composite function of independent variable s and t.

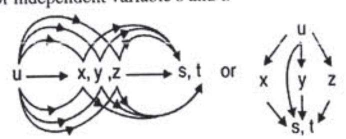


Fig. 4.9.2

Chain rule for partial derivative

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$\text{and } \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}$$

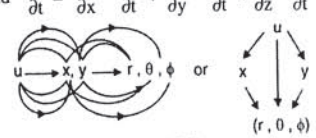


Fig. 4.9.3

Also, If $u = f(x, y)$ and x, y are functions of three variables r, θ, φ then u is a composite function of three variables r, θ, φ.

Chain rule

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$\text{and } \frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \phi}$$

Also, we can find the chain rule for composite function of several variables.

1. Implicit Function

The function of the from $f(x, y) = c$ or $f(x, y) = 0$ is called an implicit function, where c is any arbitrary constant. In implicit function $f(x, y) = c$, y is a function of x, and $\frac{\partial f}{\partial y} \neq 0$

Therefore,

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

But, $f(x, y) = c \Rightarrow \frac{df}{dx} = 0$

$$\therefore \quad 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

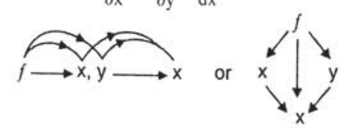


Fig. 4.9.4

$$\text{or} \quad \frac{dy}{dx} = -\left[\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right] = -\left[\frac{f_x}{f_y}\right] \quad ; \quad f_y \neq 0$$

2. Explicit function

If one variable can be expressed in terms of other variable or variables then that function is called explicit function.

$y = f(x)$ is an explicit function.

For implicit function,

$$\frac{dy}{dx} = -\frac{p}{q} \quad ; \text{ where } p = \frac{\partial f}{\partial x} \; ; \; q = \frac{\partial f}{\partial y}$$

$$r = \frac{\partial^2 f}{\partial x^2} \; ; \; s = \frac{\partial^2 f}{\partial y\,\partial x} \; ; \; t = \frac{\partial^2 f}{\partial y^2}$$

Differentiating $\frac{dy}{dx}$ w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \left[\frac{q^2r - 2pqs + p^2t}{q^3}\right]$$

$$= \frac{1}{q^3}\begin{vmatrix} r & s & p \\ s & t & q \\ p & q & 0 \end{vmatrix} \; ; \; q^3 \neq 0$$

Example 4.9.1

If $x = r\cos\theta$; $y = r\sin\theta$ then show that $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$.

Solution :

Step I : Given, $x = r\cos\theta$; $y = r\sin\theta$

Squaring x and y and then add we get,

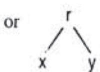
$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1}$$

$$x^2 + y^2 = r^2$$

$$r = \sqrt{x^2+y^2}$$

Here, r is a function of x and y ...(1)

$r \rightarrow x, y$ or



Step II : Differentiate r w.r.t. x, keeping y as constant

➔ **Using standard result :**

$$\left[d(\sqrt{f(x)}) = \frac{1}{2\sqrt{f(x)}}\cdot\frac{\partial}{\partial x}(f(x)) \right]$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(\sqrt{x^2+y^2})\cdot\underbrace{\frac{\partial}{\partial x}(x^2+y^2)}_{(2x)}$$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}\cdot\cancel{2}x = \frac{x}{r}$$

Squaring, we get

$$\left(\frac{\partial r}{\partial x}\right)^2 = \frac{x^2}{r^2}$$

Step III : Similarly, $\left(\frac{\partial r}{\partial y}\right)^2 = \frac{y^2}{r^2}$

Hence, by adding we get,

$$\therefore \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2+y^2}{r^2}$$

$$= \frac{\cancel{r^2}}{\cancel{r^2}} = 1\checkmark \quad \text{...Ans.}$$

Example 4.9.2

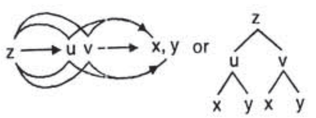
If $z = f(u,v)$ and $u = x\cos t - y\sin t$, $v = x\sin t + y\cos t$; where, t is a constant, then show that : $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$

Solution :

Step I : Given : $z = f(u,v)$

$$u = x\cos t - y\sin t \quad ...(1)$$

$$v = x\sin t + y\cos t \quad ...(2)$$



Step II : Differentiate z w.r.t. x, keeping y as constant by chain rule.

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \quad ...(3)$$

Similarly, $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \quad ...(4)$$

From Equation (1), differentiate u w.r.t. x keeping t as constnat

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\underbrace{[x\cos t - u\sin t]}_{(\cos t)}$$

and $$\frac{\partial u}{\partial y} = \frac{\partial}{\partial T}\underbrace{[x\cos t - y\sin t]}_{(-\sin t)}$$

$$\frac{\partial u}{\partial x} = \cos t; \quad \text{and} \quad \frac{\partial u}{\partial y} = -\sin t$$

From Equation (2), differentiate v w.r.t. x and y keeping t as constant

$$\frac{\partial v}{\partial x} = \sin t; \quad \text{and} \quad \frac{\partial v}{\partial y} = \cos t$$

Step III : $\therefore$ From Equations (3) and (4)

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}(\cos t) + \frac{\partial z}{\partial v}(\sin t) \quad ...(5)$$

and $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot(-\sin t) + \frac{\partial z}{\partial v}(\cos t) \quad ...(6)$$

Step IV : From Equations (5) and (6)

$x \times$ Equation (5) + $y \times$ Equation (6), we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \left[x\cdot\cos t\frac{\partial z}{\partial u} + x\cdot\sin t\frac{\partial z}{\partial v}\right] + \left[-y\sin t\frac{\partial z}{\partial u} + y\cdot\cos t\frac{\partial z}{\partial v}\right]$$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\underbrace{[x\cos t - y\sin t]}_{u} + \frac{\partial z}{\partial v}\underbrace{(x\sin t + y\cos t)}_{v}$$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} \checkmark \quad \text{...Ans.}$$

...From Equations (1) and (2)

Example 4.9.3

If $x = \cos\theta - r\sin\theta$, $y = \sin\theta + r\cos\theta$, then show that $\frac{\partial r}{\partial x} = \frac{x}{r}$.

Solution :

Step I : Given : $x = \cos\theta - r\sin\theta$;

$$y = \sin\theta + r\cos\theta$$

Squaring and adding,

$$x^2 + y^2 = (\cos\theta - r\sin\theta)^2 + (\sin\theta + r\cos\theta)^2$$

$$= \cos^2\theta + r^2\sin^2\theta - 2r\sin\cancel{\theta}\cos\theta$$

$$+ \sin^2\theta + r^2\cos^2\theta + 2r\sin\cancel{\theta}\cos\theta$$

$$= \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + r^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} = 1 + r^2(1)$$

$$\therefore x^2 + y^2 = 1 + r^2$$

$$\Rightarrow r^2 = x^2 + y^2 - 1 \quad ...(1)$$

Step II : Differentiate Equation (1) partially w.r.t. x keeping y as constnat

$$\cancel{2}r\frac{\partial r}{\partial x} = \underbrace{\frac{\partial}{\partial x}(x^2 + y^2)}_{2x} = \cancel{2}x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$= \frac{x}{r} \checkmark \quad \text{...Ans.}$$

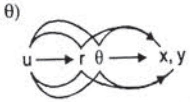
Example 4.9.4

If $u = \frac{1}{r}f(\theta)$, $x = r\cos\theta$; $y = r\sin\theta$ then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

Solution :

Given, u is a function (r, θ)

Step I : Given : $u = \frac{1}{r}f(\theta)$



$$x = r\cos\theta;$$

$$y = r\sin\theta$$

$$x^2 + y^2 = r^2 \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad ...(1)$$

Step II : $\therefore$ By chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} \quad ...(2)$$

and $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} \quad ...(3)$$

$$\because u = \frac{1}{r}f(\theta)$$

Differentiate u w.r.t. r, keeping θ as constant

$$\frac{\partial u}{\partial r} = f(\theta)\cdot\underbrace{\frac{\partial}{\partial x}\left(\frac{1}{r}\right)}_{-1/r^2}$$

$$\frac{\partial u}{\partial r} = -\frac{1}{r^2}f(\theta) \quad ...(A)$$

Differentite u w.r.t. θ, keeping r as constant

$$\frac{\partial u}{\partial \theta} = \frac{f'(\theta)}{r}$$

Step III : But differentiate Equation (r^2) w.r.t. x keeping y as constant

$$2r\frac{\partial r}{\partial x} = \underbrace{\frac{\partial}{\partial x}(x^2 + y^2)}_{2x}$$

$$2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

And differentiating Equation (r^2) w.r.t. y, keeping x as constant

$$2r\frac{\partial r}{\partial y} = \underbrace{\frac{\partial}{\partial y}(x^2 + y^2)}_{2y}$$

$$2r\frac{\partial r}{\partial y} = 2y \quad \Rightarrow \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

Differentiate θ w.r.t. x, keeping y as constant

→ Using standard result :

$$\left[\frac{d}{dx}\tan^{-1}(f(x)) = \frac{1}{1 + [f(x)]^2}f'(x)\right]$$

$$\frac{\partial \theta}{\partial x} = \underbrace{\frac{\partial}{\partial x}\tan^{-1}\left(\frac{y}{x}\right)}_{\frac{1}{1+\left(\frac{y}{x}\right)^2}}\cdot\underbrace{\frac{\partial}{\partial x}\left(\frac{y}{x}\right)}_{\left(-\frac{y}{x^2}\right)}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{(x^2 + y^2)}$$

Differentiate θ w.r.t. y, keeping x as constant

$$\frac{\partial \theta}{\partial y} = \underbrace{\frac{\partial}{\partial y}\tan^{-1}\left(\frac{y}{x}\right)}_{\frac{1}{1+\left(\frac{y}{x}\right)^2}}\cdot\underbrace{\frac{\partial}{\partial x}\left(\frac{y}{x}\right)}_{\left(\frac{1}{x}\right)}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{(x^2 + y^2)}$$

Step IV : Now, Equations (2) and (3) becomes

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{x}{r}\right) + \frac{\partial u}{\partial \theta}\left(-\frac{y}{x^2 + y^2}\right) \quad ...(4)$$

and $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{y}{r}\right) + \frac{\partial u}{\partial \theta}\left(\frac{x}{x^2 + y^2}\right) \quad ...(5)$$

Step V : By $x \times$ Equation (4) + $y \times$ Equation (5), we get,

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= x\left[\frac{x}{r}\frac{\partial u}{\partial r} - \frac{y}{x^2 + y^2}\cdot\frac{\partial u}{\partial \theta}\right] + y\left[\frac{y}{r}\frac{\partial u}{\partial r} + \frac{x}{x^2 + y^2}\cdot\frac{\partial u}{\partial \theta}\right]$$

$$= \frac{\partial u}{\partial r}\left[\frac{x^2}{r} + \frac{y^2}{r}\right] + \frac{\partial u}{\partial \theta}\left[\frac{-\cancel{xy}}{\cancel{x^2 + y^2}} + \frac{\cancel{xy}}{\cancel{x^2 + y^2}}\right]$$

$$= \frac{\partial u}{\partial r}\left(\frac{r^{\cancel{2}}}{\cancel{r}}\right) + 0$$

$$= r\frac{\partial u}{\partial r} = r\left(-\frac{\cancel{1}}{r^{\cancel{2}}}f(\theta)\right) = -\frac{1}{r}f(\theta) \quad \text{...from (A)}$$

$$= -u \checkmark \quad \text{...Ans.}$$

Example 4.9.5

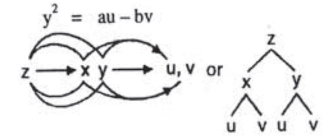
If $z = f(x, y)$ where, $x^2 = au + bv$; $y^2 = au - bv$ then prove that $u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = \frac{1}{2}\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right]$

Solution :

Step I : Given : $z = f(x, y)$

$$x^2 = au + bv \quad ...(1)$$

$$y^2 = au - bv \quad ...(2)$$



Differentiate z w.r.t. u and v, by chain rule

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad ...(3)$$

and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$ ...(4)

Step II : Differentiate (1) and (2) w.r.t u and v

$$2x\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\underbrace{(au + bv)}_{a} = a \quad \Rightarrow \frac{\partial x}{\partial u} = \frac{a}{2x}$$

$$2x\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\underbrace{(au + bv)}_{b} = b \quad \Rightarrow \frac{\partial x}{\partial v} = \frac{b}{2x}$$

$$\text{and } 2y\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\underbrace{(au - bv)}_{a} = a \quad \Rightarrow \frac{\partial y}{\partial u} = \frac{a}{2y}$$

$$2y\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\underbrace{(au - bv)}_{-b} = -b \quad \Rightarrow \frac{\partial y}{\partial u} = \frac{-b}{2y}$$

Step III : Equation (3) becomes,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\left(\frac{a}{2x}\right) + \frac{\partial z}{\partial y}\left(\frac{a}{2y}\right)$$

Multiplying both sides by u

$$\therefore u\frac{\partial z}{\partial u} = \frac{au}{2x}\frac{\partial z}{\partial x} + \frac{au}{2y}\frac{\partial z}{\partial y} \quad ...(5)$$

and Equation (4) becomes

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(\frac{b}{2x}\right) + \frac{\partial z}{\partial y}\left(-\frac{b}{2y}\right)$$

Multiplying both sides by v

$$\therefore v\frac{\partial z}{\partial v} = \frac{bv}{2x}\left(\frac{\partial z}{\partial x}\right) - \frac{bv}{2y}\left(\frac{\partial z}{\partial y}\right) \quad ...(6)$$

Step IV : Adding Equations (5) and (6),

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = \left[\frac{au}{2x}\cdot\frac{\partial z}{\partial x} + \frac{au}{2y}\cdot\frac{\partial z}{\partial y}\right] + \left[\frac{bv}{2x}\cdot\frac{\partial z}{\partial x} - \frac{bu}{2y}\cdot\frac{\partial z}{\partial y}\right]$$

$$= \left(\frac{au + bv}{2x}\right)\frac{\partial z}{\partial x} + \left(\frac{au - bv}{2y}\right)\frac{\partial z}{\partial y}$$

$$= \frac{1}{2}\left[\frac{x^2}{\cancel{x}}\frac{\partial z}{\partial x} + \frac{y^2}{\cancel{y}}\frac{\partial z}{\partial y}\right] \quad \text{...(By using (1) and (2))}$$

$$= \frac{1}{2}\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right]$$

L.H.S = RHS ✓ ...Ans.

Example 4.9.6

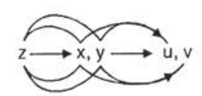
If $z = f(x, y)$ where, $x = e^u + e^{-v}$; $y = e^{-u} - e^v$, then prove that ; $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Solution :

Step I : Given : $z = f(x\ y)$

$$x = e^u + e^{-v} \quad ...(1)$$

$$y = e^{-u} - e^v \quad ...(2)$$



Step II : Differentiate z partially by chain rule,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad ...(3)$$

and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$ ...(4)

Step III : Differentiate Equation (1) and (2), we get

➔ **Using standard result**

$$...\left[\frac{d}{dx}(e^{f(x)}) = e^{f(x)}\cdot\frac{d}{dx}f(x)\right]$$

$$...\left[\frac{d}{dx}e^{ax} = e^{a(x)}\cdot a\right]$$

$$\frac{\partial x}{\partial u} = e^u \ ; \quad \frac{\partial x}{\partial v} = -e^{-v}$$

$$\frac{\partial y}{\partial u} = -e^{-u} \ ; \quad \frac{\partial y}{\partial v} = -e^v$$

Step IV : $\therefore$ Equations (3) and (4) becomes

$$\frac{\partial z}{\partial u} = e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} \quad ...(5)$$

and $\frac{\partial z}{\partial v} = -e^{-v}\frac{\partial z}{\partial x} - e^v\frac{\partial z}{\partial y}$ ...(6)

Step V : By subtracting Equations (5) and (6)

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \left[e^u\cdot\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y}\right] - \left[e^{-v}\cdot\frac{\partial z}{\partial x} - e^v\frac{\partial z}{\partial y}\right]$$

$$= \underbrace{(e^u + e^{-v})}_{x}\frac{\partial z}{\partial x} - \underbrace{(e^{-u} - e^v)}_{y}\frac{\partial z}{\partial y}$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

... By using Equation (1) and (2)

$$\therefore \quad \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} \checkmark \quad \text{...Ans.}$$

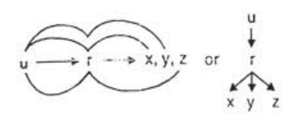
Example 4.9.7

If $u = f(r)$ and $r = \sqrt{x^2 + y^2 + z^2}$ then prove that :

$u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r}f'(r)$

Solution :

Step I : Given : $u = f(r)$ and $r = \sqrt{x^2 + y^2 + z^2}$



Step II : Differentiate u partially w.r.t. x

$$\frac{\partial u}{\partial x} = \frac{du}{dr}\cdot\frac{\partial r}{\partial x}$$

Given, $r = \sqrt{x^2 + y^2 + z^2}$

Differentiate r w.r.t. x keeping y and z as constant.

➔ **Using standard, rule :**

$$...\left[\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}}\cdot\frac{d}{dx}\cdot f(x)\right]$$

Here, $f(x) = x^2 + y^2 + z^2$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}}\cdot\underbrace{\frac{\partial}{\partial x}(x^2 + y^2 + z^2)}_{2x}$$

$$\frac{\partial r}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

Similarly, we get,

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{and} \quad \frac{\partial y}{\partial z} = \frac{z}{r}$$

Differentiate u w.r.t x, keeping y and z as constant

$$\therefore \frac{\partial u}{\partial x} = \frac{d}{dr}f(r)\cdot\frac{\partial r}{\partial x} = f'(r)\cdot\frac{x}{r}$$

Step III : Again, differentiating $\frac{\partial u}{\partial x}$ w.r.t. x, remember that r is a function of x, y, z.

➔ **Using standard rule :**

$$...\left[\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\frac{d}{dx}g(x) + g(x)\cdot\frac{d}{dx}f(x)\right]$$

Or

$$...\left[\frac{d}{dx}[I\cdot II] = I\cdot\frac{d}{dx}(II) + II\cdot\frac{d}{dx}(I)\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\underbrace{f'(r)}_{I}\underbrace{\frac{x}{r}}_{II}\right] = f'(r)\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \underbrace{\frac{x}{r}\cdot\frac{\partial}{\partial x}f'(r)\frac{\partial r}{\partial x}}_{f''(r)}$$

$$= f'(r)\frac{\partial}{\partial x}\left[\frac{x}{r}\right] + \frac{x}{r}f''(r)\frac{\partial r}{\partial x}$$

➔ **Using standard rule :**

$$...\left[\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[g(x)\cdot\frac{d}{dx}f(x) - f(x)\cdot\frac{d}{dx}g(x)\right]}{(g(x))^2}\right]$$

$$= f'(r)\left[\frac{r\cdot 1 - x\frac{\partial r}{\partial x}}{r^2}\right] + \frac{x}{r}\left(\frac{x}{r}\right)f''(r) \quad \left(\because \frac{\partial r}{\partial x} = \frac{x}{r}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r} - \frac{x}{r^2}f'(r)\frac{x}{r} + \frac{x^2}{r^2}f''(r)$$

$$= f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r} - \frac{x^2}{r^3}f'(r)$$

Similarly, we get

$$\frac{\partial^2 u}{\partial y^2} = f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r} - \frac{y^2}{r^3}f'(r)$$

and $\frac{\partial^2 u}{\partial z^2} = f''(r)\frac{z^2}{r^2} + \frac{f'(r)}{r} - \frac{z^2}{r^3}f'(r)$

Step IV : Adding the above equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \left[f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r} - \frac{x^2}{r^3}f'(r)\right] + \left[f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r} - \frac{y^2}{r^3}f'(r)\right] + \left[f''(r)\frac{z^2}{r^2} + \frac{f'(r)}{r} - \frac{z^2}{r^3}f'(r)\right]$$

$$= \frac{f''(r)}{r^2}(x^2 + y^2 + z^2) + \frac{3}{r}f'(r) - \frac{(x^2 + y^2 + z^2)f'(r)}{r^3}$$

$$= f''(r)\frac{\cancel{r^2}}{\cancel{r^2}} + \frac{3f'(r)}{r} - \frac{\cancel{r^2}}{\cancel{r^3}}f'(r)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \underbrace{\left(\frac{3}{r} - \frac{1}{r}\right)}_{\frac{2}{r}}f'(r)$$

$$= f''(r) + \frac{2}{r}f'(r) \checkmark \quad \text{...Ans.}$$

Example 4.9.8

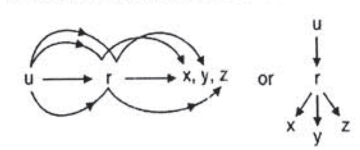
If $u^2(x^2 + y^2 + z^2) = 1$; prove that $u_{xx} + u_{yy} + u_{zz} = 0$, given, $x^2 + y^2 + z^2 = r^2$

Solution :

Step I : Given : $u^2r^2 = 1$; $u^2 = r^{-2}$ ...(1)

and $r^2 = x^2 + y^2 + z^2$ ...(2)

Here, u is a function of r and r is a function of x, y, z.



Step II : Differentiate Equation (1) partially w.r.t. x, keeping y constant and r is a function of x, y, z.

→ **Using standard result :**

$$\ldots\left[\frac{d}{dx}[f(x)]^n = n\, f(x)^{n-1}\cdot\frac{d}{dx}f(x)\right]$$

$$\therefore\ \cancel{2}u\,\frac{\partial u}{\partial x} = -\cancel{2}\,r^{-3}\underbrace{\frac{\partial r}{\partial x}}_{x/r} \qquad \text{by } r=\sqrt{x^2+y^2+z^2}$$

$$\frac{\partial r}{\partial x} = \frac{1\cdot\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

Similary,

$$\frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{y}{r}$$

$$\frac{\partial u}{\partial x} = -\frac{r^{-3}}{u}\left(\frac{x}{r}\right)$$

$$= -\frac{x}{u\,r^4}$$

$$= -\left(\frac{x}{u\,r^2\cdot r^2}\right)$$

$$\therefore\ \frac{\partial u}{\partial x} = -\frac{x}{\cancel{u}\cdot\frac{1}{u}\frac{1}{u^2}} = -x\,u^3 \qquad \ldots \text{by using (1)}$$

Differentiate $\frac{\partial u}{\partial x}$ w.r.t. x, rembember u is a function of r and then x, y, z.

→ **Using standard result**

$$\ldots\left[\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\frac{d}{dx}g(x) + g(x)\cdot\frac{d}{dx}f(x)\right]$$

Or

$$\ldots\left[\frac{d}{dx}[\mathrm{I}\cdot\mathrm{II}] = \mathrm{I}\cdot\frac{d}{dx}(\mathrm{II}) + \mathrm{II}\cdot\frac{d}{dx}(\mathrm{I})\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(-x\,u^3)$$

$$= -\left[u^3\cdot\underbrace{\frac{\partial}{\partial x}(x)}_{1} + x\cdot\underbrace{\frac{\partial}{\partial x}u^3}_{3u^2}\cdot\frac{\partial u}{\partial x}\right]$$

$$= -\left[u^3\cdot 1 + 3x\,u^2\cdot\frac{\partial u}{\partial x}\right] = -\left[u^3 + 3x\,u^2\cdot(-x\,u^3)\right]$$

$$\frac{\partial^2 u}{\partial x^2} = -\left[u^3 - 3x^2u^5\right] \qquad \ldots(3)$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = -\left[u^3 - 3\,y^2u^5\right] \qquad \ldots(4)$$

and $$\frac{\partial^2 u}{\partial z^2} = -\left[u^3 - 3z^2u^5\right] \qquad \ldots(5)$$

Step III : Adding Equations (3), (4), (5)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = -\left[u^3-3x^2u^5\right] - \left[u^3-3y^2u^5\right] - \left[u^3-3z^2u^5\right]$$

$$= -\left[3u^3 - 3u^5\underbrace{(x^2+y^2+z^2)}_{r^2}\right]$$

$$= -\left[3u^3 - 3u^5(r^2)\right]$$

$$= -\left[3u^3 - 3u^5\cdot\frac{1}{u^2}\right]$$

$$= -\left[3\cancel{u^3} - 3\cancel{u^3}\right] \ldots \text{From Equation (1)}$$

$$\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 0 \checkmark \qquad \ldots\textbf{Ans.}$$

Example 4.9.9

If $u = \log\left(\sqrt{x^2+y^2+z^2}\right)$ show that $(x^2+y^2+z^2)\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right] = 1;$

Solution :

Step I : Given : $u = \log\left(\sqrt{x^2+y^2+z^2}\right)$

Here, u is function of x, y, z

This means $u \to f(x, y, z)$ ...(1)

Step II : Differentiate Equation (1) partially w.r.t. x, keeping y and z constant

→ **Using standard results :**

$$\ldots\left[\frac{d}{dx}\log\left(\sqrt{f(x)}\right) = \frac{1}{\sqrt{f(x)}}\cdot\frac{d}{dx}\sqrt{f(x)}\cdot\frac{d}{dx}f(x)\right]$$

$$\left[\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}}\cdot\frac{d}{dx}\cdot f(x)\right]$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2+z^2}}\cdot\underbrace{\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}}_{\frac{1}{2\sqrt{x^2+y^2+z^2}}}\cdot\underbrace{\frac{\partial}{\partial x}(x^2+y^2+z^2)}_{2x}$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\cdot\frac{1\cdot\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2+z^2}$$

Again differentiate $\frac{\partial u}{\partial x}$ w.r.t. x.

→ **Using standard result :**

$$\ldots\left[\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[g(x)\cdot\frac{d}{dx}f(x) - f(x)\cdot\frac{d}{dx}g(x)\right]}{(g(x))^2}\right]$$

Or

$$\ldots\left[\frac{d}{dx}\left[\frac{\mathrm{I}}{\mathrm{II}}\right] = \frac{\left[\mathrm{II}\cdot\frac{d}{dx}(\mathrm{I}) - (\mathrm{I})\cdot\frac{d}{dx}(\mathrm{II})\right]}{(\mathrm{II})^2}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)$$

$$= \frac{(x^2+y^2+z^2)\cdot\underbrace{\frac{\partial(x)}{\partial x}}_{(1)} - (x)\underbrace{\frac{\partial}{\partial x}(x^2+y^2+z^2)}_{(2x)}}{(x^2+y^2+z^2)^2}$$

$$= \left[\frac{(x^2+y^2+z^2)\cdot 1 - x\,(2x)}{(x^2+y^2+z^2)^2}\right] = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$$

Multiplying both sides by $(x^2+y^2+z^2)$

$$\therefore\ (x^2+y^2+z^2)\frac{\partial^2 u}{\partial x^2} = (x^2+\cancel{y^2}+z^2)\cdot\frac{y^2+z^2-x^2}{(x^2+y^2+z^2)\cancel{^2}}$$

$$= \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)} \qquad \ldots(A)$$

Similarly, we get

$$(x^2+y^2+z^2)\frac{\partial^2 u}{\partial y^2} = \frac{z^2+x^2-y^2}{(x^2+y^2+z^2)} \qquad \ldots(B)$$

and $$(x^2+y^2+z^2)\frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)} \qquad \ldots(C)$$

Step III : Adding Equations (A) , (B) and (C), we get

$$\therefore\ (x^2+y^2+z^2)\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

$$= \left(\frac{y^2+z^2-x^2}{x^2+y^2-z^2}\right) + \left(\frac{z^2+x^2-y^2}{x^2+y^2+z^2}\right) + \left(\frac{x^2+y^2-z^2}{x^2+y^2+z^2}\right)$$

$$= \frac{[y^2+z^2+-\cancel{y^2}+\cancel{z^2}+\cancel{x^2}+-\cancel{y^2}+x^2+\cancel{y^2}-\cancel{z^2}]}{(x^2+y^2+z^2)}$$

$$= \frac{x^2+y^2+\cancel{z^2}}{\cancel{x^2}+y^2+z^2} = 1$$

$$\Rightarrow \text{Hence, } (x^2+y^2+z^2)\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right] = 1\checkmark \qquad \ldots\textbf{Ans.}$$

Example 4.9.10

If $\theta = e^{r-x}$ and $r = \sqrt{x^2+y^2}$ then prove that $\frac{\partial^2\theta}{\partial y^2} = \left(\frac{x^2}{r^3}+\frac{y^2}{r^2}\right)e^{r-x}$.

Solution :

Step I : Given : $\theta = e^{r-x}$ ...(1)

and $r = \sqrt{x^2+y^2}$ ...(2)

$\therefore\ r^2 = x^2+y^2$

Here, θ is a function of x and r. Also r is a function of x and y

Since, $\theta \to r, x$; r → x, y or θ → x, r ; r → x, y

Differentiate r^2 w.r.t. x and y

$$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ and } \frac{\partial r}{\partial y} = \frac{y}{r}$$

Step II : Differentiate θ w.r.t. y, keeping x as constant and note that r is a function of x and y.

→ **Using standard rule :**

$$\ldots\left[\frac{d}{dx}(e^{f(x)}) = e^{f(x)}\cdot\frac{d}{dx}f(x)\right]$$

$$\ldots\left[\frac{d}{dx}e^{ax} = e^{a(x)}\cdot a\right]$$

$$\frac{\partial\theta}{\partial y} = \underbrace{\frac{\partial}{\partial y}\cdot e^{(r-x)}}_{e^{(r-x)}}\cdot\underbrace{\frac{\partial}{\partial y}(r-x)}_{\frac{\partial r}{\partial y}}$$

$$= e^{r-x}\underbrace{\left[\frac{\partial r}{\partial y}\right]}_{(y/r)} = e^{r-x}\left[\frac{y}{r}\right]$$

Again differentiating $\frac{\partial\theta}{\partial y}$ w.r.t. y, keeping x as constant and note that r is a function of x, y

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\partial}{\partial y}\cdot\frac{\partial\theta}{\partial y} = \frac{\partial}{\partial y}\left(\underbrace{e^{r-x}\cdot\frac{y}{r}}_{\mathrm{I}\ \ \mathrm{II}}\right)$$

$$= e^{(r-x)}\cdot\frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \left(\frac{y}{r}\right)\cdot\underbrace{\frac{\partial}{\partial y}e^{r-x}}_{e^{r-x}}\cdot\underbrace{\frac{\partial}{\partial y}(r-x)}_{\frac{\partial r}{\partial y}}$$

$$= e^{r-x}\cdot\frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{y}{r}\,e^{r-x}\cdot\underbrace{\left(\frac{\partial r}{\partial y}\right)}_{\left(\frac{y}{r}\right)}$$

$$= e^{r-x}\left[\frac{r\cdot 1 - y\frac{\partial r}{\partial y}}{r^2}\right] + \frac{y}{r}\left(\frac{y}{r}\right)e^{r-x}$$

$$\left(\ldots\because\ \frac{\partial}{\partial y}\left(\frac{y}{r}\right) = \frac{r\cdot\underbrace{\frac{\partial y}{\partial y}}_{(1)} - y\cdot\underbrace{\frac{\partial r}{\partial y}}_{(y/r)}}{r^2}\right)$$

$$= e^{r-x}\cdot\left[\frac{r-y\frac{y}{r}}{r^2}\right] + \frac{y^2}{r^2}e^{r-x} = e^{r-x}\left\{\frac{r}{r^2}-\frac{y^2}{r^3}+\frac{y^2}{r^2}\right\}$$

$$= e^{r-x}\left\{\frac{r^2-y^2}{r^3}+\frac{y^2}{r^2}\right\}$$

$$\frac{\partial^2\theta}{\partial y^2} = e^{r-x}\left\{\frac{x^2}{r^3}+\frac{y^2}{r^2}\right\} ✓$$

...from Equation (2)

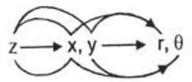
...Ans.

Example 4.9.11

If $z = f(x, y)$ where $x = r\cos\theta$; $y = r\sin\theta$ then shown that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

Solution :

Step I : Given : $z = f(x, y)$ and $x = r\cos\theta$; $y = r\sin\theta$



By chain rule, differentiate z partially, w.r.t. r and θ.

Step II : $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} \quad ...(A)$$

and $$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta} \quad ...(B)$$

Since, $x = r\cos\theta$; $y = r\sin\theta$

Differentiating x and y w.r.t r and θ, we get

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

Equation (A) and (B) becomes

$$\therefore \quad \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}(\cos\theta)+\frac{\partial z}{\partial y}(\sin\theta) \quad ...(1)$$

and $$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}(r\cos\theta) \quad ...(2)$$

Step III : Now squaring Equation (1),

➔ **Using standard formula** $\quad ...[(a+b)^2 = a^2+b^2+2ab]$

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)(\cos\theta)+\left(\frac{\partial z}{\partial y}\right)(\sin\theta)\right]^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta+\left(\frac{\partial z}{\partial y}\right)^2\cdot\sin^2\theta$$

$$+2\sin\theta\cos\theta\cdot\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) \quad ...(3)$$

And squaring Equation (2),

➔ **Using standard formula** $\quad ...[(-a+b)^2 = a^2+b^2-2ab]$

$$\left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)(-r\sin\theta)+\left(\frac{\partial z}{\partial y}\right)(r\cos\theta)\right]^2$$

$$= r^2\sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2+r^2\cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2$$

$$-2r^2\sin\theta\cos\theta\ \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}$$

Multiplying both sides by $\frac{1}{r^2}$

$$\therefore \quad \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \frac{1}{r^2}\left[\cancel{r^2}\sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2+\cancel{r^2}\cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2\right.$$

$$\left.-2\cancel{r^2}\sin\theta\cdot\cos\theta\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\right]$$

$$= \sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2+\cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2$$

$$-2\sin\theta\cos\theta\left(\frac{\partial z}{\partial x}\right)\cdot\left(\frac{\partial z}{\partial y}\right) \quad ...(4)$$

Step IV : Adding Equations (3) and (4),

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\underbrace{[\sin^2\theta+\cos^2\theta]}_{1}$$

$$+\left(\frac{\partial z}{\partial y}\right)^2\underbrace{[\sin^2\theta+\cos^2\theta]}_{1}+0$$

$$= \left(\frac{\partial z}{\partial x}\right)^2(1)+\left(\frac{\partial z}{\partial y}\right)^2(1) = \left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

Hence, $$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 ✓ \quad ...\textbf{Ans.}$$

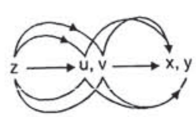
Example 4.9.12

If $z = f(u, v)$; $u = \log(x^2+y^2)$; $v = \frac{y}{x}$ then show that $x\frac{\partial z}{\partial y}-y\frac{\partial z}{\partial x} = (1+v^2)\frac{\partial z}{\partial v}$.

Solution :

Step I : Given : $z = f(u, v)$; $u = \log(x^2+y^2)$; $v = \frac{y}{x}$

i.e.



By chain rule differentiate z partially w.r.t. x and y

Step II : $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \quad ...(A)$$

and $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \quad ...(B)$$

Since, $u = \log(x^2+y^2)$; $\qquad v = \frac{y}{x}$

Differentiate u w.r.t. x keeping y as constant

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}2x = \frac{2x}{(x^2+y^2)}$$

and

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$$

Differentiate v w.r.t. x keeping y as constant

$$\frac{\partial v}{\partial x} = -\frac{y}{x^2}$$

and

$$\frac{\partial v}{\partial y} = \frac{1}{x}$$

Step III : From Equation (A),

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\left(\frac{2x}{x^2+y^2}\right)+\frac{\partial z}{\partial v}\left(\frac{-y}{x^2}\right)$$

Multiplying both sides by y

$$y\frac{\partial z}{\partial x} = y\left[\frac{\partial z}{\partial u}\left(\frac{2x}{x^2+y^2}\right)-\frac{y}{x^2}\cdot\frac{\partial z}{\partial v}\right]$$

$$= \frac{\partial z}{\partial u}\cdot\left(\frac{2xy}{x^2+y^2}\right)-\frac{y^2}{x^2}\left(\frac{\partial z}{\partial v}\right) \quad ...(1)$$

and from Equation (B)

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left(\frac{2y}{x^2+y^2}\right)+\frac{\partial z}{\partial v}\left(\frac{1}{x}\right)$$

Multiply both sides by x

$$\therefore \quad x\frac{\partial z}{\partial y} = x\left[\frac{2y}{x^2+y^2}\cdot\frac{\partial z}{\partial u}+\frac{1}{x}\cdot\frac{\partial z}{\partial v}\right]$$

$$= \left(\frac{2xy}{x^2+y^2}\right)\cdot\left(\frac{\partial z}{\partial u}\right)+\frac{\partial z}{\partial v} \quad ...(2)$$

Step IV : Equation (2) – Equation (1)

$$\therefore \quad x\frac{\partial z}{\partial y}-y\frac{\partial z}{\partial x} = \cancel{\left(\frac{2xy}{x^2+y^2}\right)\cdot\frac{\partial z}{\partial u}}+\frac{\partial z}{\partial v}-\cancel{\frac{2xy}{x^2+y^2}\cdot\frac{\partial z}{\partial u}}$$

$$+\frac{y^2}{x^2}\left(\frac{\partial z}{\partial v}\right)$$

$$= 0+\Big(1+\underbrace{\frac{y^2}{x^2}}_{V^2}\Big)\frac{\partial z}{\partial v} = (1+v^2)\frac{\partial z}{\partial v}$$

Hence,

$$x\frac{\partial z}{\partial y}-y\frac{\partial z}{\partial x} = (1+v^2)\frac{\partial z}{\partial v} ✓ \quad ...\textbf{Ans.}$$

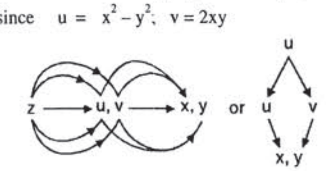
Example 4.9.13

If $z = f(u, v)$ and $u = x^2-y^2$; $v = 2xy$ then show that : $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2 = 4(u^2+v^2)^{1/2}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]$.

Solution :

Step I : **Given :** $z = f(u, v)$, $u = x^2-y^2$; $v = 2xy$

since $u = x^2-y^2$; $v = 2xy$



Step II : Differentiate u and v

$$\frac{\partial u}{\partial x} = 2x \, ; \qquad \frac{\partial v}{\partial x} = 2y$$

and $\qquad$ and

$$\frac{\partial u}{\partial y} = -2y \, ; \qquad \frac{\partial v}{\partial y} = 2x$$

Differentiate z, partially w.r.t. x, use chain rule :

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \quad ...(A)$$

And differentiate z w.r.t. y

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \quad ...(B)$$

From Equation (A), $\quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}(2x)+\frac{\partial z}{\partial v}\cdot(2y)$

Squaring

$$\left(\frac{\partial z}{\partial x}\right)^2 = \left[2x\cdot\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\cdot 2y\right]^2$$

$$= 4x^2\left(\frac{\partial z}{\partial u}\right)^2+4y^2\left(\frac{\partial z}{\partial v}\right)^2+8xy\left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial z}{\partial v}\right) \quad ...(1)$$

From Equation (B), $\quad \left(\frac{\partial z}{\partial y}\right) = \frac{\partial z}{\partial u}\cdot(-2y)+\frac{\partial z}{\partial v}(2x)$

Squaring

$$\left(\frac{\partial z}{\partial y}\right)^2 = \left[-2y\cdot\frac{\partial z}{\partial u}+2x\cdot\frac{\partial z}{\partial v}\right]^2$$

$$= 4y^2\left(\frac{\partial z}{\partial u}\right)^2+4x^2\left(\frac{\partial z}{\partial v}\right)^2-8xy\left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial z}{\partial v}\right) \quad ...(2)$$

Step III : Adding Equations (1) and (2),

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\right)^2[4x^2+4y^2]+\left(\frac{\partial z}{\partial v}\right)^2[4y^2+4x^2]+0$$

$$= [4x^2+4y^2]\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4(x^2+y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4[(x^2+y^2)^2]^{1/2}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right] \quad ...(3)$$

Step IV : Since, $u^2+v^2 = (x^2-y^2)^2+4x^2y^2$

$$= x^4+y^4-2x^2y^2+4x^2y^2$$

$$= x^4+y^4+2x^2y^2$$

$$u^2+v^2 = (x^2+y^2)^2$$

Equation (3) becomes,

$$\therefore \quad \left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2 = 4[(u^2+v^2)]^{1/2}\cdot\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right] ✓$$

...**Ans.**

Example 4.9.14

If $x = \sqrt{vw}$; $y = \sqrt{uw}$; $z = \sqrt{uv}$; and ϕ is a function of x, y, z then prove that

$$x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$$

Solution :

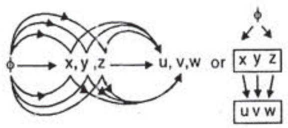
Step I : Given : ϕ is a function of x, y, z and

$$x = \sqrt{vw} \;;\; y = \sqrt{uw} \;;\; z = \sqrt{uv}$$

By chain rule, differentiate ϕ w.r.t. u, v, w, partially,

$$\therefore \quad \frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial \phi}{\partial z}\cdot\frac{\partial z}{\partial u} \quad \text{...(A)}$$

$$\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial \phi}{\partial z}\cdot\frac{\partial z}{\partial v} \quad \text{...(B)}$$



$$\text{and} \quad \frac{\partial \phi}{\partial w} = \frac{\partial \phi}{\partial x}\cdot\frac{\partial x}{\partial w} + \frac{\partial \phi}{\partial y}\cdot\frac{\partial y}{\partial w} + \frac{\partial \phi}{\partial z}\cdot\frac{\partial z}{\partial w} \quad \text{...(C)}$$

Now, differentiate x, y, z partially,

➔ **Using standard rule**

$$\left[\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}}\cdot\frac{d}{dx}\cdot f(x)\right]$$

Step II : Since,

(i) $x = \sqrt{vw}$

Differentiate x w.r.t u

$$\frac{\partial x}{\partial u} = \sqrt{vw}\,\underbrace{\frac{\partial}{\partial u}(1)}_{0} = 0$$

$$\therefore \frac{\partial x}{\partial u} = 0$$

Differentiate x w.r.t v

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\sqrt{vw} = \frac{1}{2\sqrt{vw}}\cdot\underbrace{\frac{\partial}{\partial v}(vw)}_{w} = \frac{w}{2\sqrt{vw}}$$

$$\frac{\partial x}{\partial v} = \frac{w}{2\sqrt{vw}}$$

Differentiate x w.r.t w

$$\frac{\partial x}{\partial w} = \frac{\partial}{\partial w}\sqrt{vw} = \frac{1}{2\sqrt{vw}}\cdot\underbrace{\frac{\partial}{\partial w}(vw)}_{v} = \frac{v}{2\sqrt{vw}}$$

$$\frac{\partial x}{\partial w} = \frac{v}{2\sqrt{vw}}$$

(ii) $y = \sqrt{uw}$

Differentiate y w.r.t u

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\sqrt{uw} = \frac{1}{2\sqrt{uw}}\cdot\underbrace{\frac{\partial}{\partial u}(uw)}_{w} = \frac{w}{2\sqrt{uw}}$$

$$\frac{\partial y}{\partial u} = \frac{w}{2\sqrt{uw}}$$

Differentiate y w.r.t v

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\sqrt{uw} = \sqrt{uw}\cdot\underbrace{\frac{\partial}{\partial v}(1)}_{0} = 0$$

$$\frac{\partial y}{\partial v} = 0$$

Differentiate y w.r.t w

$$\frac{\partial y}{\partial w} = \frac{\partial}{\partial w}\sqrt{uw} = \frac{1}{2\sqrt{uw}}\underbrace{\frac{\partial}{\partial w}(uw)}_{u} = \frac{u}{2\sqrt{uw}}$$

$$\frac{\partial y}{\partial w} = \frac{u}{2\sqrt{uw}}$$

(iii) $z = \sqrt{uv}$

Differentiate z w.r.t u

$$\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}\sqrt{uv} = \frac{1}{2\sqrt{uv}}\cdot\underbrace{\frac{\partial}{\partial u}(uv)}_{u} = \frac{v}{2\sqrt{uv}}$$

$$\frac{\partial z}{\partial u} = \frac{v}{2\sqrt{uv}}$$

Differentiate z w.r.t v

$$\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}\sqrt{uv} = \frac{1}{2\sqrt{uv}}\cdot\underbrace{\frac{\partial}{\partial v}(uv)}_{v} = \frac{u}{2\sqrt{uv}}$$

$$\frac{\partial z}{\partial v} = \frac{u}{2\sqrt{uv}}$$

Differentiate z w.r.t w

$$\frac{\partial z}{\partial w} = \frac{\partial}{\partial w}\sqrt{uv} = \sqrt{uv}\cdot\underbrace{\frac{\partial}{\partial w}(1)}_{0} = 0$$

$$\frac{\partial z}{\partial w} = 0$$

Step III : From Equation (A),

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}(0) + \frac{\partial \phi}{\partial y}\left(\frac{w}{2\sqrt{uw}}\right) + \frac{\partial \phi}{\partial z}\left(\frac{v}{2\sqrt{uv}}\right)$$

Multiplying both sides by u

$$u\frac{\partial \phi}{\partial u} = \frac{uw}{2\sqrt{uw}}\frac{\partial \phi}{\partial y} + \frac{uv}{2\sqrt{uv}}\frac{\partial \phi}{\partial z}$$

$$= \left[\frac{\sqrt{uw}\cdot\cancel{\sqrt{uw}}}{2\cancel{\sqrt{uw}}}\frac{\partial \phi}{\partial y}\right] + \left[\frac{\sqrt{uv}\cdot\cancel{\sqrt{uv}}}{2\cancel{\sqrt{uv}}}\frac{\partial \phi}{\partial z}\right]$$

$$\therefore \quad u\frac{\partial \phi}{\partial u} = \frac{1}{2}\sqrt{uw}\,\frac{\partial \phi}{\partial y} + \frac{\sqrt{uv}}{2}\frac{\partial \phi}{\partial z}$$

$$\Rightarrow \quad u\frac{\partial \phi}{\partial u} = \frac{y}{2}\frac{\partial \phi}{\partial y} + \frac{z}{2}\frac{\partial \phi}{\partial z} \quad \text{...(1)}$$

... (From given)

Similarly, we get

$$v\frac{\partial \phi}{\partial z} = \frac{x}{2}\frac{\partial \phi}{\partial x} + \frac{z}{2}\frac{\partial \phi}{\partial z} \quad \text{...(2)}$$

$$\text{and} \quad w\frac{\partial \phi}{\partial w} = \frac{x}{2}\frac{\partial \phi}{\partial x} + \frac{y}{2}\frac{\partial \phi}{\partial y} \quad \text{...(3)}$$

Step IV : Adding Equations (1), (2) and (3), we get

$$u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial z} + w\frac{\partial \phi}{\partial w} = \left[\frac{y}{z}\frac{\partial \phi}{\partial y} + \frac{z}{2}\frac{\partial \phi}{\partial z}\right] + \left[\frac{x}{2}\frac{\partial \phi}{\partial x} + \frac{z}{2}\frac{\partial \phi}{\partial z}\right] + \left[\frac{x}{2}\frac{\partial \phi}{\partial x} + \frac{y}{2}\frac{\partial \phi}{\partial y}\right]$$

$$= \left(\frac{x}{2} + \frac{x}{2}\right)\frac{\partial \phi}{\partial x} + \left(\frac{y}{2} + \frac{y}{2}\right)\frac{\partial \phi}{\partial y} + \left(\frac{z}{2} + \frac{z}{2}\right)\frac{\partial \phi}{\partial z}$$

Hence,

$$\Rightarrow u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial z} + w\frac{\partial \phi}{\partial w} = \left(\frac{\cancel{2}x}{\cancel{2}}\right)\frac{\partial \phi}{\partial x} + \left(\frac{\cancel{2}y}{\cancel{2}}\right)\frac{\partial \phi}{\partial y} + \left(\frac{\cancel{2}z}{\cancel{2}}\right)\frac{\partial \phi}{\partial z}$$

$$= x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z} \checkmark \quad \text{...Ans.}$$

Example 4.9.15

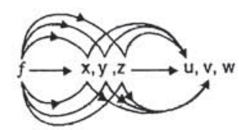
If $x = u + v + w$; $y = uv + uw + vw$; $z = uvw$ and F is function of x, y, z ; then prove that :

$$x\frac{\partial f}{\partial x} + 2y\frac{\partial f}{\partial y} + 3z\frac{\partial f}{\partial z} = u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v} + w\frac{\partial f}{\partial w}$$

Solution :

Step I : Given : $x = u + v + w$; $y = uv + vw + uw$

$z = uvw$ and f is a function of x, y, z.



Step II : Differentiate f w.r.t u, partially, by chain rule,

$$\therefore \quad \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}$$

Now, we have to find derivatives of x, y, z w.r.t u

$$\frac{\partial x}{\partial u} = 1;\ \frac{\partial y}{\partial u} = (v + w);\ \frac{\partial z}{\partial u} = vw$$

$$\therefore \quad \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot(1) + \frac{\partial f}{\partial y}(v + w) + \frac{\partial f}{\partial z}\cdot(vw)$$

Multiplying both sides by u

$$u\frac{\partial f}{\partial u} = u\frac{\partial f}{\partial x} + (uv + uw)\frac{\partial f}{\partial y} + uvw\frac{\partial f}{\partial z} \quad \text{...(1)}$$

Differentiate f w.r.t. v, partially, by chain rule,

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial v}$$

Since, $\frac{\partial x}{\partial v} = 1$; $\frac{\partial y}{\partial v} = u + w$; $\frac{\partial z}{\partial v} = uw$

$$\Rightarrow \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y}(u + w) + \frac{\partial f}{\partial z}\cdot(uw)$$

Multiplying both sides by v

$$v\frac{\partial f}{\partial v} = v\frac{\partial f}{\partial x} + (vu + vw)\frac{\partial f}{\partial y} + uvw\frac{\partial f}{\partial z} \quad \text{...(2)}$$

Similarly, we get,

$$w\frac{\partial f}{\partial w} = w\frac{\partial f}{\partial x} + (wu + wv)\frac{\partial f}{\partial y} + uvw\frac{\partial f}{\partial z} \quad \text{...(3)}$$

Step III : Adding Equations, (1), (2) and (3).

$$u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v} + w\frac{\partial f}{\partial w} = \underbrace{(u + v + w)}_{x}\frac{\partial f}{\partial x} + 2\underbrace{(uv + uw + vw)}_{y}\frac{\partial f}{\partial y} + 3\underbrace{(uvw)}_{z}\frac{\partial f}{\partial z}$$

$$= x\frac{\partial f}{\partial x} + 2y\frac{\partial f}{\partial y} + 3z\frac{\partial f}{\partial z} \checkmark \quad \text{...Ans.}$$

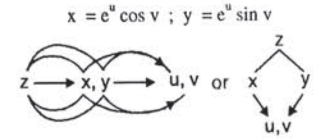
Example 4.9.16

If $z = f(x, y)$; where $x = e^u \cos v$; $y = e^u \sin v$; then prove that : $y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\cdot\frac{\partial z}{\partial y}$

Solution :

Step I : Given : $z = f(x, y)$ and

$$x = e^u \cos v \;;\; y = e^u \sin v$$



Differentiate z, w.r.t. u, partially

Step II : $\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$

Differentiate x and y w.r.t. u and v, partially

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(e^u \cos v) = e^u \cos v$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(e^u \sin v) = e^u \sin v$$

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot (e^u \cos v) + \frac{\partial z}{\partial y}(e^u \sin v)$$

Multiplying both sides by y

$$y\frac{\partial z}{\partial u} = y \cdot e^u \cos v\left(\frac{\partial z}{\partial x}\right) + y \cdot e^u \sin v\left(\frac{\partial z}{\partial y}\right) \quad ...(1)$$

Differentiate z w.r.t. v, partially, by chain rule,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Differentiate x and y w.r.t. v, keeping u as constant

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}(e^u \cos v) = e^u \cdot (-\sin v)$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(e^u \cdot \sin v) = e^u \cdot \cos v$$

$$\therefore \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^u \cdot \sin v) + \frac{\partial z}{\partial y}(e^u \cos v)$$

Multiplying both sides by x

$$\therefore x\frac{\partial z}{\partial v} = -x(e^u \sin v)\frac{\partial z}{\partial x} + x \cdot e^u \cos v \cdot \frac{\partial z}{\partial y} \quad ...(2)$$

Step III : Adding Equations (1) and (2) we get,

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}[y e^u \cos v - x e^u \sin v] + \frac{\partial z}{\partial y}[y e^u \sin v + x e^u \cos v]$$

$$\therefore y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}[e^{2u} \sin v \cancel{\cos v} - e^{2u} \cancel{\sin v} \cos v] + \frac{\partial z}{\partial y}[e^{2u}\sin^2 v + e^{2u}\cos^2 v]$$

... (by putting the values of x and y)

$$\therefore y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = 0 + e^{2u}\underbrace{(\sin^2 v + \cos^2 v)}_{1}\frac{\partial z}{\partial y}$$

$$= e^{2u} \cdot \frac{\partial z}{\partial y}$$

$$y\frac{\partial z}{\partial u} + x \cdot \frac{\partial z}{\partial v} = e^{2u} \cdot \frac{\partial z}{\partial y} \checkmark \quad ...\text{Ans.}$$

4.10 By Substitution : Examples based on Composite Function

Example 4.10.1

If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$, then prove that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$

Solution :

Step I : Given : $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$

Let, $x^2 - y^2 = l$
$y^2 - z^2 = m$
$z^2 - x^2 = n$

Hence, u is function of l, m, n and l, m, n are functions of x, y, z.

$$u \longrightarrow l, m, n \longrightarrow x, y, z \quad \text{or} \quad u \to \boxed{l, m, n} \to \boxed{x, y, z}$$

Differentiate u partially, w.r.t. x, y and z by chain rule,

Step II : $\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial x} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial x} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial x}$ ...(A)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial y} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial y} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial y} \quad ...(B)$$

and $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial z} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial z} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial z}$ ...(C)

Step III :

Since, $l = x^2 - y^2$	$m = y^2 - z^2$	$n = z^2 - x^2$	
$\frac{\partial l}{\partial x} = 2x$	$\frac{\partial m}{\partial x} = 0$	$\frac{\partial n}{\partial x} = -2x$	(use standard result of
$\frac{\partial l}{\partial y} = -2y$	$\frac{\partial m}{\partial y} = 2y$	$\frac{\partial n}{\partial y} = 0$	derivatives)
$\frac{\partial l}{\partial z} = 0$	$\frac{\partial m}{\partial z} = -2z$	$\frac{\partial n}{\partial z} = 2z$	

Step IV : $\therefore$ From Equation (A),

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}(2x) + \frac{\partial u}{\partial m}(0) + \frac{\partial u}{\partial n}(-2x)$$

Multiplying both sides by $\frac{1}{x}$

$$\frac{1}{x}\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial l} + 0 + \left(\frac{-2\,\partial u}{\partial n}\right) \quad ...(1)$$

From Equation (B)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}(-2y) + (2y) + \frac{\partial u}{\partial n}(0)$$

Multiplying both sides by $\frac{1}{y}$

$$\frac{1}{y}\frac{\partial u}{\partial y} = -2\frac{\partial u}{\partial l} + 2\frac{\partial u}{\partial m} \quad ...(2)$$

and from Equation (C),

$$\frac{\partial y}{\partial z} = \frac{\partial u}{\partial l}(0) + \frac{\partial u}{\partial m}(-2z) + \frac{\partial u}{\partial n}(2z)$$

Multiplying both sides by $\frac{1}{z}$

$$\frac{1}{z}\frac{\partial u}{\partial z} = -2\frac{\partial u}{\partial m} + 2\frac{\partial u}{\partial n} \quad ...(3)$$

Step V : Adding Equations (1), (2) and (3), we get

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z}$$

$$= 2\cancel{\frac{\partial y}{\partial l}} - 2\cancel{\frac{\partial y}{\partial n}} - 2\cancel{\frac{\partial u}{\partial l}} + 2\cancel{\frac{\partial u}{\partial m}} - 2\cancel{\frac{\partial u}{\partial m}} + \cancel{\frac{\partial y}{\partial n}} = 0$$

Hence, $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$ ✓ ...**Ans.**

Example 4.10.2

If $u = f(e^{x-y}, e^{y-z}, e^{z-x})$ then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Solution :

Step I : Given : $u = f(e^{x-y}, e^{y-z}, e^{z-x})$

Let, $l = e^{x-y}$; $m = e^{y-z}$; $n = e^{z-x}$

$$\Rightarrow \quad u \longrightarrow l, m, n \longrightarrow x, y, z$$

Differentiate u w.r.t. x, y, z respectively, by chain rule.

Step II : $\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial x} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial x} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial x}$ ...(A)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial y} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial y} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial y} \quad ...(B)$$

and $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial z} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial z} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial z}$ ...(C)

Step III : Differentiate l, m, n w.r.t. x, y, z

→ **Using standard rule**

$$...\left[\frac{d}{dx}(e^{f(x)}) = e^{f(x)} \cdot \frac{d}{dx}f(x)\right]$$

$$\left[\frac{d}{dx}e^{ax} = e^{a(x)} \cdot a\right]$$

Since, $l = e^{x-y}$;	$m = e^{y-z}$	$n = e^{z-x}$
$\frac{\partial l}{\partial x} = e^{x-y}$	$\frac{\partial m}{\partial x} = 0$;	$\frac{\partial n}{\partial x} = -e^{z-x}$
$\frac{\partial l}{\partial y} = e^{-x-y} \cdot (-1)$;	$\frac{\partial m}{\partial y} = e^{y-z}$;	$\frac{\partial n}{\partial y} = 0$
$\frac{\partial l}{\partial z} = 0$	$\frac{\partial m}{\partial z} = e^{y-z} \cdot (-1)$	$\frac{\partial n}{\partial z} = e^{z-x}$

Step IV : From Equations (A), (B) and (C) we get,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}(e^{x-y}) + \frac{\partial u}{\partial m}(0) + \frac{\partial u}{\partial n}(-e^{z-x}) \quad ...(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}(-e^{x-y}) + \frac{\partial u}{\partial m}(e^{y-z}) + \frac{\partial u}{\partial n}(0) \quad ...(2)$$

and $\frac{\partial y}{\partial z} = \frac{\partial u}{\partial l}(0) + \frac{\partial u}{\partial m}(-e^{z-x}) + \frac{\partial u}{\partial n}(e^{z-x})$ ...(3)

Adding Equations (1), (2) and (3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \left[e^{x-y}\cancel{\frac{\partial u}{\partial l}} - e^{z-}\cancel{\cdot\frac{\partial u}{\partial n}}\right] + \left[-e^{x-y}\cancel{\frac{\partial u}{\partial l}} + e^{y-}\cancel{\cdot\frac{\partial u}{\partial m}}\right] + \left[-e^{z-x}\cancel{\frac{\partial u}{\partial m}} + e^{z-x}\cancel{\frac{\partial u}{\partial n}}\right]$$

$$= 0 \checkmark \quad ...\text{Ans.}$$

Example 4.10.3

If $f(xy^2, z - 2x) = 0$ then prove that $x\frac{\partial z}{\partial x} - \frac{1}{2}y\frac{\partial z}{\partial y} = 2x$.

Solution :

Step I : Given : $f(xy^2, z - 2x) = 0$

Let, $xy^2 = l$; $z - 2x = m$

$\therefore f(l, m) = 0$

$$\Rightarrow \quad f \longrightarrow l, m \longrightarrow x, y, z \quad \text{or} \quad f \to \boxed{l, m} \to x\ y\ z$$

Step II : Differentiate f w.r.t. x partially, keeping y a[s] constant and note that z is a function of x and y

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial l} \cdot \frac{\partial l}{\partial x} + \frac{\partial f}{\partial m} \cdot \frac{\partial m}{\partial x} \quad ...(A)$$

and $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial l} \cdot \frac{\partial l}{\partial y} + \frac{\partial f}{\partial m} \cdot \frac{\partial m}{\partial y}$ ...(B)

Step III : Since, $f = 0 \Rightarrow \frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

and $l = xy^2$; $m = z - 2x$

Differentiate l and m w.r.t. x, keeping y as consta[nt] and note that z is a function of x, y.

$$\frac{\partial l}{\partial x} = y^2; \quad \frac{\partial m}{\partial x} = \frac{\partial z}{\partial x} - 2$$

Also, differentiate l and m w.r.t y, keeping x as constant.

$$\frac{\partial l}{\partial y} = 2xy; \quad \frac{\partial m}{\partial y} = \frac{\partial z}{\partial y}$$

Step IV : Now from the Equation (A)

$$0 = \frac{\partial f}{\partial l} \cdot y^2 + \frac{\partial f}{\partial m} \cdot \left(\frac{\partial z}{\partial x} - 2\right)$$

$$\frac{\partial f}{\partial l} y^2 = -\frac{\partial f}{\partial m}\left[\frac{\partial z}{\partial x} - 2\right]$$

$$\Rightarrow \quad \frac{\partial f/\partial l}{\frac{\partial f}{\partial m}} = -\left[\frac{\frac{\partial z}{\partial x} - 2}{y^2}\right] \quad \ldots(1)$$

Step V : and from Equation (B),

$$0 = \frac{\partial f}{\partial l} \cdot (2xy) + \frac{\partial f}{\partial m} \cdot \left(\frac{\partial z}{\partial y}\right)$$

$$\Rightarrow \quad \frac{\partial f}{\partial l}(2xy) = -\frac{\partial f}{\partial m} \cdot \left(\frac{\partial z}{\partial y}\right)$$

$$\Rightarrow \quad \frac{\partial f/\partial l}{\partial f/\partial m} = -\left[\frac{\partial z/\partial y}{2xy}\right] \quad \ldots(2)$$

Step VI : From Equations (1) and (2) ; L.H.S are same, So, R.H.S. must be same, we get

$$\left[\frac{\frac{\partial z}{\partial x} - 2}{\cancel{y^2}}\right] = \left[\frac{\frac{\partial z}{\partial y}}{2x\cancel{y}}\right]$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} - 2 = \frac{y}{2x}\frac{\partial z}{\partial y}$$

$$\Rightarrow \quad x\frac{\partial z}{\partial x} - 2x = \frac{y}{2}\frac{\partial z}{\partial y}$$

$$\Rightarrow \quad \mathbf{x\frac{\partial z}{\partial x} - \frac{1}{2}y\frac{\partial z}{\partial y} = 2x} \quad ✓ \quad \ldots\textbf{Ans.}$$

Example 4.10.4

If $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $lx + my + nz = 0$ then prove that,

$$\frac{dx}{\frac{ny}{b^2} - \frac{mz}{c^2}} = \frac{dy}{\frac{lz}{c^2} - \frac{nx}{a^2}} = \frac{dz}{\frac{mx}{a^2} - \frac{ly}{b^2}}$$

Solution :

Step I : Let, $f = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$

and $\phi = lx + my + nz = 0$

Step II : Differentiate f function

$f \to x, y, z$

$f \to x\ y\ z$

$$df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy + \frac{\partial f}{\partial z} \cdot dz$$

Since, $f = 0 \Rightarrow df = 0$

Differentiate f w.r.t x, y and z

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = \frac{2x}{a^2};$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = \frac{2y}{b^2};$$

$$\text{and } \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = \frac{2z}{c^2}$$

$$0 = \frac{2x}{a^2}dx + \frac{2y}{b^2}dy + \frac{2z}{c^2}dz = 0$$

Take 2 common and cancel.

$$\Rightarrow \quad \frac{x}{a^2}dx + \frac{y}{b^2}dy + \frac{z}{c^2}dz = 0 \quad \ldots(1)$$

Now, differentiate ϕ function, w.r.t. x, y, z

$\phi \to x, y, z$

$\phi \to x, y, z$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

Step III : Since $\phi = 0 \Rightarrow d\phi = 0;$

$$\frac{\partial \phi}{\partial x} = l; \quad \frac{\partial \phi}{\partial y} = m; \quad \frac{\partial \phi}{\partial z} = n.$$

$$\therefore \quad 0 = l\,dx + m\,dy + n\,dz$$

$$\Rightarrow \quad l\,dx + m\,dy + n\,dz = 0 \quad \ldots(2)$$

Step IV : Solving Equations (1) and (2) by Cramer's rule,

$$\left\{\begin{array}{l} \because \text{ If } a_1x + b_1y + c_1z = 0 \text{ and } a_2x + b_2y + c_2z = 0 \\ \text{then } \dfrac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \dfrac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \dfrac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \end{array}\right\}$$

$$\therefore \quad \frac{dx}{\begin{vmatrix} \frac{y}{b^2} & \frac{z}{c^2} \\ m & n \end{vmatrix}} = \frac{-dy}{\begin{vmatrix} \frac{x}{a^2} & \frac{z}{c^2} \\ l & n \end{vmatrix}} = \frac{dz}{\begin{vmatrix} \frac{x}{a^2} & \frac{y}{b^2} \\ l & m \end{vmatrix}}$$

Hence,

$$\frac{\mathbf{dx}}{\frac{\mathbf{ny}}{\mathbf{b^2}} - \frac{\mathbf{mz}}{\mathbf{c^2}}} = \frac{\mathbf{dy}}{\frac{\mathbf{lz}}{\mathbf{c^2}} - \frac{\mathbf{nx}}{\mathbf{a^2}}} = \frac{\mathbf{dz}}{\frac{\mathbf{mx^2}}{\mathbf{a^2}} - \frac{\mathbf{ly}}{\mathbf{b^2}}} \quad ✓ \quad \ldots\textbf{Ans.}$$

Example 4.10.5

If $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$ then prove that

$$\frac{\mathbf{dx}}{\mathbf{bny - cmz}} = \frac{\mathbf{dy}}{\mathbf{clz - anx}} = \frac{\mathbf{dz}}{\mathbf{amx - bly}}$$

Solution :

Step I : Let, $f = ax^2 + by^2 + cz^2 - 1 = 0$

and $\phi = lx + my + nz = 0$

Step II : Differentiate f function, $f \to x, y, z$

$f \to x, y, z$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Since, $f = 0 \Rightarrow df = 0;$

$$\frac{\partial f}{\partial x} = 2ax; \quad \frac{\partial f}{\partial y} = 2by; \quad \frac{\partial f}{\partial z} = 2cz$$

$$\Rightarrow \quad 0 = 2ax\,dx + 2by\,dy + 2cz\,dz$$

$$\Rightarrow \quad ax\,dx + by\,dy + cz\,dz = 0 \quad \ldots(1)$$

and differentiate ϕ function,

$\phi \to x, y, z$

$\phi \to x, y, z$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

But $\phi = 0 \Rightarrow d\phi = 0;$ $\frac{\partial \phi}{\partial x} = l;$ $\frac{\partial \phi}{\partial y} = m;$ $\frac{\partial \phi}{\partial z} = n$

$$\therefore \quad 0 = l\,dx + m\,dy + n\,dz$$

$$\Rightarrow \quad l\,dx + m\,dy + n\,dz = 0 \quad \ldots(2)$$

Step III : Solving Equations (1) and (2) by Cramer's rule,

Cramer's rule

$$\ldots\left[\text{If } a_1x + b_1y + c_1z = 0; \quad a_2x + b_2y + c_2z = 0\right.$$

$$\left.\text{Then, } \frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}\right]$$

$$\therefore \quad \frac{dx}{\begin{vmatrix} by & m \\ cz & n \end{vmatrix}} = -\frac{dy}{\begin{vmatrix} ax & l \\ cz & n \end{vmatrix}} = \frac{dz}{\begin{vmatrix} ax & by \\ l & m \end{vmatrix}}$$

Hence,

$$\therefore \quad \frac{\mathbf{dx}}{\mathbf{n\,by - c\,mz}} = \frac{\mathbf{dy}}{\mathbf{c\,lz - anx}} = \frac{\mathbf{dz}}{\mathbf{amx - bly}} \quad ✓$$

...Ans.

Example 4.10.6

If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that

$$\mathbf{x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0}$$

Solution :

Step I : Given : $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$

Let, $l = \frac{y-x}{xy}$; $m = \frac{z-x}{xz}$

$u = f(l, m)$

$u \to l, m \to x, y, z$

Step II : $\therefore$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial x} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial x} \quad \ldots(A)$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial y} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial y} \quad \ldots(B)$$

and

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial z} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial z} \quad \ldots(C)$$

Step III : Since, $l = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$;

$$m = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$$

→ Using **standard** results $\quad \ldots\left[\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\right]$

$$\ldots\left[\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[g(x) \cdot \frac{d}{dx}f(x) - f(x) \cdot \frac{d}{dx}g(x)\right]}{(g(x))^2}\right]$$

Or

$$\ldots\left[\frac{d}{dx}\left[\frac{I}{II}\right] = \frac{\left[II \cdot \frac{d}{dx}(I) - (I) \cdot \frac{d}{dx}(II)\right]}{(II)^2}\right]$$

Differentiate m and l w.r.t x, y, z

$$\frac{\partial l}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{x}\right) = \frac{-1}{x^2}; \quad \frac{\partial m}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\frac{\partial l}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{y}\right) = \frac{1}{y^2}; \quad \frac{\partial m}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{x} - \frac{1}{y}\right) = 0$$

$$\frac{\partial l}{\partial z} = 0; \quad \frac{\partial m}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{x} - \frac{1}{z}\right) = \frac{1}{z^2}$$

Step IV : From Equation (A),

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \cdot \left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial m} \cdot \left(-\frac{1}{x^2}\right)$$

Multiplying both sides by x^2

$$\therefore \quad x^2\frac{\partial u}{\partial x} = x^2\left[-\frac{1}{\cancel{x^2}} \cdot \frac{\partial u}{\partial l} - \frac{1}{\cancel{x^2}}\frac{\partial u}{\partial m}\right] = -\frac{\partial u}{\partial l} - \frac{\partial u}{\partial m} \quad \ldots(1)$$

From Equation (B),

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial m}(0)$$

Multiplying both sides by y^2

$$y^2\frac{\partial u}{\partial x} = \cancel{y}\left(\frac{1}{\cancel{y}}\cdot\frac{\partial u}{\partial l}\right) = \frac{\partial u}{\partial l} \qquad ...(2)$$

and from Equation (C),

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}(0) + \frac{\partial u}{\partial m}\left(\frac{1}{z^2}\right)$$

Multiplying both sides by z^2

$$\Rightarrow z^2\frac{\partial u}{\partial z} = \cancel{z}\left(\frac{1}{\cancel{z}}\cdot\frac{\partial u}{\partial m}\right) = \frac{\partial u}{\partial m} \qquad ...(3)$$

Step V : Adding Equations (1), (2) and (3)

$$\therefore x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z}$$

$$= -\cancel{\frac{\partial u}{\partial l}} - \cancel{\frac{\partial u}{\partial m}} + \cancel{\frac{\partial u}{\partial l}} + \cancel{\frac{\partial u}{\partial m}} = 0$$

Hence,

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0 \checkmark \qquad ...\text{Ans.}$$

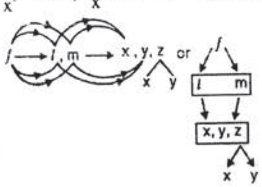
Example 4.10.7 :

If $f\left(\frac{z}{x^3}, \frac{y}{x}\right) = 0$ then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$

Solution :

Step I : Given : $f\left(\frac{z}{x^3}, \frac{y}{x}\right) = 0$

Let, $\frac{z}{x^3} = l$; $\frac{y}{x} = m \Rightarrow f(l, m) = 0$



Step II : Differentiate f partially, note that z is a function of x and y,

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial f}{\partial m}\cdot\frac{\partial m}{\partial x} \qquad ...(A)$$

$$\text{and } \frac{\partial f}{\partial y} = \frac{\partial f}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial f}{\partial m}\cdot\frac{\partial m}{\partial y} \qquad ...(B)$$

Step III : But $f = 0 \Rightarrow \frac{\partial f}{\partial x} = 0$; $\frac{\partial f}{\partial y} = 0$ ...(C)

and $l = \frac{z}{x^3}$; $m = \frac{y}{x}$

→ **Using standard results of derivatives**

$$...\left[\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[g(x)\cdot\frac{d}{dx}f(x) - f(x)\cdot\frac{d}{dx}g(x)\right]}{(g(x))^2}\right]$$

Or

$$...\left[\frac{d}{dx}\left[\frac{I}{II}\right] = \frac{\left[II\cdot\frac{d}{dx}(I) - (I)\cdot\frac{d}{dx}(II)\right]}{(II)^2}\right]$$

Differentiate l and m w.r.t. x and y.

$$\frac{\partial l}{\partial x} = \frac{x^6\cdot\frac{\partial z}{\partial x} - z\underbrace{\left(\frac{\partial}{\partial x}\cdot x^3\right)}_{3x^2}}{x^6}$$

$$\frac{\partial l}{\partial x} = \frac{x^3\cdot\frac{\partial z}{\partial x} - z\,3x^2}{x^6} = \frac{\frac{\partial z}{\partial x}}{x^3} - 3z\cdot\frac{1}{x^4}$$

$$\text{and } \frac{\partial l}{\partial y} = \frac{1}{x^3}\cdot\frac{\partial z}{\partial y}$$

$$\text{Also } \frac{\partial m}{\partial x} = y\cdot\underbrace{\frac{\partial}{\partial x}\left(\frac{1}{x}\right)}_{-1/x^2} = -\frac{y}{x^2} \text{ and } \frac{\partial m}{\partial y} = \frac{1}{x}$$

Step IV : ∴ Equations (A) and (B) becomes

$$0 = \frac{\partial f}{\partial l}\cdot\left[\frac{1}{x^3}\cdot\frac{\partial z}{\partial x} - 3z\frac{1}{x^4}\right] + \frac{\partial f}{\partial m}\left(-\frac{y}{x^2}\right)$$

$$\Rightarrow \frac{\partial f}{\partial m}\left(\frac{y}{x^2}\right) = \frac{\partial f}{\partial l}\left[\frac{1}{x^3}\cdot\frac{\partial z}{\partial x} - 3z\frac{1}{x^4}\right]$$

$$\frac{\frac{\partial f}{\partial m}}{\partial f/\partial l} = \frac{\left[\frac{1}{x^3}\cdot\frac{\partial z}{\partial x} - 3z\frac{1}{x^4}\right]}{(y/x^2)} \qquad ...(1)$$

Step V : From Equation (B)

$$0 = \frac{\partial f}{\partial l}\cdot\left[\frac{1}{x^3}\cdot\frac{\partial z}{\partial y}\right] + \frac{\partial f}{\partial m}\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{\partial f}{\partial m}\left(\frac{1}{x}\right) = -\frac{\partial f}{\partial l}\left[\frac{1}{x^3}\cdot\frac{\partial z}{\partial y}\right]$$

$$\frac{\frac{\partial f}{\partial m}}{\partial f/\partial l} = \frac{-\frac{1}{x^3}\cdot\frac{\partial z}{\partial y}}{(1/x)} \qquad ...(2)$$

Step VI : From Equations (1) and (2), L.H.S are equal, so R.H.S must be equal, we get,

$$\frac{\left[\frac{1}{x^3}\cdot\frac{\partial z}{\partial x} - 3z\frac{1}{x^4}\right]}{(y/x^2)} = \frac{-\frac{1}{x^3}\cdot\frac{\partial z}{\partial x}}{(1/x)}$$

$$\frac{1}{x^3}\cdot\frac{\cancel{x^2}}{y}\cdot\frac{\partial z}{\partial x} - 3z\cdot\frac{\cancel{x^2}}{y}\cdot\frac{1}{\cancel{x^4}2} = -\frac{x}{\cancel{x^3}2}\cdot\frac{\partial z}{\partial x}$$

$$\Rightarrow \frac{1}{xy}\frac{\partial z}{\partial x} - \frac{3z}{x^2y} = -\frac{1}{x^2}\frac{\partial z}{\partial y}$$

⇒ Multiplying both side by x^2y, we get,

$$\frac{\cancel{x^2y}}{\cancel{xy}}\frac{\partial z}{\partial x} - \frac{\cancel{x^2y}\,3z}{\cancel{x^2y}} = -\cancel{x^2}y\frac{1}{\cancel{x^2}}\frac{\partial z}{\partial y}$$

$$x\frac{\partial z}{\partial x} - 3z = -y\frac{\partial z}{\partial y}$$

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z \checkmark \qquad ...\text{Ans.}$$

Example 4.10.8

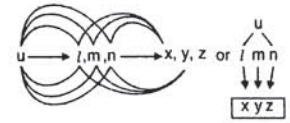
If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ then prove that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

Solution :

Step I : Given : $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$

Let, $\frac{x}{y} = l$, $\frac{y}{z} = m$, $\frac{z}{x} = n$



Differentiate u w.r.t. x, y, z by chain rule.

Step II : $\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$ ...(A)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z} \qquad ...(B)$$

$$\text{and } \frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z} \qquad ...(C)$$

Step III :

Differentiate l, m, n w.r.t x, y, z ...

Since, $l = \frac{x}{y}$	$m = \frac{y}{z}$	$n = \frac{z}{x}$	
$\frac{\partial l}{\partial x} = \frac{1}{y}$	$\frac{\partial m}{\partial x} = 0$	$\frac{\partial n}{\partial x} = -\frac{z}{x^2}$	(∵ Use standard results of derivatives)
$\frac{\partial l}{\partial y} = -\frac{x}{y^2}$	$\frac{\partial m}{\partial y} = \frac{1}{z}$	$\frac{\partial n}{\partial y} = 0$	
$\frac{\partial l}{\partial z} = 0$	$\frac{\partial m}{\partial z} = \frac{-y}{z^2}$	$\frac{\partial n}{\partial z} = \frac{1}{x}$	

Step IV : Equation (A) becomes,

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial m}(0) + \frac{\partial u}{\partial n}\left(\frac{-z}{x^2}\right)$$

Multiplying both sides by x

$$\therefore x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial l} - \frac{z}{x}\frac{\partial u}{\partial n} \qquad ...(1)$$

Step V : Equation (B), becomes

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial m}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial n}(0)$$

Multiplying both sides of y

$$\therefore y\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial l}\left(\frac{x}{y}\right) + \frac{y}{z}\frac{\partial u}{\partial m} \qquad ...(2)$$

Step VI : Equation (C) becomes,

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}(0) + \frac{\partial u}{\partial m}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial n}\left(\frac{1}{x}\right)$$

Multiplying both sides by z

$$\therefore z\frac{\partial u}{\partial x} = -\frac{y}{z}\cdot\frac{\partial u}{\partial m} + \frac{z}{x}\cdot\frac{\partial u}{\partial n} \qquad ...(3)$$

Step VII : Adding Equations (1), (2) and (3) we get,

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\left[\cancel{\frac{x}{y}} - \cancel{\frac{x}{y}}\right] + \frac{\partial u}{\partial m}\left[\cancel{\frac{y}{z}} - \cancel{\frac{y}{z}}\right] + \frac{\partial u}{\partial n}\left[-\cancel{\frac{z}{x}} + \cancel{\frac{z}{x}}\right]$$

Hence,

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0 \checkmark \qquad ...\text{Ans.}$$

Example 4.10.9

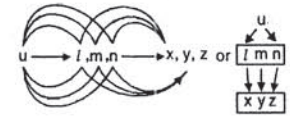
If $u = f(2x - 3y, 3y - 4z ; 4z - 2x)$ then prove that :

$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$

Solution :

Step I : Given : $u = f(2x - 3y,\ 3y - 4z\ ;\ 4z - 2x)$

Let, $2x - 3y = l$, $3y - 4z = m$; $4z - 2x = n$



Step II : Differentiate u w.r.t. x keeping y and z a constant by chain rule,

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$

Differentiate l, m, n w.r.t. x

$$\frac{\partial l}{\partial x} = 2;\ \frac{\partial l}{\partial y} = -3;\ \frac{\partial l}{\partial z} = 0;$$

$$\frac{\partial m}{\partial x} = 0;\ \frac{\partial m}{\partial y} = 3;\ \frac{\partial n}{\partial z} = -4;$$

$$\frac{\partial n}{\partial x} = -2;\ \frac{\partial n}{\partial y} = 0;\ \frac{\partial n}{\partial z} = 4.$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}(2) + \frac{\partial u}{\partial m}(0) + \frac{\partial u}{\partial n}(-2)$$

Multiplying both sides by $\frac{1}{2}$

$$\therefore \quad \frac{1}{2}\frac{\partial u}{\partial x} = \frac{1}{\cancel{2}}\left[\cancel{2}\cdot\frac{\partial u}{\partial l} - \cancel{2}\cdot\frac{\partial u}{\partial n}\right] = \frac{\partial u}{\partial l} - \frac{\partial u}{\partial n} \qquad \ldots(1)$$

Step III : Differentiate u partially w.r.t. y keeping x and z as constant, by chain rule,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$\therefore \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}(-3) + \frac{\partial u}{\partial m}(3) + \frac{\partial u}{\partial n}(0)$$

Multiplying both sides by $\frac{1}{3}$

$$\therefore \quad \frac{1}{3}\frac{\partial u}{\partial z} = \frac{1}{\cancel{3}}\left[-\cancel{3}\cdot\frac{\partial u}{\partial l} + \cancel{3}\cdot\frac{\partial u}{\partial m}\right] = \frac{-\partial u}{\partial l} + \frac{\partial u}{\partial m} \qquad \ldots(2)$$

Step IV : Differentiate u partially w.r.t. z, keeping x and y as constant

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$

$$\therefore \quad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}(0) + \frac{\partial u}{\partial m}(-4) + \frac{\partial u}{\partial n}(4)$$

Multiplying both sides by $\frac{1}{4}$

$$\frac{1}{4}\frac{\partial u}{\partial z} = \frac{1}{\cancel{4}}\left[-\cancel{4}\cdot\frac{\partial u}{\partial m} + \cancel{4}\cdot\frac{\partial u}{\partial n}\right] = -\frac{\partial u}{\partial m} + \frac{\partial u}{\partial n} \qquad \ldots(3)$$

Step V : Adding Equations (1), (2) and (3) we get,

Hence,

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = \left[\cancel{\frac{\partial u}{\partial l}} - \cancel{\frac{\partial u}{\partial n}}\right] + \left[-\cancel{\frac{\partial u}{\partial l}} + \cancel{\frac{\partial u}{\partial m}}\right] + \left[-\cancel{\frac{\partial u}{\partial m}} + \cancel{\frac{\partial u}{\partial n}}\right] = 0 \checkmark \qquad \ldots\textbf{Ans.}$$

4.11 Total Derivative (Implicit Functions)

If $u = f(x, y)$ and $x = \phi_1(t)$; $y = \phi_2(t)$ then u is a function of independent single variable t then u is called composite function of single variable t.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$\text{or} \quad du = \frac{\partial u}{\partial x}\cdot dx + \frac{\partial u}{\partial y}\cdot dy$$

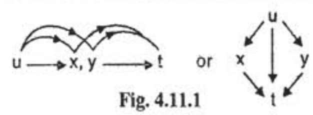


Fig. 4.11.1

is called total derivative of u w.r.t t.

Similarly, for function of three variables. If $u = f(x, y, z)$ and $x = \phi_1(r)$; $y = \phi_2(r)$; $z = \phi_3(r)$ then u is a composite function of single independent variable r .

$$\frac{du}{dr} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dr} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dr} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dr}$$

$$\text{or} \quad du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

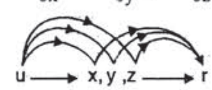


Fig. 4.11.2

Example 4.11.1

If $y^{x^y} = \sin x$ then find $\frac{dy}{dx}$.

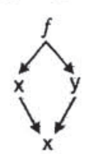
Solution :

Step I : Given : $y^{x^y} = \sin x$

Taking log on both sides, we get

→ Using standard result : ...$[\log a^x = x \log a]$

$$\therefore x^y \cdot \log y = \log \sin x$$



and let, $\quad f \equiv x^y \log y - \log \sin x = 0$

$f(x, y) = 0,\quad$ f is implicit function in x and y

Differentiate f partially w.r.t. x

Step II :

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

$$f = 0, \quad \frac{df}{dx} = 0$$

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx}\cdot\frac{\partial f}{\partial y} = -\frac{\partial f}{\partial x}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \qquad \ldots(1)$$

Now, differentiate f w.r.t. x

→ Using standard rule :

$$\ldots\left[\frac{d}{dx}[f(x)]^n = n\, f(x)^{n-1}\cdot\frac{d}{dx}f(x)\right]$$

$$\frac{\partial f}{\partial x} = \log y \cdot \underbrace{\frac{\partial}{\partial x}(x^y)}_{yx^{y-1}} - \underbrace{\frac{\partial}{\partial x}\cdot\log \sin x}_{\left(\frac{\cos x}{\sin x}\right)} = yx^{y-1}\log y - \frac{\cos x}{\sin x}$$

$$= y\,x^{y-1}\log y - \cot x$$

Step III : differentiate f w.r.t. y by using

$$d(a^x) = a^x \log a$$

$$\frac{\partial f}{\partial y} = x^y\cdot\underbrace{\frac{\partial}{\partial y}\cdot\log y}_{1/y} + \log y\cdot\underbrace{\frac{\partial}{\partial y}\cdot x^y}_{x^y\cdot\log x}$$

$$x^y\cdot\frac{1}{y} + \log y\cdot x^y \log x$$

Equation (1) becomes

$$\frac{dy}{dx} = -\left[\frac{y\,x^{y-1}\cdot\log y - \cot x}{x^y\cdot\frac{1}{y} + x^y\log y\cdot\log x}\right] \checkmark \quad \ldots\textbf{Ans.}$$

Example 4.11.2

Find $\frac{dy}{dx}$ if $(\cos x)^y = (\sin y)^x$.

Solution :

Step I : Given : $(\cos x)^y = (\sin y)^x$

Taking log on both sides

→ Use standard rule : ...$[\log a^x = x \log a]$

$$y \log \cos(x) = x \log \sin y$$

$$\therefore f(x, y) = y \log \cos x - x \log \sin y$$

$$\therefore f(x, y) = 0\text{; f is implicit function.}$$

Step II : $\because \frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$

Differentiate f w.r.t. x and y by **using standard rules of derivatives**

$$\therefore \quad \frac{dy}{dx} = \frac{y\cdot\overbrace{\frac{\partial}{\partial x}\cdot\log\cos x}^{-\frac{\sin x}{\cos x}} - \log\sin y\cdot\overbrace{\frac{\partial}{\partial x}\cdot x}^{1}}{\log\cos x\cdot\underbrace{\frac{\partial y}{\partial y}}_{1} - x\underbrace{\frac{\partial}{\partial y}\log\sin y}_{\frac{\cos y}{\sin y}}}$$

$$\therefore \frac{dy}{dx} = \frac{-\left[\frac{y(-\sin x)}{\cos x} - \log\sin y\right]}{\log\cos x - \frac{x}{\sin y}(\cos y)} = \frac{\mathbf{y\tan x\log\sin y}}{\mathbf{\log\cos x - x\cot y}} \checkmark \quad \ldots\textbf{Ans.}$$

Example 4.11.3

If $y^x + x^y = (x + y)^{x+y}$ find $\frac{dy}{dx}$.

Solution :

Step I :

Given : $y^x + x^y = (x + y)^{x+y}$

Taking log on both sides,

$$\log y^x + \log x^y = \log(x + y)^{x+y}$$

→ Using formula : ...$[\log a^b = b \log a]$

$$\Rightarrow x \log y + y \log x = (x + y)\log(x + y)$$

$$\therefore f = x\log y + y\log x - (x + y)\log(x + y)$$

Step II : $\because$ We know $\frac{dy}{dx} = -\left[\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right] \qquad \ldots(1)$

Differentiate f w.r.t. x, keeping y as constant

$$\text{Now,}\ \frac{\partial f}{\partial x} = \log y\cdot\underbrace{\frac{\partial x}{\partial x}}_{1} + y\cdot\underbrace{\frac{\partial}{\partial x}\log x}_{\frac{1}{x}}$$

$$-\left[(x + y)\cdot\underbrace{\frac{\partial}{\partial x}\log(x + y)}_{1/(x+y)} + \log(x + y)\cdot\underbrace{\frac{\partial}{\partial x}(x + y)}_{1}\right]$$

$$= \log y + y\frac{1}{x} - \left(\cancel{x+y}\cdot\frac{1}{\cancel{x+y}} + \log(x + y)\right)$$

$$\frac{\partial f}{\partial x} = \log y + \frac{y}{x} - 1 - \log(x + y) \qquad \ldots(A)$$

and differentiate f w.r.t. x, keeping x as constant

$$\frac{\partial f}{\partial y} = x\underbrace{\frac{\partial}{\partial y}\cdot\log y}_{\frac{1}{y}} + \log x\cdot\underbrace{\frac{\partial}{\partial y}y}_{1}$$

$$-\left[(x+y)\cdot\underbrace{\frac{\partial}{\partial y}\log(x+y)}_{\frac{1}{x+y}} + \log(x+y)\cdot\underbrace{\frac{\partial}{\partial y}(x+y)}_{1}\right]$$

$$= x\cdot\frac{1}{y}+\log x-\left((x+y)\cdot\frac{1}{x+y}+\log(x+y)\right)$$

$$\frac{\partial f}{\partial y} = \frac{x}{y}+\log x-1-\log(x+y) \quad \text{...(B)}$$

Step III : By using Equations (A) and (B), Equation (1) becomes,

$$\frac{dy}{dx} = -\left[\frac{\log y+\frac{y}{x}-1-\log(x+y)}{\frac{x}{y}+\log x-1-\log(x+y)}\right] \checkmark \quad \text{...Ans.}$$

Example 4.11.4

If $x = r\cos\theta$; $y = r\sin\theta$, where r and θ are functions of t, then prove that : $x\frac{dy}{dt}-y\frac{dx}{dt} = r^2\frac{d\theta}{dt}$

Solution :

Step I : Given, $x = r\cos\theta$; $y = r\sin\theta$ and r, θ are functions of t.

i.e.

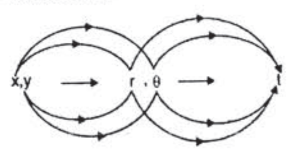


Fig. P. 4.11.4

Differentiate x, y w.r.t. t

Step II :
$$\frac{dx}{dt} = \frac{\partial x}{\partial r}\cdot\frac{dr}{dt}+\frac{\partial x}{\partial\theta}\cdot\frac{d\theta}{dt} \quad \text{...(1)}$$

and
$$\frac{dy}{dt} = \frac{\partial y}{\partial r}\cdot\frac{dr}{dt}+\frac{\partial y}{\partial\theta}\cdot\frac{d\theta}{dt} \quad \text{...(2)}$$

Step III : Since, $x = r\cos\theta$; $y = r\sin\theta$

Differentiate x w.r.t. r and θ

Also differentiate y w.r.t. r and θ

$$\frac{\partial x}{\partial r} = \cos\theta\ ;$$
$$\frac{\partial y}{\partial r} = \sin\theta$$

and
$$\frac{\partial x}{\partial\theta} = -r\sin\theta\ ;$$
$$\frac{\partial y}{\partial\theta} = r\cos\theta$$

Step IV : From Equation (1),

$$\frac{dx}{dt} = \cos\theta\cdot\frac{dr}{dt}+(-r\sin\theta)\cdot\frac{d\theta}{dt}$$

Multiplying both sides by y, we get

$$\Rightarrow y\frac{dx}{dt} = y\cos\theta\frac{dr}{dt}-y\cdot r\sin\theta\frac{d\theta}{dt} \quad \text{...(3)}$$

And from Equation (2),

$$\frac{dy}{dt} = \sin\theta\frac{dr}{dt}+r\cos\theta\frac{d\theta}{dt}$$

Multiplying both sides by x, we get

$$\Rightarrow x\frac{dy}{dt} = x\sin\theta\frac{dr}{dt}+x\cdot r\cdot\cos\theta\cdot\frac{d\theta}{dt} \quad \text{...(4)}$$

Step V : Equation (4) – Equation (3),

$$x\frac{dy}{dt}-y\frac{dx}{dt} = (x\sin\theta-y\cos\theta)\cdot\frac{dr}{dt}+(x\,r\cos\theta+y\,r\sin\theta)\frac{d\theta}{dt}$$

$$= (r\cos\theta\sin\theta-r\sin\theta\cos\theta)\frac{dr}{dt}+[x^2+y^2]\frac{d\theta}{dt}$$

$$= r^2\frac{d\theta}{dt} \checkmark \quad \text{...Ans.}$$

$(\because x = r\cos\theta;\ y = r\sin\theta \Rightarrow x^2+y^2 = r^2)$

Example 4.11.5

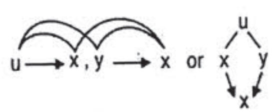
If $u = \log(xy)$ and $x^3+y^3+3xy = 0$ then find $\frac{du}{dx}$.

Solution :

Step I : Given : $u = \log(x\ y)$ and $x^3+y^3+3xy = 0$

By differentiating u w.r.t. x

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \quad \text{...(1)}$$



Step II :

Differentiate u w.r.t. x, keeping y as constant and w.r.t. y keeping x as constant

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\log(x\cdot y) = \frac{1}{xy}\cdot\underbrace{\frac{\partial}{\partial x}(x\,y)}_{y}$$

$$= \frac{1}{xy}\cdot(y) = \frac{1}{x}$$

and
$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\log(x\,y) = \frac{1}{xy}\cdot\underbrace{\frac{\partial}{\partial y}(x\,y)}_{x}$$

$$= \frac{1}{xy}(x) = \frac{1}{y}$$

Equation (1) becomes $\frac{du}{dx} = \frac{1}{x}+\frac{1}{y}\frac{dy}{dx}$ ...(2)

Step III : Let, $f = x^3+y^3+3xy$

We know,
$$\frac{dy}{dx} = -\left[\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right] = -\left[\frac{3x^2+3y}{3y^3+3x}\right]$$

$$= -\left[\frac{x^2+y}{y^2+x}\right]$$

($\because$ differentiating f w.r.t. x and y by using **standard results**)

Step IV : Equation (2) becomes

$$\frac{du}{dx} = \frac{1}{x}-\frac{1}{y}\left[\frac{x^2+y}{y^2+x}\right] \checkmark \quad \text{...Ans.}$$

Example 4.11.6

If $\phi(x, y, z) = 0$; prove that $\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial x}{\partial z}\right)_y\left(\frac{\partial y}{\partial x}\right)_z = -1$

Solution :

Step I : Given : $\phi(x, y, z) = 0$

$$\left(\frac{\partial z}{\partial y}\right)_x = -\left[\frac{\frac{\partial\phi}{\partial y}}{\frac{\partial\phi}{\partial z}}\right] \quad \text{...(1)}$$

$$\left(\frac{\partial x}{\partial z}\right)_y = -\left[\frac{\frac{\partial\phi}{\partial z}}{\frac{\partial\phi}{\partial x}}\right] \quad \text{...(2)}$$

and
$$\left(\frac{\partial y}{\partial x}\right)_z = -\left[\frac{\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial y}}\right] \quad \text{...(3)}$$

($\because x = f(y, z)$ **or** $y = f(x, z)$ **or** $z = f(x, y)$ so, we can write partial derivative notation)

Step II :

$$\therefore \left(\frac{\partial z}{\partial y}\right)_x\cdot\left(\frac{\partial x}{\partial z}\right)_y\cdot\left(\frac{\partial y}{\partial x}\right)_z =$$

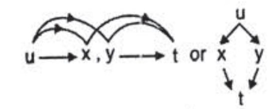
$$-\left[\frac{\cancel{\frac{\partial\phi}{\partial y}}}{\cancel{\frac{\partial\phi}{\partial z}}}\right]\left[-\frac{\cancel{\frac{\partial\phi}{\partial z}}}{\cancel{\frac{\partial\phi}{\partial x}}}\right]\left[-\frac{\cancel{\frac{\partial\phi}{\partial x}}}{\cancel{\frac{\partial\phi}{\partial y}}}\right] = -1 \checkmark \quad \textbf{Hence the result.}$$

Example 4.11.7

If $u = \sin\left(\frac{x}{y}\right)$ where $x = e^t$; $y = t^2$ find $\frac{du}{dt}$.

Solution :

Step I : Given : $u = \sin\left(\frac{x}{y}\right)$; $x = e^t$; $y = t^2$



$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \quad \text{...(1)}$$

Step II : $\therefore \frac{du}{dx} = \cos\left(\frac{x}{y}\right)\cdot\left(\frac{1}{y}\right)\cdot(e^t)+\cos\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)\cdot(2t)$

→ **Using standard results :**

... [$d\ \sin\cdot f(x) = \cos f(x)\cdot d(f(x))$ and $d(\cos f(x)) = -\sin f(x)\cdot d(f(x))$]

$$\therefore \frac{du}{dt} = \frac{1}{y}\cos\left(\frac{x}{y}\right)e^t-\frac{x}{y^2}\cos\left(\frac{x}{y}\right)(2t)$$

$$= \cos\frac{x}{y}\left[\frac{e^t}{y}-\frac{2xt}{y^2}\right]$$

Put, $x = e^t$; $y = t^2$

$$= \cos\left(\frac{e^t}{t^2}\right)\left[\frac{e^t}{t^2}-\frac{2e^t t}{t^4}\right]$$

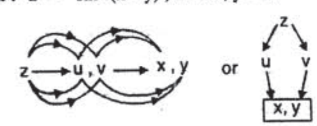
$$\therefore \frac{du}{dt} = \frac{e^t}{t^2}\cdot\cos\left(\frac{e^t}{t}\right)\left[1-\frac{2}{t}\right] \checkmark \quad \text{...Ans.}$$

Example 4.11.8

If $z = \sin^{-1}(x-y)$; $x = 3t$; $y = 4t^3$, then prove tha $\frac{dz}{dt} = \frac{3}{\sqrt{1-t^2}}$

Solution :

Given : $z = \sin^{-1}(x-y)$; $x = 3t$; $y = 4t^3$



Differentiate z w.r.t. u, by chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

→ **Using standard result :**

$$\ldots\left[\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}} \text{ and } d(x^n)=n\,x^{n-1}\right]$$

$$\frac{dz}{dt} = \frac{\partial}{\partial x}(\sin^{-1}(x-y))+\frac{\partial}{\partial y}\cdot\sin^{-1}(x-y)\frac{d}{dt}(4t^3)$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}\cdot(3)+\frac{1}{\sqrt{1-(x-y)^2}}(-1)\cdot(12t^2)$$

$$= \frac{3-12t^2}{\sqrt{1-(x^2+y^2-2xy)}} = \frac{3(1-4t^2)}{\sqrt{1-9t^2-16t^6+24t^4}}$$

($\because$ Put x ; y values)

$$= \frac{3(1-4t^2)}{\sqrt{1-8t^2-t^2-16t^6+16t^4+8t^4}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2-t^2(1+16t^4-8t^2)}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2-t^2(1-4t^2)}}$$

$$= \frac{3\cancel{(1-4t^2)}}{\cancel{(1-4t^2)}\sqrt{1-t^2}} = \frac{3}{\sqrt{1-t^2}} \checkmark$$ ...Ans.

Example 4.11.9

If y log cos x = x (log sin y), find $\frac{dy}{dx}$

Solution :

Given : Let, $f = y\log\cos x = x\log\sin y$ ...(1)

Here, differentiate f w.r.t. x and y, using standard result $d(f(g(x))) = f'(g(x))\cdot\frac{d}{dx}g(x)$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\left[\frac{y\frac{1}{\cos x}(-\sin x)-\log(\sin y)}{\log\cos x-\frac{x}{\sin y}\cdot\cos y}\right]$$

$$= \frac{y\tan x+\log\sin y}{\log\cos x - x\cot y}$$

Example 4.11.10

If $u = \tan^{-1}\left(\frac{x}{y}\right)$ where $x^2+y^2=a^2$, find $\frac{du}{dx}$.

Solution :

Step I : Given : $u = \tan^{-1}\left(\frac{x}{y}\right)$

Let, $f \equiv x^2+y^2-a^2=0$

Here, $u \to x,\ y \to x$

or $u \longrightarrow x\ y \longrightarrow x$

Step II : $$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx}+\frac{\partial u}{\partial y}\frac{dy}{dx}$$ ...(1)

Differentiate 'u' w.r.t. x and y,

→ **Using standard result :**

$$\ldots\left[\frac{d}{dx}\tan^{-1}(f(x))=\frac{1}{1+[f(x)]^2}f'(x)\right]$$

$$\frac{\partial u}{\partial x} = \underbrace{\frac{\partial}{\partial x}\tan^{-1}\left(\frac{x}{y}\right)}_{\frac{1}{1+\left(\frac{x}{y}\right)^2}}\cdot\underbrace{\frac{\partial}{\partial x}\left(\frac{x}{y}\right)}_{\frac{1}{y}}$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y} = \frac{y}{x^2+y^2}$$ ...(2)

$$\frac{\partial u}{\partial y} = \underbrace{\frac{\partial}{\partial y}\tan^{-1}\left(\frac{x}{y}\right)}_{\frac{1}{1+\left(\frac{x}{y}\right)^2}}\cdot\underbrace{\frac{\partial}{\partial y}\left(\frac{x}{y}\right)}_{\left(-\frac{x}{y^2}\right)}$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(-\frac{x}{y^2}\right) = -\frac{x}{x^2+y^2}$$ ...(3)

$\therefore$ We know, $$\frac{dy}{dx} = -\left[\frac{\partial f/\partial x}{\partial f/\partial y}\right] = -\left[\frac{+2x}{2y}\right] = \frac{-x}{y}$$ ...(4)

($\because$ by differentiating f w.r.t. x and y)

Step III : By using Equations (2), (3) and (4), Equation (1) becomes

$$\frac{du}{dx} = \frac{y}{x^2+y^2} - \frac{x}{x^2+y^2}\left(\frac{-x}{y}\right)$$

$$= \frac{y\cdot y}{y(x^2+y^2)} + \frac{x^2}{(x^2+y^2)}$$

$$\frac{du}{dx} = \frac{\cancel{(y^2+x^2)}}{\cancel{(x^2+y^2)}\,y} = \frac{1}{y} \checkmark$$ ...Ans.

Example 4.11.11

If f(x, y) = 0 ; ϕ(x, z) = 0 then prove that :

$$\frac{\partial\phi}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\frac{dy}{dz} = \frac{\partial f}{\partial x}\cdot\frac{\partial\phi}{\partial z}$$

Solution :

Step I : Given : implicit function,

$f(x, y) = 0$ and $\phi(x, z) = 0$

$\therefore$ $f \to x,\ y \to x$ and $\phi \to x,\ z \to x$

$f \longrightarrow x, y$ and $\phi \longrightarrow x, z$

Step II : Since, by chain rule

$$df = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

and $$d\phi = \frac{\partial\phi}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial\phi}{\partial z}\cdot\frac{dz}{dx}$$

Since, $f=0,\ \phi=0 \Rightarrow df=0\ ;\ d\phi=0$

$$\therefore \frac{dy}{dx} = -\left[\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right]$$

and $$\frac{dz}{dx} = -\left[\frac{\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial z}}\right]$$

Step III : $\therefore$ $$\frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dz}{dx}\right)} = \frac{-\left[\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right]}{-\left[\frac{\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial z}}\right]};$$

Here, $y \longrightarrow x \longrightarrow z$ Therefore, $$\frac{dy}{dz} = \frac{\frac{\partial f}{\partial x}\cdot\frac{\partial\phi}{\partial z}}{\frac{\partial\phi}{\partial x}\cdot\frac{\partial f}{\partial y}}$$

$$\Rightarrow \frac{\partial\phi}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\frac{dy}{dz} = \frac{\partial f}{\partial x}\cdot\frac{\partial\phi}{\partial z} \checkmark$$

Hence the result.

Example 4.11.12

If $x^x\cdot y^y\cdot z^z = c$ then show that $\frac{\partial^2 z}{\partial x\partial y} = -(x\log e x)^{-1}$ at the point $x = y = z$.

Solution :

Step I : $\because$ Here, $\frac{\partial^2 z}{\partial x\partial y}$ indicates that z is an implicit function of x and y

$z \longrightarrow x, y$ or

Given : $x^x\cdot y^y\cdot z^z = c$

Taking log on both sides,

$$\log(x^x\cdot y^y\cdot z^z) = \log c$$

$$\Rightarrow \log x^x+\log y^y+\log z^z = \log c$$

Use : $\log(ABC) = \log A+\log B+\log C$

and $\log a^b = b\log_a$

$$\therefore x\log x+y\log y+z\log z = \log c$$ ...(1)

Step II : Differentiate Equation (1) w.r.t x keeping y as constant and remember z is a function of x and y.

$$\therefore x\cdot\frac{1}{x}+\log x+0+z\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x}+\log z\cdot\frac{\partial z}{\partial x}=0$$

$$\Rightarrow [1+\log x]+(1+\log z)\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\left(\frac{1+\log x}{1+\log z}\right)$$ ...(2)

Similarly, we get

$$\frac{\partial z}{\partial y} = -\left(\frac{1+\log y}{1+\log z}\right)$$ ...(3)

Step III : Differentiate $\frac{\partial z}{\partial y}$ w.r.t. x keeping y constant and remember that z is a function of x and y

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[-\left(\frac{1+\log y}{1+\log z}\right)\right]$$

$$= -(1+\log y)\cdot\left[\frac{-1\left(\frac{1}{z}\cdot\frac{\partial z}{\partial x}\right)}{(1+\log z)^2}\right]$$

→ **Using standard rule of derivatives :**

$$\ldots\left[d\left(\frac{1}{x}\right)d = \frac{-1}{x^2}\right]$$

Here, x means $(1+\log z)$

$$= \frac{(1+\log y)}{z(1+\log z)^2}\left[\frac{-(1+\log x)}{(1+\log z)}\right]$$

... From Equation (2)

$$\therefore \frac{\partial^2 z}{\partial x\partial y} = -\frac{(1+\log x)(1+\log y)}{z(1+\log z)^3}$$

Step IV : At $x = y = z$

Put $x = x\ \ y = x$ and $z = x$ in above equation

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log x)(1+\log x)}{x(1+\log x)^3} = -\frac{1}{x(1+\log x)}$$

$$= -\frac{1}{x[\log e + \log x]} = -\frac{1}{x \log e x}$$

($\because$ Use : $\log(a) + \log(b) = \log(ab)$)

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log e x)^{-1}$$

Example 4.11.13

If $x^3 y - \sin z + z^3 = 0$ then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution :

Step I : Given : $x^3 y - \sin z + z^3 = 0$ ...(1)

Since $z \to x, y$ or

The given Partial derivatives indicates that z is an implicit function of x and y

Step II : $\therefore$ Differentiating Equation (1) partially, w.r.t. x keeping y as constant and z is a function of x and y

$$3x^2 y - \cos z \frac{\partial z}{\partial x} + 3z^2 \frac{\partial z}{\partial x} = 0$$

$$\therefore 3x^2 y = [\cos z - 3z^2] \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{3x^2 y}{(\cos z - 3z^2)} \quad ...(2)$$

Step III : Similarly, differentiate Equation (1) partially w.r.t. y keeping x as constant and z is a function of x and y.

$$x^3 - \cos z \frac{\partial z}{\partial y} + 3z^2 \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow x^3 = (\cos z - 3z^2) \frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{x^3}{(\cos z - 3z^2)} \quad ...(3)$$

Exercise 3

Ex. 1 If $u = x^2 + y^2 + z^2$, and $x = e^{2t}$, $y = e^{2t} \cdot \cos 3t$, $z = e^{2t} \cdot \sin 3t$ then find $\frac{du}{dt}$. **(Ans. : $8e^{4t}$)**

Ex. 2 If $u = x^2 y$ then find $\frac{du}{dx}$ when x, y are connected by $x^2 + xy + y^2 = 1$

Ex. 3 Find $\frac{du}{dx}$ if $u = \sin(x^2 + y^2)$, where $a^2 x^2 + b^2 y^2 = c^2$

Ex. 4 Find $\frac{dy}{dx}$, for (i) $x^y + y^x = c$ (ii) $(\cos x)^y = (\sin y)^x$

Ex. 5 If $f(x, y) = 0$, $\phi(y, z) = 0$ then show that $\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$

Ex. 6 If u is a function of x and y, where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Ex. 7 If $z = \log(u^2 + v)$, $u = e^{x^2+y^2}$, $v = x^2 + y$ then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Ex. 8 If $V = f(r, s)$ and $r = x\cos\theta - y\sin\theta$; $s = x\sin\theta + y\cos\theta$, show that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = r\frac{\partial v}{\partial r} + s\frac{\partial v}{\partial s}$

Ex. 9 If $z = f(x, y)$, $x = u\cosh v$, $y = u\sinh v$, prove that $\left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$

Ex. 10 If $u = f(v, w)$ where, $v = x^2 + y^2$; $w = 2xy$, then prove that, $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 2\sqrt{v^2 - w^2}\left(\frac{\partial u}{\partial v}\right)$

Ex. 11 If z is a function of x and y, where $x = e^u \cos v$, $y = e^u \sin v$ then prove that

(i) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

Ex. 12 If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, then find $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z}$ **Ans. : 0**

Ex. 13 If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0$

Ex. 14 If $w = f(x^2 + 2yz, y^2 + 2zx)$, prove that $(y^2 - zx)\frac{\partial w}{\partial x} + (x^2 - yz)\frac{dw}{dy} + (z^2 - xy)\frac{\partial w}{\partial z} = 0$

Ex. 15 If $u = f(x^m - y^n, y^n - z^n, z^n - x^n)$, then find $\frac{1}{x^{n-1}}\frac{\partial u}{\partial x} + \frac{1}{y^{n-1}}\frac{\partial u}{\partial y} + \frac{1}{z^{n-1}}\frac{\partial u}{\partial z}$ **Ans. : 0**

Ex. 16 If $z = f(u, v)$, where $u = lx + my$, $V = ly - mx$, prove that $\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial y^2}\right)$

Ex. 17 If $w = z\sin^{-1}\left(\frac{y}{x}\right)$, where $x = 3u^2 + 2v$; $y = 4u - 2v^3$; $z = 2u^2 - 3v^2$

Find $\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} + \frac{\partial w}{\partial z}$

4.12 Homogeneous Functions

Introduction

Sometime we come across the functions which are having same degree terms. It is quite difficult to find partial derivatives of such functions. Euler's theorem on homogenous function in two and three variables or more can overcome this difficulty. In this chapter we will learn Euler's theorem and its deductions on homogenous functions in two or more variables.

4.13 Definition of Homogeneous Functions

A polynomial in x and y is said to be homogeneous function of degree n in x and y if all its terms are of the same degree n (i..e the sum of the indices of x and y in each term is same).

Generalizing this property for non-polynomials, consider a function of f (x, y) in two variables or function of f (x, y, z) in three variables. The result is again extended for functions of several variables.

An expression of the form,

$$u = f(x, y)$$
$$= a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + ... + a_n y^n \quad ...(4.13.1)$$

is called a homogeneous expression in x and y of degree 'n'.

$\therefore$ Equation (4.13.1) can be written as,

$$u = x^n \left[a_0 + a_1\left(\frac{y}{x}\right)^1 + a_2\left(\frac{y}{x}\right)^2 + ... + a_n\left(\frac{y}{x}\right)^n\right]$$

$$\Rightarrow \mathbf{u = x^n \phi\left(\frac{y}{x}\right)}$$

OR From Equation (4.13.1),

$$u = y^n\left[a_0\left(\frac{x}{y}\right)^n + a_1\left(\frac{x}{y}\right)^{n-1} + a_2\left(\frac{x}{y}\right)^{n-2} + ... + a_n\right]$$

$$\Rightarrow \mathbf{u = y^n \phi\left(\frac{x}{y}\right)}$$

OR By putting $x = xt$, $y = yt$ in Equation (10.2.1)

$u = [a_0 x^n t^n + a_1 x^{n-1} t^{n-1} yt + a_2 x^{n-2} t^{n-2} y^2 t^2 + ... + a_n y^n t^n]$

$= [a_0 x^n t^n + a_1 x^{n-1} t^n y^1 + a_2 x^{n-2} y^2 t^n + ... + a_n y^n t^n]$

$= t^n [a_0 x^n + a_1 x^{n-1} y^1 + a_2 x^{n-2} y^2 + ... + a_n y^n]$

$$u = t^n f(x, y) = t^n u$$

Therefore, we can define homogenous function as follows :

A function $u = f(x, y)$ is said to be homogeno function of degree n in x and y if

$$u = x^n \phi\left(\frac{y}{x}\right) \text{ or } y^n \phi\left(\frac{x}{y}\right)$$

OR by putting, $x = xt$, $y = yt$, we get, $u = u t^n$

The same method can be extended for a functions of mo than two variables.

Note : Here, n need not be an integer, n may be positi real number, negative real number or 0.

Ex. 1 $u = x^3 y - 3x^2 y^2 + 7xy^3 + 5y^4$ is homogeneous function of degree n = 4

Ex. 2 $u = \cos^{-1}\left(\frac{y}{x}\right)$ is an homogeneous function in and y of degree n = 0

$$u = x^0 \cos^{-1}\left(\frac{y}{x}\right)$$

Ex. 3 $u = \frac{\sqrt{x} + \sqrt{y}}{x^2 + y^2}$; Put $x = xt$, $y = yt$

$$u = \frac{\sqrt{xt} + \sqrt{yt}}{(xt)^2 + (yt)^2} = \frac{t^{1/2}}{t^2}\left[\frac{\sqrt{x} + \sqrt{y}}{x^2 + y^2}\right]$$
$$= t^{-3/2} u$$

is a homogeneous function of degree n = – 3/2 x and y.

Ex. 4 Here, the function, $u = \cos^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$ is n homogeneous function because, by putting x = y = yt, not getting $t^n u$ form

$$u = \cos^{-1}\left[\frac{t^2(x^2 + y^2)}{t(x + y)}\right] = \cos^{-1}\left[\frac{t(x^2 + y^2)}{(x + y)}\right] \neq u$$

But, $\cos(u) = t'\left(\frac{x^2 + y^2}{x + y}\right)$ is homogeneo function of degree n = 1 in x and y.

$\therefore u = \cos^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$ is not homogeneous function but it is function of homogenous expression.

Ex. 5 $u = \tan^{-1}\left(\frac{x - y}{x^2 + y^2}\right)$ is also not a homogene function but it is function of homogene expression of degree n = – 1.

$$\left(\because f(u) = \tan u = \frac{1}{t}\left(\frac{x - y}{x^2 + y^2}\right) = t^{-1} \cdot \left(\frac{x - y}{x^2 + y^2}\right)\right)$$

Note : Always alert about the homogeneous functio and a function of homogeneous expression in and y.

4.14 Euler's Theorem for Homogeneous Functions

Euler's Theorem on Homogeneous Function of two variables

Statement

If u is a homogenous function of degree n in x and y then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu$$

Proof : Given, u is a homogenous function in x and y of degree n

$\therefore$ By definition of homogeneous function,

$$u = x^n\,\phi\left(\frac{y}{x}\right) \qquad \ldots(1)$$

$$u \to x, y$$

Differentiate u partially w.r.t. x, keeping y as constant

➔ **Using standard rules of derivatives :** $\ldots[\,d(fg) = f\,dg + g\,df]$

Here, $f = x^n$ and $g = \phi\left(\frac{y}{x}\right)$

$$\therefore \frac{\partial u}{\partial x} = n\,x^{n-1}\,\phi\left(\frac{y}{x}\right)+x^n\cdot\phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

Multiplying both sides by x

$$x\frac{\partial u}{\partial x} = n\cdot x^n\,\phi\left(\frac{y}{x}\right) - x^{n-1}\cdot y\,\phi'\left(\frac{y}{x}\right) \qquad \ldots(2)$$

Differentiate u partially w.r.t. y, keeping x as constant

$$\frac{\partial u}{\partial y} = x^n\,\phi'\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right)$$

Multiplying both sides by y

$$\therefore\; y\frac{\partial u}{\partial y} = x^{n-1}\cdot y\,\phi'\left(\frac{y}{x}\right) \qquad \ldots(3)$$

Adding Equation (2) and (3) we get,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = n\,\underbrace{x^n\phi\left(\frac{y}{x}\right)}_{u} - \cancel{x^{n-1}}\cdot y\,\phi'\left(\frac{y}{x}\right) + \cancel{x^{n-1}}\cdot\phi'\left(\frac{y}{x}\right)$$

$$= nu \qquad \ldots\text{From Equation (1)}$$

Hence, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu$

In general : If u is a homogenous function in x, y, z, … of degree n then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}+\ldots\ldots = nu$$

Type I : Examples Based on Euler's Theorem on Homogeneous Function

Example 4.14.1

Verify Euler's theorem on homogeneous function; when $u = \log\left(\frac{xy+yz}{x^2+y^2+z^2}\right)$

Solution :

Step I : Given : $u = \log\left(\frac{xy+yz}{x^2+y^2+z^2}\right) \qquad \ldots(1)$

Put, $x = xt$; $y = yt$; $z = zt$ in u, we get,

$$u = \log\left[\frac{(xt)(yt)+(yt)(zt)}{(xt)^2+(yt)^2+(zt)^2}\right] = t^0\cdot\underbrace{\log\left(\frac{xy+yz}{x^2+y^2+z^2}\right)}_{u}$$

$$= t^0 u$$

This shows that u is homogeneous function in x, y, z of degree n = 0.

Step II : Therefore, by Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = nu = 0 \qquad \ldots(2)$$

Step III : Now, from Equation (1),

$$u = \log(xy+yz) - \log(x^2+y^2+z^2) \ldots(3)$$

Differentiate Equation (3) partially w.r.t. x, keeping y, z as constant.

➔ **Using standard rules of derivatives :** $\ldots\left[\frac{d}{dx}[\log f(x)] = \frac{1}{f(x)}\cdot\frac{d}{dx}f(x)\right]$

$$\frac{\partial u}{\partial x} = \frac{1}{xy+yz}\cdot\underbrace{\frac{\partial}{\partial x}(xy-yz)}_{y} - \frac{1}{x^2+y^2+z^2}\underbrace{\frac{\partial}{\partial x}(x^2+y^2+z^2)}_{(2x)},$$

$$= \frac{y}{xy+yz} - \frac{1\cdot 2x}{x^2+y^2+z^2};$$

Multiplying both sides by x

$$x\frac{\partial u}{\partial x} = \frac{xy}{xy+yz} - \frac{2x^2}{x^2+y^2+z^2} \qquad \ldots(4)$$

Differentiate Equation (3) partially w.r.t. y, keeping x and z as constant.

$$\frac{\partial u}{\partial y} = \frac{1}{(xy+yz)}\underbrace{\frac{\partial}{\partial y}(xy+yz)}_{(x+z)} - \frac{1}{x^2+y^2+z^2}\underbrace{\frac{\partial}{\partial y}(x^2+y^2+z^2)}_{2y}$$

$$= \frac{(x+z)}{xy+yz} - \frac{1\cdot 2y}{x^2+y^2+z^2};$$

Multiplying both sides by y

$$y\frac{\partial u}{\partial y} = \frac{yx+yz}{xy+yz} - \frac{2y^2}{x^2+y^2+z^2} \qquad \ldots(5)$$

and differentiate Equation (3) partially w.r.t. z, keeping x and y as constant.

$$\frac{\partial u}{\partial z} = \frac{1}{xy+yz}\underbrace{\frac{\partial}{\partial z}(xy+yz)}_{y} - \frac{1}{x^2+y^2+z^2}\underbrace{\frac{\partial}{\partial z}(x^2+y^2+z^2)}_{2z}$$

$$= \frac{y}{xy+yz} - \frac{1\cdot 2z}{x^2+y^2+z^2};$$

Multiplying both sides by z

$$z\frac{\partial u}{\partial z} = \frac{zy}{xy+yz} - \frac{2z^2}{x^2+y^2+z^2} \qquad \ldots(6)$$

Step IV : Adding Equations (4), (5) and (6), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = \left(\frac{\cancel{xy+yz}}{\cancel{xy+yz}}+1\right) - 2\frac{\cancel{(x^2+y^2+z^2)}}{\cancel{(x^2+y^2+z^2)}}$$

$$= 2-2 = 0 \qquad \ldots(7)$$

Hence, from Equations (2) and (7), it is clear that, **Euler's theorems is verified.**

Example 4.14.2

Verify Euler's theorem on homogeneous function for $u = (\sqrt{x}+\sqrt{y}+\sqrt{z})$

Solution :

Step I : Given : $u = (\sqrt{x}+\sqrt{y}+\sqrt{z}) \qquad \ldots(1)$

Put $x = xt$; $y = yt$; $z = zt$ in equation (1)

$$u = (\sqrt{xt}+\sqrt{yt}+\sqrt{zt})$$

$$= t^{1/2}\underbrace{(\sqrt{x}+\sqrt{y}+\sqrt{z})}_{u} = t^{1/2}u$$

Hence, u is homogeneous function in x, y, z of degree $n = \frac{1}{2}$.

Step II : Therefore, by Euler's theorem on homogenous function

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = nu = \frac{1}{2}(\sqrt{x}+\sqrt{y}+\sqrt{z}) \qquad \ldots(2)$$

Step III : Now, differentiate Equation (1) partially w.r.t. x, y and z

➔ **Using standard formula :** $\ldots\left[\,d(\sqrt{x}) = \frac{1}{2\sqrt{x}}\right]$

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x}} \Rightarrow x\frac{\partial u}{\partial x} = \frac{x}{2\sqrt{x}} = \frac{\sqrt{x}\cancel{\sqrt{x}}}{2\cancel{\sqrt{x}}} = \frac{1}{2}\sqrt{x}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{y}} \Rightarrow y\frac{\partial u}{\partial y} = \frac{y}{2\sqrt{y}} = \frac{\sqrt{y}\cancel{\sqrt{y}}}{2\cancel{\sqrt{y}}} = \frac{1}{2}\sqrt{y}$$

$$\text{and } \frac{\partial u}{\partial z} = \frac{1}{2\sqrt{z}} \Rightarrow z\frac{\partial u}{\partial z} = \frac{z}{2\sqrt{z}} = \frac{\sqrt{z}\cancel{\sqrt{z}}}{2\cancel{\sqrt{z}}} = \frac{1}{2}\sqrt{z}$$

Step IV : Adding, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$

$$= \frac{1}{2}(\sqrt{x}+\sqrt{y}+\sqrt{z}) \qquad \ldots(3)$$

From Equations (2) and (3), it is clear that **Euler's theorem is verified.**

Example 4.14.3

Verify Euler's theorem on homogeneous function for $u = x^4y^2\sin^{-1}\left(\frac{y}{x}\right)$.

Solution :

Step I : Given : $u = x^4y^2\sin^{-1}\left(\frac{y}{x}\right) \qquad \ldots(1)$

Put, $x = xt$; $y = yt$; $z = zt$ in u,

$$u = (xt)^4(yt)^2\sin^{-1}\left(\frac{yt}{xt}\right)$$

$$= t^6\underbrace{x^4y^2\sin^{-1}\left(\frac{y}{x}\right)}_{u} = t^6u$$

Hence, u is homogeneous function of x and y of degree n = 6

Step II : Therefore, by Euler's theorem on homogenous function

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu = 6x^4y^2\sin^{-1}\left(\frac{y}{x}\right) \qquad \ldots(2)$$

Step III : Now, differentiate Equation (1) partially, w.r.t. x keeping y as constant.

→ **Using standard rule of derivatives**

$$\ldots\left[\frac{d}{dx}\{f(x)\cdot g(x)\} = f(x)\cdot\frac{d}{dx}g(x) + g(x)\cdot\frac{d}{dx}f(x)\right]$$

And $[d[x]^n = n(x)^{n-1}]$

$$\frac{\partial u}{\partial x} = y^2\left[x^4\cdot\frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\cdot\left(-\frac{y}{x^2}\right) + 4x^3\sin^{-1}\left(\frac{y}{x}\right)\right]$$

$$\left(\because \frac{d}{dx}\sin^{-1}(f(x)) = \frac{1}{\sqrt{1-(f(x))^2}}\cdot\frac{d}{dx}f(x)\right)$$

Multiplying both sides by x

$$x\cdot\frac{\partial u}{\partial x} = y^2\left[\frac{x^4\cdot \cancel{x}\cdot \cancel{x}}{\sqrt{x^2-y^2}}\left(-\frac{y}{x^2}\right) + 4x^4\sin^{-1}\left(\frac{y}{x}\right)\right]$$

$$= y^2\left[4x^4\sin^{-1}\left(\frac{y}{x}\right) - \frac{y\cdot x^4}{\sqrt{x^2-y^2}}\right] \quad \ldots(3)$$

and Differentiate u w.r.t. y, partially, keeping x as constant.

→ **Using standard rules of derivatives**

$$\ldots\left[\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\frac{d}{dx}g(x) + g(x)\cdot\frac{d}{dx}f(x)\right]$$

$$\frac{\partial u}{\partial y} = x^4\left[y^2\cdot\frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\cdot\left(\frac{-1}{x}\right) + 2y\sin^{-1}\left(\frac{y}{x}\right)\right]$$

$$\left(\because \frac{d}{dx}\sin^{-1}(f(x)) = \frac{1}{\sqrt{1-(f(x))^2}}\cdot\frac{d}{dx}f(x)\right)$$

Multiplying both sides by y

$$y\frac{\partial u}{\partial y} = \frac{x^4y^3}{\sqrt{x^2-y^2}} + 2y^2x^4\sin^{-1}\left(\frac{y}{x}\right) \quad \ldots(4)$$

Step IV : Adding Equations (3) and (4),

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 6x^4y^2\sin^{-1}\left(\frac{y}{x}\right) \quad \ldots(5)$$

From Equations (2) and (5), **Euler's theorem is verified.**

Example 4.14.4

Verify Euler's theorem for homogeneous functions :
$f(x, y, z) = 3x^2yz + 5xy^2z + 4z^4$

Solution :

Step I : Given : $f(x, y, z) = 3x^2yz + 5xy^2z + 4z^4 \quad \ldots(1)$

Obviously, it is homogeneous function of degree $n = 4$. (Put $x = xt,\ y = yt,\ z = zt$)

$\therefore$ By Euler's theorem on homogeneous function,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf = 4f \quad \ldots(2)$$

Step II : Now, differentiate Equation (1) partially, w. r. t x, y, z respectively,

$$\frac{\partial f}{\partial x} = 6xyz + 5y^2z$$

Multiplying both sides by x

$$\Rightarrow x\frac{\partial f}{\partial x} = 6x^2yz + 5y^2xz$$

$$\frac{\partial f}{\partial y} = 3x^2z + 10xyz$$

Multiplying both sides by y

$$\Rightarrow y\frac{\partial f}{\partial y} = 3x^2yz + 10xy^2z$$

and $\frac{\partial f}{\partial z} = 3x^2\cdot y + 5xy^2 + 16z^3$

Multiplying both sides by z

$$\Rightarrow z\frac{\partial f}{\partial z} = 3x^2yz + 5xy^2z + 16z^4$$

Step III : Adding, we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = (6x^2yz + 5y^2xz)$$
$$+ (3x^2yz + 10xy^2z) + (3x^2yz + 5xy^2z + 16z^4)$$
$$= 12x^2yz + 20xy^2z + 16z^4$$
$$= 4\underbrace{[3x^2yz + 5xy^2z + 4z^4]}_{} = 4f$$

...from Equations (1) to (3)

Hence, from equations (2) and (3) it is clear that **Euler's theorem is verified.**

Example 4.14.5

Find the value of : $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$, where $u = \frac{x^3y^3z^3}{x^3+y^3+z^3}$

Solution :

Step I : Given, $u = \frac{x^3y^3z^3}{x^3+y^3+z^3}$

Put, $x = xt, y = yt, z = zt$

$$\therefore u = \frac{(tx)^3(ty)^3(zx)^3}{(xt)^3+(ty)^3+(tz)^3}$$

$$= \frac{t^9}{t^3}\underbrace{\left[\frac{x^3\cdot y^3\cdot z^3}{x^3+y^3+z^3}\right]}_{u}$$

$$\Rightarrow u = \frac{t^9}{t^3}u$$

$$u = t^6u$$

This shows that u is homogeneous function in x and y of degree $n = 6$

$\therefore$ By Euler's theorem, on homogeneous function

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 6\left(\frac{x^3y^3z^3}{x^3+y^3+z^3}\right) \checkmark \quad \ldots\text{Ans.}$$

Hence proved.

Example 4.14.6

Find the value of :

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ **if** $u = \frac{\sqrt{xy}}{\sqrt{x}+\sqrt{y}}$

(ii) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ **if** $u = \frac{\sqrt{x}+\sqrt{y}}{xy}$

Solution :

Step I :

(i) Given : $u = \frac{\sqrt{xy}}{\sqrt{x}+\sqrt{y}}$

Put $x = xt;\ y = yt;$

we get, $u = \frac{\sqrt{(xt)(yt)}}{\sqrt{xt}+\sqrt{yt}} = \frac{t\sqrt{xy}}{t^{1/2}(\sqrt{x}+\sqrt{y})} = t^{1/2}\cdot u$

This shows that u is homogeneous function in x and y of degree $n = \frac{1}{2}$.

Step II : By Euler's theorem on homogeneous function

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = \frac{1}{2}\left(\frac{\sqrt{xy}}{\sqrt{x}+\sqrt{y}}\right)$$

Step III : (ii) Given : $u = \frac{\sqrt{x}+\sqrt{y}}{xy}$

Put $x = xt;\ y = yt \Rightarrow u = \frac{\sqrt{xt}+\sqrt{yt}}{(xt)(yt)}$

$$u = t^{-3/2}\underbrace{\left(\frac{\sqrt{x}+\sqrt{y}}{xy}\right)}_{u} = t^{-3/2}\cdot u$$

This shows that u is homogenous function of degree, $n = -\frac{3}{2}$ in x and y

Step IV : By Euler's theorem an homogenous function

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = -\frac{3}{2}\left(\frac{\sqrt{x}+\sqrt{y}}{xy}\right) \checkmark \quad \ldots\text{Ans.}$$

Type II : Examples Based on Euler's Theorem on Homogeneous Function for Three Variables

Example 4.14.7

If $z = x^4y^2\sin^{-1}\left(\frac{x}{y}\right) + \log x - \log y$ then evaluate :
$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$

Solution :

Step I : Given : $z = x^4y^2\sin^{-1}\left(\frac{x}{y}\right) + \log x - \log y$

→ **Use : $\log A - \log B = \log\left(\frac{A}{B}\right)$**

$$= x^4y^2\sin^{-1}\left(\frac{x}{y}\right) + \log\left(\frac{x}{y}\right)$$

$$z = u + v \quad \ldots(1)$$

Step II : Differentiate z w.r.t. x and y

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$$

and $\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \ldots(2)$

Step III : Now,

Here, $u = x^4y^2\sin^{-1}\left(\frac{x}{y}\right)$

Put, $x = xt,\ y = yt$ in u

$$u = (xt)^4(yt)^2\sin^{-1}\left(\frac{xt}{yt}\right)$$

$$= \underbrace{t^6x^4y^2\sin^{-1}\left(\frac{x}{y}\right)}_{u} \qquad \therefore u = t^6u$$

Hence, u is homogeneous function in x and y of degree $n = 6$

Step IV : By Euler's theorem for homogeneous function

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 6u \quad \ldots(3)$$

Step V : and $v = \log\left(\frac{x}{y}\right)$

Put, $x = xt, y = yt$ in v

$$v = \log\left(\frac{xt}{yt}\right) = \underbrace{t^0\log\left(\frac{x}{y}\right)}_{v} = t^0v$$

Hence, v is also homogeneous function in x and y of degree $n = 0$

$\therefore$ By Euler's theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y} = nv = 0 \qquad ...(4)$$

Step VI : Adding Equations (3) and (4), we get

$$x\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\right)+y\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\right) = (6u+0) = 6u$$

and by using Equation (2), we get

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = 6u = 6\cdot x^4 y^2 \sin^{-1}\left(\frac{x}{y}\right) \checkmark \text{ ...Ans.}$$

Example 4.14.8

If $u=\frac{x^2y^2z^2}{x^2+y^2+z^2}+\cos\left(\frac{xy+yz}{x^2+y^2+z^2}\right)$ then show that :

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=4\left(\frac{x^2y^2z^2}{x^2+y^2+z^2}\right)$$

Solution :

Step I : Given : $u=\frac{x^2y^2z^2}{x^2+y^2+z^2}+\cos\left(\frac{xy+yz}{x^2+y^2+z^2}\right)$

$$\therefore \quad u = v+w \qquad ...(1)$$

Here, $v = \frac{x^2y^2z^2}{x^2+y^2+z^2}$

Put $x = xt, y = yt, z = zt$ in v

$$v = \frac{(xt)^2(yt)^2(zt)^2}{(xt)^2+(yt)^2+(zt)^2}$$

$$= \underbrace{\frac{t^6}{t^2}\left[\frac{x^2y^2z^2}{x^2+y^2+z^2}\right]}_{v} = t^4 v$$

$\Rightarrow$ v is homogeneous functions in x, y and z of degree n = 4.

Step II : $\therefore$ By Euler's theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial z}+z\frac{\partial v}{\partial z} = nv = 4\left(\frac{x^2y^2z^2}{x^2+y^2+z^2}\right) \qquad ...(2)$$

Step III : and $w = \cos\left(\frac{xy+yz}{x^2+y^2+z^2}\right)$

put $x = xt, y = yt, z = zt$ in w

$$\therefore w = \cos\left(\frac{(xt)(yt)+(yt)(zt)}{(xt)^2+(yt)^2+(zt)^2}\right)$$

$$= \underbrace{t^0\cos\left(\frac{xy+yz}{x^2+y^2+z^2}\right)}_{w} = t^0 w$$

This shows that w is homogeneous function in x, y, z of degree n = 0.

Step IV : $\therefore$ By Euler's theorem,

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial z}+z\frac{\partial w}{\partial z} = nw = 0 \qquad ...(3)$$

Step V : Adding Equations (2) and (3)

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)+z\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right)$$

$$= 4\left(\frac{x^2y^2z^2}{x^2+y^2+z^2}\right)+0$$

$$= 4\left(\frac{x^2y^2z^2}{x^2+y^2+z^2}\right) \qquad ...(4)$$

But, from Equation (1), $u = v + w$;

Differentiate Equation (1) partially,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}$$

$$; \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}+\frac{\partial w}{\partial y};$$

$$\text{and} \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z}+\frac{\partial w}{\partial z} \qquad ...(5)$$

Step VI : By using Equation (5), Equation (4) becomes

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=4\left[\frac{x^2y^2z^2}{x^2+y^2+z^2}\right] \checkmark$$

...Hence Proved

Example 4.14.9

If $T = \sin\left[\frac{xy}{x^2+y^2}\right]+\sqrt{x^2+y^2}+\frac{x^2y}{x+y}$ then find the value of $x\frac{\partial T}{\partial x}+y\frac{\partial T}{\partial y}$

Solution :

Step I : Given : $T = \sin\left[\frac{xy}{x^2+y^2}\right]+\sqrt{x^2+y^2}+\frac{x^2y}{x+y}$

$$T = U+V+W \qquad ...(1)$$

Where, $U = \sin\left(\frac{xy}{x^2+y^2}\right)$; $V = \sqrt{x^2+y^2}$;

$$W = \frac{x^2y}{x+y}$$

Put, $x = xt$, $y = yt$ in U, V, W, we get

$$U = \sin\left(\frac{xt\cdot yt}{(xt)^2+(yt)^2}\right);$$

$$V = \sqrt{(xt)^2+(yt)^2} \quad \text{and} \ ; \quad W = \frac{(xt)^2(yt)}{(xt)+(yt)}$$

We get,

$$U = \underbrace{t^0\sin\left(\frac{xy}{x^2+y^2}\right)}_{u} \ ; \ V = \underbrace{t\sqrt{x^2+y^2}}_{v} \ ;$$

$$W = \underbrace{t^2\left(\frac{x^2y}{x+y}\right)}_{w}$$

$$\Rightarrow \ U = t^0 U \ ; \quad V = tV \ ; \quad W = t^2 W$$

This means U, V, W are homogenous functions in x, y of degrees n = 0, 1, 2 respectively.

Step II : By Euler's theorem for homogenous functions

$$\left.\begin{aligned} x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y} &= nU = 0; \\ x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y} &= nV = 1V; \\ \text{and} \quad x\frac{\partial W}{\partial x}+y\frac{\partial W}{\partial y} &= nW = 2W; \end{aligned}\right\} \qquad ...(2)$$

Step III : Adding Equation (2),

$$x\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial x}+\frac{\partial W}{\partial x}\right)+y\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial y}\right)$$

$$= 0+1V+2W \qquad ...(3)$$

Step IV : But from (1), $T = U + V + W$

Differentiate T w.r.t x and y,

$$\frac{\partial T}{\partial x} = \frac{\partial U}{\partial x}+\frac{\partial V}{\partial x}+\frac{\partial W}{\partial x}$$

$$\text{and} \quad \frac{\partial T}{\partial y} = \frac{\partial U}{\partial y}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial y}$$

Equation (3) becomes, $x\frac{\partial T}{\partial x}+y\frac{\partial T}{\partial y} = V+2W$

$$= \left(\sqrt{x^2+y^2}\right)+2\left(\frac{x^2y}{x+y}\right)$$

Now at $(3, 4) \equiv (x, y)$

$$\therefore \quad x\frac{\partial T}{\partial x}+y\frac{\partial T}{\partial y} = (5)+2\left(\frac{36}{7}\right) = \frac{107}{7} \checkmark \quad \text{...Ans.}$$

Example 4.14.10

If $u=\frac{1}{x^2}+\frac{1}{y^2}+\frac{(\log x-\log y)}{x^2+y^2}$ then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+2u=0$

Solution :

Step I : Given : $u = \underbrace{\frac{1}{x^2}+\frac{1}{y^2}+\left[\frac{\log x-\log y}{x^2+y^2}\right]}_{\textit{Cross multiplication}}$

$$= \frac{x^2+y^2}{x^2y^2}+\frac{\log\left(\frac{x}{y}\right)}{x^2+y^2}$$

$$u = v+w \qquad ...(1)$$

Where, $v = \frac{x^2+y^2}{x^2y^2}$; $w = \frac{\log\left(\frac{x}{y}\right)}{x^2+y^2}$

Put $x = xt$, $y = yt$ in v, w we get

$$v = \frac{(xt)^2+(yt)^2}{(xt)^2(yt)^2} \ ; \quad w = \frac{\log\left(\frac{xt}{yt}\right)}{(xt)^2+(yt)^2}$$

$$v = \frac{t^2(x^2+y^2)}{t^4(x^2\cdot y^2)} \ ; \quad w = \frac{\log\left(\frac{x}{y}\right)}{t^2(x^2+y^2)}$$

$$v = t^{-2}\underbrace{\left(\frac{x^2+y^2}{x^2\cdot y^2}\right)}_{v}; \quad w = t^{-2}\underbrace{\frac{\log\left(\frac{x}{y}\right)}{t^2(x^2+y^2)}}_{w}$$

$$v = t^{-2}v \quad \text{and} \quad w = t^{-2}w$$

This shows that v and w are homogeneous functions in x, y of degree n = – 2.

Step II : $\therefore$ By Euler's theorem on homogeneous functions

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y} = nv = -2v$$

$$\text{and} \quad x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y} = nw = -2w$$

Adding these two equations, we get

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right) = -2v-2w$$

$$= -2(v+w) \qquad ...(2)$$

Step III : From Equation (1), $u = v + w$

Differentiate u partially w.r.t. x and y

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}+\frac{\partial w}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}$$

Step IV : Equation (2) becomes,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = -2u$$

$$\Rightarrow \ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+2u = 0$$

OR

Here, given, $u = \frac{1}{x^2}+\frac{1}{y^2}+\left[\frac{\log\left(\frac{x}{y}\right)}{x^2+y^2}\right]$

This shows that, u is a homogeneous function in x and y of degree n = – 2.

. By Eulers theorem on homogeneous function,

$y\frac{\partial u}{\partial y} = nu = -2u$ ✓ ...Hence Proved. ...Ans.

e 4.14.11

$\frac{}{u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, **where u is an**

eneous function of degree n in x, y, z then show

$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\,nu.$

n :

$$: \frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \qquad ...(1)$$

ntiate u partially w.r.t. x keeping y and z as constant,

$$\frac{\cdot 2x - x^2\frac{\partial u}{\partial x}}{(a^2+u)^2} - \frac{y^2}{(b^2+u)^2}\cdot\frac{\partial u}{\partial x} - \frac{z^2}{(c^2+u)^2}\cdot\frac{\partial u}{\partial x} = 0$$

$$\frac{(a^2+u)}{(a^2+u)^2} - \frac{x^2}{(a^2+u)^2}\cdot\frac{\partial u}{\partial x} - \frac{y^2}{(a^2+u)^2}\cdot\frac{\partial u}{\partial x} - \frac{z^2}{(c^2+u)^2}\cdot\frac{\partial u}{\partial x}\Big] = 0$$

$$\frac{2x}{(a^2+u)} - \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x} = 0$$

$$\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x} = \frac{2x}{(a^2+u)}$$

$$R\cdot\frac{\partial u}{\partial x} = \frac{2x}{(a^2+u)} \qquad ...(2)$$

rly, $R\frac{\partial u}{\partial y} = \frac{2y}{(b^2+u)}$ ...(3)

$$R\frac{\partial u}{\partial z} = \frac{2z}{(c^2+u)} \qquad ...(4)$$

e, $R = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}$ ...(5)

II : Squaring and adding Equations (2), (3) and (4), we get,

$$\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] = \frac{4x^2}{(a^2+u)^2} + \frac{4y^2}{(b^2+u)^2} + \frac{4z^2}{(c^2+u)^2}$$

$$= 4\underbrace{\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]}_{R}$$

$$= 4R \qquad \text{...from Equation (5)}$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{4R}{R^2} = \frac{4}{R} \qquad ...(6)$$

Step III : **Now given,** u is a homogeneous function in x, y, z of degree n.

$\therefore$ By Euler's theorem, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

Step IV : By using Equations (2), (3) and (4), above equation becomes,

$$x\cdot\frac{2x}{R(a^2+u)} + y\cdot\frac{2y}{R(b^2+u)} + z\cdot\frac{2z}{R(c^2+u)} = nu$$

$$\Rightarrow \frac{2}{R}\underbrace{\left[\frac{x^2}{(a^2+u)} + \frac{y^2}{(b^2+u)} + \frac{z^2}{(c^2+u)}\right]}_{1} = nu$$

$$\Rightarrow \frac{2}{R}\cdot 1 = nu \quad \text{...using Equation (1)}$$

Multiplying both sides by 2,

$$\Rightarrow 2\cdot\frac{2}{R} = 2\,nu$$

$$\frac{4}{R} = 2\,nu \qquad \text{...Put this value in Equation (6)}$$

We get, $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2nu$ ✓

...Hence Proved.

Example 4.14.12

If $z = f(x, y)$, $u = x^n\phi\left(\frac{y}{x}\right)$, $v = x^n\psi\left(\frac{y}{x}\right)$, then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$

Solution :

Step I : **Given :** $u = x^n\phi\left(\frac{y}{x}\right)$; $v = x^n\psi\left(\frac{y}{x}\right)$

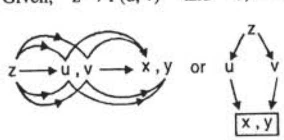
i.e. u and v are homogeneous function in x and y of degree n.

Step II : $\therefore$ By Euler's theorem on homogeneous functions in x and y of degree n

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad ...(1)$$

and $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv$ ...(2)

Step III : Given, $z \to f(u, v)$ and $u, v \to f(x, y)$



Differentiate z w.r.t. x, partially by chain rule.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$\Rightarrow$ Multiplying both sides by x,

$$x\frac{\partial z}{\partial x} = x\frac{\partial u}{\partial x}\cdot\frac{\partial z}{\partial u} + x\frac{\partial v}{\partial x}\cdot\frac{\partial z}{\partial v} \qquad ...(3)$$

and differentiate z w.r.t. y, partially by chain rule.

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$\Rightarrow$ Multiplying both sides by y,

$$y\frac{\partial z}{\partial y} = y\frac{\partial u}{\partial y}\cdot\frac{\partial z}{\partial u} + y\frac{\partial v}{\partial y}\cdot\frac{\partial z}{\partial v} \qquad ...(4)$$

Step IV : Adding Equations (3) and (4), we get,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] + \frac{\partial z}{\partial v}\left[x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right]$$

$$= \frac{\partial z}{\partial u}(nu) + \frac{\partial z}{\partial v}(nv)$$

...using Equations (1) and (2)

$\therefore$ **$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n\left[u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right]$** ✓ **...Hence Proved.**

Example 4.14.13

If $z = \log(x^2+y^2) + \frac{x^2+y^2}{x+y} - 2\log(x+y)$ then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x^2+y^2}{x+y}$

Solution :

Step I : Given, $z = \log(x^2+y^2) + \left(\frac{x^2+y^2}{x+y}\right) - 2\log(x+y)$

(Since, $n\log f(x) = \log[f(x)]^n$

i.e. $z = \log(x^2+y^2) - \log(x+y)^2 + \left(\frac{x^2+y^2}{x+y}\right)$

➔ **Using standard formulae :** ...$\log(A) - \log(B) = \log\left(\frac{A}{B}\right)$

$$z = \log\left(\frac{x^2+y^2}{(x+y)^2}\right) + \left(\frac{x^2+y^2}{x+y}\right)$$

i.e. $z = u + v$ ...(1)

where, $u = \log\left(\frac{x^2+y^2}{(x+y)^2}\right)$ and $v = \frac{x^2+y^2}{x+y}$

Step II : Put $x = xt$; $y = yt$

$$u = \log\left[\frac{t^2(x^2+y^2)}{t^2(x+y)^2}\right] \quad \text{and}$$

$$v = \frac{t^2(x^2+y^2)}{t(x+y)}$$

i.e. $u = t^0\underbrace{\log\left(\frac{x^2+y^2}{(x+y)^2}\right)}_{u}$ and

$$v = t^1\underbrace{\left(\frac{x^2+y^2}{x+y}\right)}_{v}$$

$\Rightarrow$ $u = t^0 u$ and $v = t^1 v$

This shows that u is homogenous function of degree n = 0 and v also homogenous function of degree n = 1 in x and y

Step III : By Euler's theorem on homogeneous function,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{and} \quad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \quad \text{and}$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1\cdot v = \left(\frac{x^2+y^2}{x+y}\right) \qquad ...(2)$$

Step IV : Now differentiate Equation (1), w.r.t. x and y

$\therefore \frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$ Multiplying both sides by x,

$$\Rightarrow x\frac{\partial z}{\partial x} = x\frac{\partial u}{\partial x} + x\frac{\partial v}{\partial x} \qquad ...(3)$$

and $\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ Multiplying both sides by y,

$$\Rightarrow y\frac{\partial z}{\partial y} = y\frac{\partial u}{\partial y} + y\frac{\partial v}{\partial y} \qquad ...(4)$$

Adding Equation (3) and (4)

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) + \left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 0 + \left(\frac{x^2+y^2}{x+y}\right)$$

...from Equation (2)

Hence, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \left(\frac{x^2+y^2}{x+y}\right)$ ✓ **...Hence Proved.**

Example 4.14.14

If $u = \frac{x^3y^3z^3}{x^3+y^3+z^3} + \log\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 6\cdot\left(\frac{x^3y^3z^3}{x^3+y^3+z^3}\right)$

Solution :

Step I : Given, $u = \frac{x^3y^3z^3}{x^3+y^3+z^3} + \log\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$

i.e. $u = v + w$ ...(1)

where, $u = \dfrac{x^3 y^3 z^3}{x^3+y^3+z^3}$

and $w = \log\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$

Step II : Put $x = xt$; $y = yt$ and $z = zt$;

$$v = \frac{(xt)^3 (yt)^3 (zt)^3}{(xt)^3+(yt)^3+(zt)^3};$$

and
$$w = \log\left(\frac{(xt)(yt)+y(t)(zt)+(zt)(xt)}{(xt)^2+(yt)^2+(zt)^2}\right)$$

i.e.
$$v = t^6 \underbrace{\left(\frac{x^3 y^3 z^3}{x^3+y^3+z^3}\right)}_{v}$$

and
$$w = t^0 \underbrace{\log\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)}_{w}$$

$$v = t^6 v \text{ and } w = t^0 w$$

This shows that v and w are homogeneous function in x and y of degree n = 6 and n = 0 respectively.

Step III : By Euler's theorem on homogeneous function, we get

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z} = nv = 6v \quad \text{...(2)}$$

and
$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z} = nw = 0 \quad \text{...(3)}$$

Step IV : Now, differentiate Equation (1) w.r.t. x, y and z

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}+\frac{\partial w}{\partial x} \quad \text{Multiplying both sides by x,}$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} = x\frac{\partial v}{\partial x}+x\frac{\partial w}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}+\frac{\partial w}{\partial y} \quad \text{Multiplying both sides by y,}$$

$$\Rightarrow \quad y\frac{\partial u}{\partial y} = y\frac{\partial v}{\partial y}+y\frac{\partial w}{\partial y}$$

and
$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z}+\frac{\partial w}{\partial z} \quad \text{Multiplying both sides by z,}$$

$$\Rightarrow \quad z\frac{\partial u}{\partial z} = z\frac{\partial v}{\partial z}+z\frac{\partial w}{\partial z}$$

Adding, we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = \left(x\frac{\partial v}{\partial x}+x\frac{\partial w}{\partial x}\right)+\left(y\frac{\partial v}{\partial y}+y\frac{\partial w}{\partial y}\right)+\left(z\frac{\partial v}{\partial z}+z\frac{\partial w}{\partial z}\right)$$

$$= \left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}\right)+\left(x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}\right)$$

$$= 6v + 0$$

$$= 6v \quad \text{...from (2) and (3)}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = 6\left(\frac{x^3y^3z^3}{x^3+y^3+z^3}\right) \checkmark \text{...Hence Proved.}$$

Example 4.14.15

If $u = \cos\left(\dfrac{xy}{x^2+y^2}\right)+\left(\sqrt{x^2+y^2}\right)+\left(\dfrac{xy^2}{x+y}\right)$, then find the value of $xu_x + yu_y$ at (3, 4).

Solution :

Step I :

Given,
$$u = \cos\left(\frac{xy}{x^2+y^2}\right)+\left(\sqrt{x^2+y^2}\right)+\left(\frac{xy^2}{x+y}\right) \quad \text{...(1)}$$

i.e.
$$u = R + S + T \quad \text{...(2)}$$

where,
$$R = \cos\left(\frac{xy}{x^2+y^2}\right);$$

$$S = \sqrt{x^2+y^2}\ ;\ T = \frac{xy^2}{x+y}$$

Step II : Let, $x = xt$, $y = yt$,

$$\therefore \quad R = \cos\left(\frac{xt\cdot yt}{(xt)^2+(yt)^2}\right)$$

$$= t^0 \underbrace{\cos\left(\frac{xy}{x^2+y^2}\right)}_{R} = t^0 R$$

is a **homogeneous function** of **degree n = 0**

$$S = \sqrt{(xt)^2+(yt)^2} = t\underbrace{\sqrt{x^2+y^2}}_{S} = t\,S$$

is a **homogeneous function** of **degree n = 1**

and

$$T = \frac{(xt)\cdot(yt)^2}{(xt)+(yt)} = t^2\underbrace{\left(\frac{xy^2}{x+y}\right)}_{T} = t^2\,T$$

is a **homogeneous function of degree n = 2**

Step III : By Euler's theorem for homogeneous function,

$$x\frac{\partial R}{\partial x}+y\frac{\partial R}{\partial y} = nR = 0; \quad \text{...(A)}$$

$$x\frac{\partial S}{\partial x}+y\frac{\partial S}{\partial y} = nS = S \quad \text{...(B)}$$

and
$$x\frac{\partial T}{\partial x}+y\frac{\partial T}{\partial y} = nT = 2T \quad \text{...(C)}$$

Adding Equation (A), (B) and (C) we get,

$$x\left(\frac{\partial R}{\partial x}+\frac{\partial S}{\partial x}+\frac{\partial T}{\partial x}\right)+y\left(\frac{\partial R}{\partial y}+\frac{\partial S}{\partial y}+\frac{\partial T}{\partial y}\right) = 0 + S + 2T$$

$$= S + 2T \quad \text{...(3)}$$

Step IV : Differentiate Equation (1) w.r.t. x

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial R}{\partial x}+\frac{\partial S}{\partial x}+\frac{\partial T}{\partial x}$$

Multiplying by x, we get

$$x\frac{\partial u}{\partial x} = x\left(\frac{\partial R}{\partial x}+\frac{\partial S}{\partial x}+\frac{\partial T}{\partial x}\right) \quad \text{...(D)}$$

Differentiate Equation (1) w.r.t. y and then multiplying by y

and
$$y\frac{\partial u}{\partial y} = y\left(\frac{\partial R}{\partial y}+\frac{\partial S}{\partial y}+\frac{\partial T}{\partial y}\right) \quad \text{...(E)}$$

Adding Equations (D) and (E),

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = x\left[\frac{\partial R}{\partial x}+\frac{\partial S}{\partial x}+\frac{\partial T}{\partial x}\right]+y\left[\frac{\partial R}{\partial y}+\frac{\partial S}{\partial y}+\frac{\partial T}{\partial y}\right] \quad \text{...(4)}$$

Step V :

By using Equation (3), Equation (4) becomes,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = S + 2T = \left(\sqrt{x^2+y^2}\right)+2\left(\frac{xy^2}{x+y}\right)$$

At $(x, y) \equiv (3, 4)$

$$\therefore \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \left(\sqrt{25}\right)+2\left(\frac{48}{7}\right) = 5+\left(\frac{96}{7}\right)$$

$$= \frac{131}{7} \checkmark \quad \text{..Ans.}$$

4.15 Deductions on Euler's Theorem (Corollary : 1 on Euler's Theorem)

Corollary 1 : If u is a homogenous function in x and y of degree n then,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)\,u$$

Proof : Since, by Euler's theorem on homogenous function in x and y of degree n,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu \quad \text{...(1)}$$

Differentiating Equation (1) partially w.r.t. x, keeping y as constant,

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\,\partial y} = n\frac{\partial u}{\partial x}$$

Multiplying by x

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\,\partial y} = nx\frac{\partial u}{\partial x} \quad \text{...(2)}$$

Differentiating Equation (1) partially w.r.t. y, keeping x as constant.

and
$$x\frac{\partial^2 u}{\partial y\,\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y} = n\frac{\partial u}{\partial y}$$

Multiplying by y,

$$xy\frac{\partial^2 u}{\partial y\,\partial x}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y} = ny\frac{\partial u}{\partial y} \quad \text{....(3)}$$

Adding Equation (2) and (3) we get,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\underbrace{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}}_{nu} = n\underbrace{\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)}_{nu}$$

$$\Rightarrow \quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+nu = n(nu)$$

...($\because$ use Equation (1))

$$\therefore \quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$$

$$= n^2 u - nu = n(n-1)\,u$$

Note : If u is homogenous function of degree n in x, y, z and u = f (V, W, T).
Where, V, W, T are the first order partial derivatives w.r.t. x, y, z respectively then

$$V\cdot\frac{\partial f}{\partial V}+W\cdot\frac{\partial f}{\partial W}+T\cdot\frac{\partial f}{\partial T} = \frac{n}{n-1}\cdot u$$

Type I : Examples Based on Euler's Theorem on Homogeneous Function Corollary : 1

Example 4.15.1

If $u = y^2 e^{\frac{y}{x}} + x^2\tan^{-1}\left(\dfrac{x}{y}\right)$ then show that

$x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y} = 2u$ and $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\,\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2} = 2u$

Solution : Step I : Given : $u = y^2 e^{\frac{y}{x}} + x^2\tan^{-1}\left(\dfrac{y}{x}\right)$

Put, $x = xt$; $y = yt$ in u, we get

$$u = (yt)^2\,e^{\left(\frac{yt}{xt}\right)}+(xt)^2\tan^{-1}\left[\frac{yt}{xt}\right]$$

$$= t^2\underbrace{\left[y^2 e^{\frac{y}{x}}+x^2\tan^{-1}\left(\frac{y}{x}\right)\right]}_{u}$$

$$u = t^2 u$$

This shows that u is homogeneous functions in x and y of degree n = 2.

Step II : $\therefore$ By Euler's theorem, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 2u$$

and by deduction of Euler's theorem, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$= 2(2-1)u = 2u \checkmark$$

...Hence Proved.

Example 4.15.2

If $Z = x f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$ then prove that $x^2 Z_{xx} + 2xy Z_{xy} + y^2 Z_{yy} = 0$

Soln. :

Step I : Given : $Z = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$

$$\therefore Z = U + V \quad ...(1)$$

Where, $U = xf\left(\frac{y}{x}\right)$ and $V = g\left(\frac{y}{x}\right)$

Put $x = xt$; $y = yt$ in U and V, we get

$$U = (xt)\, f\left(\frac{yt}{xt}\right) \quad \text{and} \quad V = g\left(\frac{yt}{xt}\right)$$

$$\Rightarrow \quad U = \underbrace{t\, x f\left(\frac{y}{x}\right)}_{u}; \quad \text{and} \quad V = t^0 \underbrace{g\left(\frac{y}{x}\right)}_{v}$$

$$U = t\,U \text{ and } V = t^0 V$$

This shows that U and V are homogeneous functions in x and y of degree n = 1 and n = 0 respectively.

Step II : $\therefore$ By Euler's theorem on homogeneous functions.

$$x^2 U_{xx} + 2xy\, U_{xy} + y^2 U_{yy} = n(n-1)U$$
$$= 1(1-1)U = 0$$

and $$x^2 V_{xx} + 2xy\, V_{xy} + y^2 V_{yy} = n(n-1)V$$
$$= 0(0-1)V = 0$$

Adding we get,

$$x^2(U_{xx} + V_{xx}) + 2xy(U_{xy} + V_{xy}) + y^2(U_{yy} + V_{yy}) = 0 \quad ...(2)$$

Step III : But from Equation (1),

$$Z = U + V$$

Differentiate Z w.r.t. x and y two times partially,

$Z_x = U_x + V_x$; $Z_{xx} = U_{xx} + V_{xx}$

and $Z_y = U_y + V_y$; $Z_{yy} = U_{yy} + V_{yy}$

and $Z_{xy} = U_{xy} + V_{xy}$

Step IV : Equation (2) becomes,

$$x^2 Z_{xx} + 2xy\, Z_{xy} + y^2 Z_{yy} = 0 \checkmark$$

...Hence Proved.

Example 4.15.3

If $u = \frac{x^3 + y^3}{y\sqrt{x}} + \frac{1}{x^7}\sin^{-1}\left(\frac{x^2 + y^2}{2xy}\right)$ then find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ at point (1, 1).

Solution :

Step I : Given :

$$u = \frac{x^3 + y^3}{y\sqrt{x}} + \frac{1}{x^7}\cdot\sin^{-1}\left(\frac{x^2 + y^2}{2xy}\right)$$

$$\Rightarrow \quad u = v + w$$

Where, $v = \frac{x^3 + y^3}{y\sqrt{x}}$; $w = \frac{1}{x^7}\sin^{-1}\left(\frac{x^2 + y^2}{2xy}\right)$

By putting $x = xt$; $y = yt$, we get

$$v = \frac{(xt)^3 + (yt)^3}{(yt)\sqrt{xt}} = \frac{t^3(x^3 + y^3)}{t^{3/2}(y\sqrt{x})} = t^{3-\frac{3}{2}}\underbrace{\left[\frac{x^3 + y^3}{y\sqrt{x}}\right]}_{v}$$

$$= t^{\frac{3}{2}}\, v$$

And $$w = \frac{1}{(xt)^7}\cdot\sin^{-1}\left(\frac{x^2t^2 + y^2t^2}{2(xt)(yt)}\right)$$

$$= t^{-7}\underbrace{\frac{1}{x^7}\cdot\sin^{-1}\left(\frac{x^2 + y^2}{2xy}\right)}_{w} = t^{-7}w$$

we get v is homogeneous function of degree $n = \frac{3}{2}$ and w is also homogeneous function of degree n = – 7

Step II : $\therefore$ By Euler's theorem on homogeneous function,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} \equiv nv = \frac{3}{2}v \quad ...(1)$$

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n(n-1)v$$

$$= \frac{3}{2}\left(\frac{3}{2} - 1\right)v \quad ...(2)$$

and $$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = nw = -7w \quad ...(3)$$

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\,\partial y} + y^2\frac{\partial^2 w}{\partial y^2}$$

$$= n(n-1)w = -7(-7-1)w = 56w \quad ...(4)$$

Step III : Adding above Equations, (1) and (2). also (3) and (4)

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} + x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}$$

$$= \frac{3}{2}\left(\frac{3}{2} - 1\right)v + \frac{3}{2}v = \frac{3}{2}\left(\frac{3-2}{2}v + \frac{3}{2}v\right)$$

$$= \frac{3}{2}\left(\frac{1}{2}\right)v + \frac{3}{2}v = \frac{3}{4}v + \frac{3}{2}v$$

$$= \frac{9}{4}v \quad ...(5)$$

and $$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\,\partial y} + y^2\frac{\partial^2 w}{\partial y^2} + x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y}$$

$$= -7(-7-1)w - 7w = 49w \quad ...(6)$$

Step IV : Adding Equation (5) and (6) we get

$$x^2\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right] + 2xy\left[\frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial x\,\partial y}\right]$$
$$+ y^2\left[\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right] + x\left[\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right] + y\left[\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right]$$
$$= \frac{9}{4}v + 49w \quad ...(7)$$

Step V : But , $u = v + w$ ...(given)

Differentiate u w.r.t. x,

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}; \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}$$

and differentiate u w.r.t. y,

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}; \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}$$

and differentiate $\frac{\partial u}{\partial y}$ w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial x\,\partial y}$$

Step VI : Equation (7) becomes,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= \frac{9}{4}v + 49w \quad ...(8)$$

Step VII : At point (1, 1)

Since, $v = \frac{x^3 + y^3}{x\sqrt{y}}$; $w = \frac{1}{x^7}\sin^{-1}\left(\frac{x^2 + y^2}{2xy}\right)$

$\therefore v(1, 1) = 2$; $w(1, 1) = 1\,(\pi/2)$

Equation (4) becomes,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= \frac{9}{4}(2) + 49\left(\frac{\pi}{2}\right) = \frac{9}{2} + 49\left(\frac{\pi}{2}\right) \checkmark$$

...Ans.

Example 4.15.4

If $z = x^8 f\left(\frac{y}{x}\right) + y^{-8}\phi\left(\frac{x}{y}\right)$, then prove that :

$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 64z + 8y^{-8}\phi\left(\frac{x}{y}\right) - 8x^8 f\left(\frac{y}{x}\right)$

Solution :

Step I : Given, $z = x^8 f\left(\frac{y}{x}\right) + y^{-8}\phi\left(\frac{x}{y}\right)$

$$\text{i.e. } z = u + v \quad ...(1)$$

where, $u = x^8 f\left(\frac{y}{x}\right)$ - is a **homogeneous function** in x and y of degree **n = 8**.

Also, $v = y^{-8}\phi\left(\frac{x}{y}\right)$ is a **homogeneous function** in x and y of degree **n = – 8**

Step II : By Euler's theorem on homogeneous function,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 8(7)u$$
$$= 56u \quad ...(A)$$

and $$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n(n-1)v$$
$$= -8(-9)v = 72v \quad ...(B)$$

Adding Equation (A) and (B), we get

$$x^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial x\partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$= 56u + 72v$$

$$x^2\frac{\partial^2}{\partial x^2}(u+v) + 2xy\frac{\partial^2}{\partial x\partial y}(u+v) + y^2\frac{\partial^2}{\partial y^2}(u+v)$$

$$= 56u + 72v$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 56u + 72v$$

(By using Equation (1

Now, R.H.S. $= 64z + 8y^{-8}\phi\left(\frac{x}{y}\right) - 8x^8 f\left(\frac{y}{x}\right)$

$$= 64(u+v) + 8v - 8u$$

$$= 64u + 64v + 8v - 8u$$

$$= 56u + 72v = \text{L.H.S.}$$

Hence,

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2}$$

$$= 64z + 8y^{-8}\phi\left(\frac{x}{y}\right) - 8x^8 f\left(\frac{y}{x}\right) \checkmark \quad \text{...Ans.}$$

Type II : Examples Based on Euler's Theorem on Homogeneous Function Corollary : 1

Example 4.15.5

If $Z = x^n f\left(\frac{y}{x}\right) + x^{-n}\phi\left(\frac{x}{y}\right)$; then prove that

$x^2 Z_{xx} + 2xy Z_{xy} + y^2 Z_{yy} + x Z_x + y Z_y = n^2 Z$. **OR**

If $u = x^3 f\left(\frac{y}{x}\right) + y^{-3}\phi\left(\frac{x}{y}\right)$ then prove that

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 9u$

Solution :

Step I : Given : $Z = x^n f\left(\frac{y}{x}\right) + x^{-n}\phi\left(\frac{x}{y}\right) \Rightarrow Z$

$$= U + V \quad \text{...(1)}$$

Where, $U = x^n f\left(\frac{y}{x}\right)$; $V = x^{-n}\phi\left(\frac{x}{y}\right)$

i.e. U and V are homogeneous functions in x and y of degree n and – n respectively.

Step II : $\therefore$ By Euler's theorem on homogeneous functions.

$$x U_x + y U_y = nU \quad \text{...(2)}$$

and $$x V_x + y V_y = -nV \quad \text{...(3)}$$

and $$x^2 U_{xx} + 2xy U_{xy} + y^2 U_{yy} = n(n-1)U \quad \text{...(4)}$$

$$x^2 V_{xx} + 2xy V_{xy} + y^2 V_{yy} = n(n-1)V$$

$$= -n(-n-1)V = n(n+1)V \quad \text{...(5)}$$

Step III : Adding Equations (2), (3), (4) and (5) we get,

$$x^2(U_{xx} + V_{xx}) + 2xy(U_{xy} + V_{xy}) + y^2(U_{yy} + V_{yy})$$
$$+ x(U_x + V_x) + y(U_y + V_y)$$
$$= n(n-1)U + n(n+1)V + nU - nV$$
$$= n^2U - nU + n^2V + nV + nU - nV$$
$$= n^2(U + V) \quad \text{...(6)}$$

Step IV : Now from Equation (1),

$$Z = U + V$$

Differentiate Z w.r.t. x and y

$$Z_x = U_x + V_x \quad Z_y = U_y + V_y$$

$$Z_{xy} = U_{xy} + V_{xy}$$

$$Z_{xx} = U_{xx} + V_{xx} \quad Z_{yy} = U_{yy} + V_{yy}$$

Step V : Equation (6) becomes,

$$x^2 Z_{xx} + 2xy Z_{xy} + y^2 Z_{yy} + x Z_x + y Z_y = n^2 Z \checkmark$$

...Hence Proved.

Example 4.15.6

If $U = \frac{x^3 + y^3}{y\sqrt{x}}$, Find the value of

$x^2 U_{xx} + 2xy U_{xy} + y^2 U_{yy} + x U_x + y U_y$ at the point (1, 2).

Solution : Step I : Given : $U = \frac{x^3 + y^3}{y\sqrt{x}}$

Put, $x = xt$; $y = yt$, in U we get

$$U = \frac{(xt)^3 + (yt)^3}{(yt)(\sqrt{xt})} = \frac{t^3}{t^{3/2}}\underbrace{\left[\frac{x^3 + y^3}{y\sqrt{x}}\right]}_{u} = t^{3/2}U$$

Hence U is homogeneous function in x and y of degree n = 3/2.

Step II : $\therefore$ By Euler's theorem,

$$x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = nU = \frac{3}{2}U \quad \text{...(1)}$$

and $$x^2 U_{xx} + 2xy U_{xy} + y^2 U_{yy} = n(n-1)U$$

$$= \frac{3}{2}\left(\frac{3}{2} - 1\right)U = \frac{3}{4}U \quad \text{...(2)}$$

Step III : Adding Equations (1) and (2) we get,

$$x^2 U_{xx} + 2xy U_{xy} + y^2 U_{yy} + x U_x + y U_y = \left(\frac{3}{4} + \frac{3}{2}\right)U = \frac{9}{4}U$$

$$= \frac{9}{4}\left[\frac{x^3 + y^3}{y\sqrt{x}}\right]_{(1,2)} = \frac{9}{4}\left[\frac{1+8}{2\sqrt{1}}\right] = \frac{81}{8} \checkmark \quad \text{...Ans.}$$

4.16 Corollary 2 on Euler's Theorem

Corollary 2 :

If $z = f(u)$ is homogenous function of degree n is x and y then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$

Proof :

Let $u = f^{-1}(x, y)$ is not homogenous function but it is function of homogenous expression

Let, $z = f(u) = \phi(x, y)$ is a homogenous function in x and y of degree n

Therefore, by Euler's theorem on homogenous function.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = nf(u)$$

$$\because z = f(u)$$

$$\frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = f'(u)\frac{\partial u}{\partial y}$$

Hence, $$x f'(u) \cdot \frac{\partial u}{\partial x} + y f'(u)\frac{\partial u}{\partial y} = nf(u)$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

In general : If f(u) is a homogeneous function in x and y of degree n them

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + \dots = n \cdot \frac{f(u)}{f'(u)}$$

Type I : Examples Based on Euler's Theorem on Homogeneous Expression Corollary : 2

Example 4.16.1

If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ then prove that

$2x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = \tan u$.

Solution :

Step I : Given : $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

Put $x = xt$, $y = yt$ in Equation u, we get

$$u = \sin^{-1}\left[\frac{xt + yt}{\sqrt{xt}+\sqrt{yt}}\right] = \sin^{-1}\left(t^{1/2}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\right) \text{ is}$$

not homogeneous function.

But, $$f(u) = \sin u = t^{1/2}\underbrace{\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)}_{\phi(x,\,y)}$$

is homogeneous function in x and y of degree n = 1/2.

Step II : $\therefore$ By Euler's theorem on homogeneous expression in x and y,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = \frac{1}{2}\frac{\sin u}{\cos u}$$

$$\Rightarrow 2x \cdot \frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = \tan u \checkmark \quad \text{...Hence Proved.}$$

Example 4.16.2

If $u = \sin(\sqrt{x} + \sqrt{y})$; prove that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}[\sqrt{x} + \sqrt{y}] \cdot \cos(\sqrt{x} + \sqrt{y})$.

Solution : Step I : Given : $u = \sin(\sqrt{x} + \sqrt{y})$

Put, $x = xt$; $y = yt$ in u

$$\therefore u = \sin(\sqrt{xt} + \sqrt{yt}) = \sin\left(t^{\frac{1}{2}}(\sqrt{x} + \sqrt{y})\right)$$

u is not homogeneous function

But, $f(u) = \sin^{-1}u = t^{\frac{1}{2}}(\sqrt{x} + \sqrt{y})$

is homogeneous function in x and y of degree $n = \frac{1}{2}$.

Step II : $\therefore$ By Euler's theorem on homogeneous expression in x and y, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = \frac{1}{2}\frac{\sin^{-1}u}{\left(\frac{1}{\sqrt{1-u^2}}\right)}$$

$$\left(\because \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\underbrace{\sin^{-1}u}_{\sqrt{x}+\sqrt{y}} \cdot \sqrt{1-u^2}$$

$$= \frac{1}{2}(\sqrt{x} + \sqrt{y}) \cdot \sqrt{1 - \sin^2(\sqrt{x} + \sqrt{y})}$$

$$= \frac{1}{2}(\sqrt{x} + \sqrt{y}) \cdot \sqrt{\cos^2(\sqrt{x} + \sqrt{y})}$$

$$= \frac{1}{2}(\sqrt{x} + \sqrt{y}) \cdot \cos(\sqrt{x} + \sqrt{y})$$

...Hence Proved.

Example 4.16.3

If $u = \frac{x^2y^2z^2}{x^2+y^2+z^2} + \cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$

then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Solution :

Step I : Given :

$$u = \frac{x^2y^2z^2}{x^2+y^2+z^2} + \cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$$

$$\Rightarrow u = v + w \quad \text{...(1)}$$

where, $$v = \frac{x^2y^2z^2}{x^2+y^2+z^2};$$

$$w = \cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$$

Put, $x = xt$, $y = yt$; $z = zt$ in v and w

We get, $$v = \frac{(xt)^2(yt)^2(zt)^2}{(xt)^2+(yt)^2+(zt)^2}$$

and $v = \cos^{-1}\left(\frac{xt + yt + zt}{\sqrt{xt} + \sqrt{yt} + \sqrt{zt}}\right)$

$$v = \frac{t^6}{t^2}\underbrace{\left(\frac{x^2 y^2 z^2}{x^2 + y^2 + z^2}\right)}_{v} = t^4 v$$

and $w = \cos^{-1}\left[\frac{t}{t^{1/2}}\left(\frac{x + y + z}{\sqrt{x} + \sqrt{y} + \sqrt{z}}\right)\right]$

This shows that, v is homogeneous function in x and y of degree n = 4 but, w is not homogeneous function. So, $f(w) = \cos w = \frac{t}{t^{1/2}}\left(\frac{x + y + z}{\sqrt{x} + \sqrt{y} + \sqrt{z}}\right)$ is homogeneous function of degree $n = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$

Step II : By Euler's theorem on homogeneous function in x and y, we get

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = nv$$

and $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = n\frac{f(w)}{f'(w)}$

$\therefore x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 4v$

and $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = \frac{1}{2}\frac{\cos w}{(-\sin w)} = -\frac{1}{2}\cot w$

Step III : Adding, we get

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) + z\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial z}\right) = 4v - \frac{1}{2}\cos w \quad ...(2)$$

Step IV : But from Equation (1),

$$u = v + w$$

Differentiate w.r.t. x, y and z we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \; ; \; \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}$$

and $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial z}$

Step V : Equation (2) becomes,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 4v - \frac{1}{2}\cot w$$

$$= 4\underbrace{\left(\frac{x^2 y^2 z^2}{x^2 + y^2 + z^2}\right)}_{v} - \frac{1}{2}\cot^{-1}\underbrace{\left[\frac{x + y + z}{\sqrt{x} + \sqrt{y} + \sqrt{z}}\right]}_{w} \checkmark \text{...Ans.}$$

Type II : Example Based on Euler's Theorem on Homogeneous Function Corollary : 2

Example 4.16.4

Prove that : $\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 0$**, if** $x = e^u \tan v$**;** $y = e^u \sec v$

Solution :

Step I : Here we have to find,

$$u = f(x, y) \quad \text{and} \quad v = f(x, y)$$

Given, $x = e^u \tan v$ and $y = e^u \sec v$

Therefore, $y^2 - x^2 = e^{2u} \cdot \sec^2 v - e^{2u}\tan^2 v$

$= e^{2u}(\sec^2 v - \tan^2 v) = e^{2u} \quad (\because \sec^2 x - \tan^2 x = 1)$

$\Rightarrow e^{2u} = (y^2 - x^2)$

Step II : Put = xt ; y = yt

$$f(u) = e^{2u} = t^2\underbrace{(y^2 - x^2)}_{f(u)} = t^2 f(u)$$

This shows that f (u) is homogeneous function in x and y of degree n = 2.

Step III : By Euler's theorem on homogenous expression

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 2\frac{e^{2u}}{2e^{2u}} = 1 \quad ...(A)$$

Step IV : Further from x and y,

$$\frac{x}{y} = \frac{e^u \cdot \tan v}{e^u \cdot \sec v} = \sin v$$

$\Rightarrow$ Put $x = xt, y = yt \Rightarrow v = \sin^{-1}\left(\frac{xt}{yt}\right)$

$$v = t^0 \sin^{-1}\left(\frac{x}{y}\right) = t^0 v$$

This shows that v is homogenous function in x and y of degree n = 0

Step V : By Euler's theorem on homogenous function,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 0 \quad ...(B)$$

Step VI : From Equation (A) and (B), we get

$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = (1)\cdot(0) = 0$$

✓ **...Hence Proved.**

4.17 Corollary 3 on Euler's Theorem

Corollary 3 : If f(u) is a homogenous function in x and y of degree n then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = G(u)[G'(u) - 1]$$

Where, $G(u) = n\frac{f(u)}{f'(u)}$

Proof : Since f(u) is a homogenous function in x and y of degree n.

By corollary 2, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = G(u) \quad ...(1)$

Now differentiating Equation (1) partially w.r.t. x, keeping y as constant.

→ **Use standard rule of differentiation :**

$$...x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = G'(u)\cdot\frac{\partial u}{\partial x}$$

Multiplying by x

$$\therefore \; x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} + x\frac{\partial u}{\partial x} = xG'(u)\cdot\frac{\partial u}{\partial x} \quad ...(2)$$

Now, differentiate Equation (1) partially w.r.t. y keeping x as constant

$$\therefore \; x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = G'(u)\cdot\frac{\partial u}{\partial y}$$

Multiplying by y

$$xy\cdot\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = yG'(u)\cdot\frac{\partial u}{\partial y} \quad ...(3)$$

Adding Equation (2) and (3), we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= G'(u)\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + G(u) = G'(u)\cdot G(u)$$

...(From Equation (1))

Hence, $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = G(u)[G'(u) - 1]$

Where, $G(u) = n\frac{f(u)}{f'(u)}$

Type I : Examples Based on Euler's Theorem on Homogeneous Expression Corollary : 3

Example 4.17.1

If $u = \sin^{-1}\sqrt{\frac{x^3 + y^3}{x - y}}$ **then find the value of** $x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy}$.

Solution :

Step I : Given : $u = \sin^{-1}\left(\sqrt{\frac{x^3 + y^3}{x - y}}\right)$

Put $x = xt ; y = yt$ in u

$$u = \sin^{-1}\left(\sqrt{\frac{(xt)^3 + (yt)^3}{xt - yt}}\right) = \sin^{-1}\left(\frac{t^{3/2}}{t^{1/2}}\sqrt{\frac{x^3 + y^3}{x - y}}\right)$$

This shows that u is not homogeneous function.

$(\because u \neq t^n u$

But, $\sin u = \frac{t^{3/2}}{t^{1/2}}\left(\sqrt{\frac{x^3 + y^3}{x - y}}\right)$

$= t\sqrt{\frac{x^3 + y^3}{x - y}}$ is homogeneous function in x and y o degree n = 1.

Here, $f(u) = \sin u$

Step II : By Euler's theorem on homogeneous expressio in x and y

$$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = G(u)[G'(u) - 1]$$

where, $G(u) = n\frac{f(u)}{f'(u)}$

$\therefore G(u) = n\frac{f(u)}{f'(u)} = 1\frac{\sin u}{\cos u} = \tan u$

and $G'(u) = \sec^2 u \quad (\because d(\tan x) = \sec^2 x$

$$\therefore x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = \tan u\underbrace{[\sec^2 u - 1]}_{\tan^2 u}$$

$= \tan u\,[\tan^2 u] = \tan^3 u$ ✓

$(\because \tan^2 x = \sec^2 x -$

...An

Example 4.17.2

If $u = \sin^{-1}\left[\frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}\right]^{1/2}$ **then show that**

$$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = \frac{\tan u}{144}[13 + \tan^2 u]$$

Solution :

Step I : Given : $u = \sin^{-1}\left[\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2}$.

By putting $x = xt; \quad y = yt$

It is observed that u is not homogeneous function.

But, $\quad f(u) = \sin u = \left[\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2}$

Put $x = xt; \quad y = yt$ in f(u), we get

$$f(u) = \sin u = \left[\frac{(xt)^{1/3}+(yt)^{1/3}}{(xt)^{1/2}+(yt)^{1/2}}\right]^{1/2} = t^{\frac{1}{6}-\frac{1}{4}} f(x,y)$$

$$\therefore \quad = \sin u = t^{-\frac{1}{12}} f(x, y)$$

This shows that f(u) is homogeneous function of degree $n = -\dfrac{1}{12}$.

Step II : $\therefore$ By Euler's theorem, on homogeneous expression in x and y,

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = G(u)\,[G'(u) - 1] \quad ...(1)$$

where, $\quad G(u) = n\dfrac{f(u)}{f'(u)} = -\dfrac{1}{12}\cdot\dfrac{\sin u}{\cos u} = -\dfrac{1}{12}\tan u$

and $\quad G'(u) = -\dfrac{1}{12}\sec^2 u$

Step III : $\therefore$ Equation (1) becomes,

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = -\frac{1}{12}\tan u\left[-\frac{1}{12}\sec^2 u - 1\right]$$

$$= \frac{1}{144}\tan u\,[\sec^2 u + 12]$$

$$= \frac{1}{144}\tan u\,[\sec^2 u + 12] = \frac{1}{144}\cdot\tan u\,[\tan^2 u + 1 + 12]$$

$$(\because \sec^2 x = 1 + \tan^2 x)$$

$$= \frac{1}{144}\tan u\,[13 + \tan^2 u] \checkmark$$

...Hence Proved.

Example 4.17.3

If $u = \sin^{-1}\left(\sqrt{x^2+y^2}\right)$ then prove that :
$x^2 u_{xx} + 2\,xy\, u_{xy} + y^2 u_{yy} = \tan^3 u$.

Solution :

Step I : Given : $u = \sin^{-1}\left(\sqrt{x^2+y^2}\right)$

It is observed that u is not homogeneous function.

But, $\quad f(u) = \sin u = \sqrt{x^2+y^2}$

Put $x = xt\ ; \ y = yt$ in f(u)

$$f(u) = \sin u = \sqrt{(xt)^2+(yt)^2} = t\underbrace{\sqrt{x^2+y^2}}_{f(u)}$$

is homogeneous function in x and y of degree n = 1.

Step II : $\therefore$ By Euler's theorem,

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = G(u)\,[G'(u) - 1] \quad ...(1)$$

where,, $\quad G(u) = n\dfrac{f(u)}{f'(u)} = 1\cdot\dfrac{\sin u}{\cos u}$

$$= \tan u \quad \text{and} \quad G'(u) = \sec^2 u$$

Step III : $\therefore$ Equation (1) becomes,

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \tan u\,\underbrace{[\sec^2 u - 1]}_{\tan^2 u}$$

$$= \mathbf{\tan u\,[\tan^2 u] = \tan^3 u} \checkmark$$

...Hence Proved.

Example 4.17.4

If $u = \sin^{-1}(x^3+y^3)^{2/5}$, then find the value of $x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy}$.

Soln. :

Step I : Given : $u = \sin^{-1}(x^3+y^3)^{2/5}$

Put, $x = xt\ ;\ y = yt$ in u, we get

$$u = \sin^{-1}((xt)^3+(yt)^3)^{2/5}$$
$$= \sin^{-1}(t^{6/5}(x^3+y^3)^{2/5})$$

Here, u is not homogeneous function, but

$$f(u) = \sin u = t^{6/5}(x^3+y^3)^{2/5}$$

is homogeneous function in x and y of degree n = 6/5.

Step II : $\therefore$ By Euler's theorem on homogeneous expression in x and y we get,

$$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = G(u)\cdot[G'(u) - 1] \quad ...(1)$$

Here, $\quad G(u) = n\dfrac{f(u)}{f'(u)} = \dfrac{6}{5}\cdot\dfrac{\sin u}{\cos u}$

$$G(u) = \frac{6}{5}\tan u \quad \text{and} \quad G'(u) = \frac{6}{5}\sec^2 u$$

Step III : $\therefore$ Equation (1) becomes,

$$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = \frac{6}{5}\tan u\left[\frac{6}{5}\sec^2 u - 1\right]$$

$$= \mathbf{\frac{36}{25}\tan u\left[\sec^2 u - \frac{5}{6}\right]} \checkmark \quad \text{...Ans.}$$

Example 4.17.5

If $u = \tan^{-1}\left[\dfrac{\sqrt{x^3+y^3}}{\sqrt{x}+\sqrt{y}}\right]$ then show that
$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = -2\sin^3 u\cdot\cos u$.

Solution : Step I : Given : $u = \tan^{-1}\left[\dfrac{\sqrt{x^3+y^3}}{\sqrt{x}+\sqrt{y}}\right]$

Put $x = xt, y = yt$ in u, we get,

$$u = \tan^{-1}\left[\sqrt{\frac{(xt)^3+(yt)^3}{\sqrt{xt}+\sqrt{yt}}}\right]$$
$$= \tan^{-1}\left[t\,\frac{\sqrt{x^3+y^3}}{\sqrt{x}+\sqrt{y}}\right]$$

is not homogeneous function, but

$$f(u) = \tan u = t\left[\frac{\sqrt{x^3+y^3}}{\sqrt{x}+\sqrt{y}}\right] \text{ is homogeneous}$$

in x and y of degree n = 1.

Step II : $\therefore$ By Euler's theorem on homogeneous function

$$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = G(u)\,[G'(u) - 1] \ ...(1)$$

Here, $\quad G(u) = n\dfrac{f(u)}{f'(u)} = 1\cdot\dfrac{\tan u}{\sec^2 u} = \dfrac{\sin u}{\cancel{\cos}\, u\cdot\dfrac{1}{\cos^{\cancel{2}} u}}$

$$\left(\because\ d(\tan x) = \sec^2 x \ \text{and}\ \sec^2 u = \frac{1}{\cos^2 u}\right)$$

$\therefore \quad G(u) = \sin u\cos u = \dfrac{1}{2}\cdot 2\sin u\cos u = \dfrac{1}{2}\sin 2u$

and $\quad G'(u) = \dfrac{1}{2}\cdot 2\cos 2u = \cos 2u$

Step III : Equation (1) becomes,

$$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = \sin u\ \cos u\,[\cos 2u - 1]$$
$$= -\sin u\cdot\cos u\,\underbrace{[1 - \cos 2u]}_{2\sin^2 u}$$
$$= -\sin u\cos u\cdot(2\sin^2 u)$$
$$= -2\sin^3 u\cos u \quad \text{...Hence Proved.}$$

Example 4.17.6

If $u = \tan^{-1}\left[\dfrac{x^3+y^3}{x-y}\right]$ then prove that
$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = [1 - 4\sin^2 u]\sin 2u$.

Soln. : Step I : Given : $u = \tan^{-1}\left[\dfrac{x^3+y^3}{x-y}\right]$

Put $x = xt\ ;\ y = yt$ in u, we get

$$u = \tan^{-1}\left[\frac{(xt)^3+(yt)^3}{(xt)-(yt)}\right] = \tan^{-1}\left[t^2\left(\frac{x^3+y^3}{x-y}\right)\right]$$

u is not homogeneous function.

But, $f(u) = \tan u = t^2\left(\dfrac{x^3+y^3}{x-y}\right)$ is homogeneous function in x and y of degree n = 2.

Step II : $\therefore$ By Euler's theorem on homogeneous expression we get,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y\,\partial x} + y^2\frac{\partial^2 u}{\partial y^2} = G(u)\,[G'(u) - 1] \ ...(1)$$

Here, $\quad G(u) = n\dfrac{f(u)}{f'(u)} = 2\dfrac{\tan u}{\sec^2 u} = 2\,\dfrac{\sin u}{\cancel{\cos}\,u\cdot\dfrac{1}{\cos^{\cancel{2}} u}}$

$$G(u) = 2\sin u\cos u = \sin 2u \quad \text{and}$$
$$G'(u) = 2\cos 2u$$

Step III : Equation (1) becomes,

$$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = \sin 2u\,[2\cos 2u - 1]$$
$$= \sin 2u\,[2(1 - 2\sin^2 u) - 1]$$
$$= \sin 2u\,[2 - 4\sin^2 u - 1]$$
$$= \sin 2u\,[1 - 4\sin^2 u] \checkmark \quad \text{...Ans.}$$

Example 4.17.7

If $u = \operatorname{cosec}^{-1}\cdot\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$ then show that :
$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = \dfrac{1}{144}\tan u\,[13 + \tan^2 u]$

Solution :

Step I : Given : $\quad u = \operatorname{cosec}^{-1}\cdot\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$

Put $x = xt\ ;\ y = yt$ in u, we get

$$u = \operatorname{cosec}^{-1}\sqrt{\frac{(xt)^{1/2}+(yt)^{1/2}}{(xt)^{1/3}+(yt)^{1/3}}}$$
$$= \operatorname{cosec}^{-1}\left[t^{\frac{1}{4}-\frac{1}{6}}\left(\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right)\right]$$
$$u = \operatorname{cosec}^{-1}\left[t^{\frac{1}{12}}\left(\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right)\right]$$

u is not homogeneous function.

But, $f(u) = \operatorname{cosec} u = t^{\frac{1}{12}}\underbrace{\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}}_{f(u)}$

is homogeneous function in x and y of degree $n = \dfrac{1}{12}$

Step II : By Euler's theorem on homogeneous expression in x and y, we get

$$x^2 u_{xx} + 2xy\ u_{xy} + y^2 u_{yy} = G(u)\,[G'(u) - 1] \quad ...(1)$$

where, $G(u) = \frac{n\,f(u)}{f'(u)} = \frac{1}{12}\ \frac{\operatorname{cosec} u}{-\operatorname{cosec} u \cdot \cot u} = -\frac{1}{12}\tan u$

$\ldots\left(\because \frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cdot \cot x \text{ and } \frac{1}{\cot x} = \tan x\right)$

and $G'(u) = -\frac{1}{12}\sec^2 u$

$\ldots\left(\because \frac{d}{dx}\tan x = \sec^2 x\right)$

Step III : Equation (1) becomes,

$$x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = -\frac{1}{12}\tan u\left[-\frac{1}{12}\sec^2 u - 1\right]$$

$$= \frac{1}{144}\tan u\,[\sec^2 u + 12]$$

$$= \frac{1}{144}\tan u\,[\tan^2 u + 1 + 12]$$

$$= \frac{1}{144}\tan u\,[\tan^2 u + 13] \checkmark$$

...Hence Proved.

Example 4.17.8

If $u = \log(x^3 + y^3 - x^2y - xy^2)$ then prove that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$ (ii) $x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = -3$

Solution :

Step I : Given : $u = \log[x^3 + y^3 - x^2y - xy^2]$

Put $x = xt$; $y = yt$ in u, we get

$u = \log[(xt)^3 + (yt)^3 - (xt)^2(yt) - (xt)(yt)^2]$

$= \log[t^3(x^3 + y^3 - x^2y - xy^2)]$

U is not homogeneous function.

But **$f(u) = e^u$** $= t^3[x^3 + y^3 - x^2y - xy^2]$ is homogeneous function in x and y of degree n = 3.

Step II : $\therefore$ By Euler's theorem homogenous expression in x and y, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 3\frac{e^u}{e^u} = 3$$

and $x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = G(u)\,[G'(u) - 1]$ ...(1)

where, $G(u) = n\frac{f(u)}{f'(u)} = 3\frac{e^u}{e^u} = 3$ and $G'(u) = 0$

Equation (1) becomes,

$x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = 3[0 - 1] = -3$ ✓

...Hence Proved.

Example 4.17.9

If $u = \cos^{-1}\left[\frac{x^3y^2 + 4y^3x^2}{\sqrt{x^4 + 6y^4}}\right]$ then find the value of

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ (ii) $x^2U_{xx} + 2_{xy}U_{xy} + y^2U_{yy}$

Solution :

Step I : Given : $u = \cos^{-1}\left[\frac{x^3y^2 + 4y^3x^2}{\sqrt{x^4 + 6y^4}}\right]$

Put, $x = xt$; $y = yt$ in u

$$\therefore\ u = \cos^{-1}\left[\frac{x^3y^2t^5 + 4x^2y^3t^5}{\sqrt{x^4t^4 + 6y^4t^4}}\right];$$

$$u = \cos^{-1}\left[t^3\left(\frac{x^3y^2 + 4x^2y^3}{\sqrt{x^4 + 6y^4}}\right)\right]$$

Hence, u is not homogenous function.

But, $f(u) = \cos u = t^3\left(\frac{x^3y^2 + 4x^2y^3}{\sqrt{x^4 + 6y^4}}\right)$; is homogenous function in x and y of degree n = 3

Step II : $\therefore$ By Euler's theorem on homogeneous expression in x and y, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 3\frac{\cos u}{(-\sin u)}$$

$$= -3\cot u \qquad \textbf{...Ans.}$$

and, $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$

where, $g(u) = n\frac{f(u)}{f'(u)} = -3\cot u$

and $g'(u) = -3(-\operatorname{cosec}^2 u) = 3\operatorname{cosec}^2 u$

$\therefore x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = -3\cot u\,[3\operatorname{cosec}^2 u - 1]$ ✓

...Ans.

Example 4.17.10

State and prove Euler's theorem for a functions of two variables. If $y = x\cos u$ then find the value of $x^2u_{xx} + 2xy \cdot u_{xy} + y^2u_{yy}$.

Solution : (Note : For proof of Euler's theorem please see the section 4.14)

Step I : Now, Given, $y = x\cos u$

$$\Rightarrow \frac{y}{x} = \cos u \quad \text{or} \quad f(u) = \cos u = \frac{y}{x} = \left(\frac{yt}{xt}\right)$$

$$= t^0\left(\frac{y}{x}\right)$$

is a homogenous function in x and y of degree n = 0

($\therefore$ Here, put $x = xt$, $y = yt$)

Step II : Therefore, by Euler's corollary,

$$x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = G(u)\,[G'(u) - 1]$$

Where, $G(u) = n\frac{f(u)}{f'(u)}$

Step III : Here, $f(u) = \cos u$;

$f'(u) = -\sin u$

$\therefore\ G(u) = n\frac{f(u)}{f'(u)} = 0$;

$G'(u) = 0$ ....(n = 0 here)

Hence, $x^2u_{xx} + 2xy \cdot u_{xy} + y^2u_{yy} = 0$ ✓ **...Ans.**

Example 4.17.11

State and prove Euler's theorem for homogenous functions in two variables and hence find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}$ for $u = e^{x+y} + \log(x^3 + y^3 - x^2y - xy^2)$

Solution : Step I :

For Euler's theorem proof, please seep Section 4.14.

Given, $u = e^{x+y} + \log(x^3 + y^3 - x^2y - xy^2)$

i.e. $u = v + w$ ...(1)

where, $v = e^{xt+yt} \neq v$

So, $v = e^{x-y}$ is not homogenous function but

$f(v) = \log v = x + y = (xt + yt) = t(x + y) = t'f(v)$

be a homogenous function in x and y of degree n = 1 (put $x = xt$, $y = yt$ we get, $f(v) = \log v = t(x + y)$) and $w = \log(x^3 + y^3 - x^2y - xy^2)$ is also **not** homogenous function but $f(w) = e^w = (x^3 + y^3 - x^2y - xy^2)$ is homogenous function in x and y of degree n = 3.

($\because f(w) = t^3 f(w)$)

Step II : By Euler's theorem on homogenous function is x and y.

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n\frac{f(v)}{f'(v)} = 1 \cdot \frac{\log v}{(1/v)} = v \cdot \log v \quad \text{...(A)}$$

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = g(v)\,(g'(v) - 1)$$

$$= v \cdot \log v\left(\log v + \frac{\cancel{t}}{\cancel{t}} \cdot \cancel{t} - \cancel{t}\right)$$

$$= v\log v\,(\log v) = v(\log v)^2 \quad \text{...(B)}$$

and $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = n\frac{f(w)}{f'(w)} = 3 \cdot \frac{e^w}{e^w} = 3$ ...(C)

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\,\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = g(w)\,[g'(w) - 1]$$

...($\because$ Here $g(w) = 3$, $g'(w) = 0$)

$$= 3\,[0 - 1] = 3 \quad \text{...(D)}$$

Step III : Adding Equations (A), (B), (C), (D) we get,

$$x^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial^2 x}\right) + 2xy\left(\frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial x\,\partial y}\right)$$

$$+ y^2\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right) + x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right)$$

$= v \cdot (\log v)^2 + (-3) + v \cdot \log v + (3)$

$$\Rightarrow x^2\frac{\partial^2}{\partial x^2}(v + w) + 2xy\frac{\partial^2}{\partial x\,\partial y}(v + w)$$

$$+ y^2\frac{\partial^2}{\partial y^2}(v + w) + x\left(\frac{\partial}{\partial x}\right)(v + w) + y\frac{\partial}{\partial y}(v + w)$$

$= v\log v\ (1 + \log v)$

$$\Rightarrow x\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$= v \cdot \log(v)\ [1 + \log v]$

$= e^{x+y} \cdot (x + y)(1 + x + y)$ ✓ **...Ans.**

4.18 Change of Variables (Using Equivalent Operator)

Change of Variables (Using Equivalent Operator) :

If $u = f(x, y)$ where $x = \phi(r, \theta)$, $y = \Psi(r, \theta)$

Then $u \longrightarrow (x, y) \longrightarrow (r, \theta)$

It is sometimes necessary to change expressions involving

$u, x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ Etc. into u, r, θ

$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial^2 u}{\partial r^2}, \frac{\partial^2 u}{\partial \theta^2}$ etc.

For that we can use chain rule composite formulas

e. g. $u \longrightarrow (x, y) \longrightarrow (r, \theta)$

$$\therefore\ \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

and $\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}$

or $u \longrightarrow (r, \theta) \longrightarrow (x, y)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

and $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial x}{\partial y} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$

Similarly, we can get higher order partial derivatives

$$\frac{\partial^2 u}{\partial r^2}, \frac{\partial^2 u}{\partial \theta^2}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2} \text{ etc.}$$

Illustrative Examples

Example 4.18.1

If $z = f(u, v)$ and $u = ax + by$ $v = ay - bx$

P.T. $\rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (a^2 + b^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$

Soln. :

Given, $z \longrightarrow (u, v) \longrightarrow (x, y)$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = a\frac{\partial z}{\partial u} - b\frac{\partial z}{\partial v}$$

$$\frac{\partial}{\partial x} = a\frac{\partial}{\partial u} - b\frac{\partial}{\partial v} \quad \ldots(1)$$

Also $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$\therefore \quad \frac{\partial z}{\partial y} = b\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}$$

$$\frac{\partial}{\partial y} = b\frac{\partial}{\partial u} + a\frac{\partial}{\partial v} \quad \ldots(2)$$

Now, $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(a\frac{\partial}{\partial u} - b\frac{\partial}{\partial v}\right)\left(a\frac{\partial z}{\partial u} - b\frac{\partial z}{\partial v}\right)$$

...using Equation (1)

$$\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial u^2} - 2ab\frac{\partial^2 z}{\partial u\, \partial v} + b^2\frac{\partial^2 x}{\partial v^2} \quad \ldots(A)$$

as $$\left(\frac{\partial^2 z}{\partial u\, \partial v} = \frac{\partial^2 z}{\partial v\, \partial u}\right)$$

Also, $$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$= \left(b\frac{\partial}{\partial u} + a\frac{\partial}{\partial v}\right)\left(b\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}\right)$$

...using Equation (2)

$$\frac{\partial^2 z}{\partial y^2} = b^2\frac{\partial^2 z}{\partial u^2} + 2ab\frac{\partial^2 z}{\partial u\, \partial v} + a^2\frac{\partial^2 z}{\partial v^2} \quad \ldots(B)$$

(A) + (B) gives

$$\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial y^2} = (a^2 + b^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right) \quad \text{...Hence proved.}$$

Example 4.18.2

If $x + y = 2e^{\theta}\cos\phi$, $x - y = 2ie^{\theta}\sin\phi$,

show that : $\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy\frac{\partial^2 u}{\partial x\, \partial y}$

Soln. :

$$x + y = 2e^{\theta}\cos\phi \quad \ldots(1)$$

$$x - y = 2ie^{\theta}\sin\phi \quad \ldots(2)$$

By adding Equation (1) and (2)

$$2x = 2e^{\theta}(\cos\phi + i\sin\phi)$$

$$x = e^{\theta}e^{i\phi} = e^{\theta + i\phi}$$

By subtracting Equation (1) and (2)

$$2y = 2e^{\theta}(\cos\phi - i\sin\phi)$$

$$y = e^{\theta}e^{-i\phi} = e^{\theta - i\phi}$$

Regard u as a function of x, y where x and y are functions of θ, ϕ, hence u becomes composite function of θ, ϕ i.e. $u \rightarrow x, y \rightarrow \theta, \phi$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}e^{\theta + i\phi} + \frac{\partial u}{\partial y}e^{\theta - i\phi}$$

$$\frac{\partial u}{\partial \theta} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial \theta}(u) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(u)$$

i.e. $$\frac{\partial}{\partial \theta} \equiv x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) + y\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= x\left\{x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\, \partial y}\right\} + y\left\{x\frac{\partial^2 u}{\partial y\, \partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\right\}$$

$$\frac{\partial^2 u}{\partial \theta^2} = x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + 2xy\frac{\partial^2 u}{\partial x\, \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial \theta^2} = x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\, \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \quad \ldots(3)$$

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \phi} = \frac{\partial u}{\partial x}e^{\theta + i\phi}\, i + \frac{\partial u}{\partial y} \cdot e^{\theta - i\phi}(-i)$$

$$\frac{\partial u}{\partial \phi} = i\left[x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right]$$

$$\frac{\partial}{\partial \phi}(u) = i\left[x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right](u) \text{ i.e. } \frac{\partial}{\partial \phi} \equiv i\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial \phi^2} = \frac{\partial}{\partial \phi}\left(\frac{\partial u}{\partial \phi}\right)$$

$$= \left[i\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\right]\left[i\left(x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right)\right]$$

$$= i^2\left\{x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right) - y\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right)\right\}$$

$$= -\left\{x\left[x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} - y\frac{\partial^2 u}{\partial x\, \partial y}\right] - y\left[x\frac{\partial^2 u}{\partial y\, \partial x} - y\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\right]\right\}$$

$$= -\left\{x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} - 2xy\frac{\partial^2 u}{\partial x\, \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y}\right\}$$

$$\frac{\partial^2 u}{\partial \phi^2} = -x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\, \partial y} - y^2\frac{\partial^2 u}{\partial y^2} - x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} \quad \ldots(4)$$

By adding Equations (3) and (4),

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy\frac{\partial^2 u}{\partial x\, \partial y}$$

Exercise

Ex. 1 Verify Euler's theorem

(i) $u = x^2 yz - 4y^2 z^2 + 2x z^3$

(ii) $u = \sqrt{x} + \sqrt{y}$

(iii) $u = x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right)$

(iv) $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

Ex. 2 If $u = \log\left(\frac{x^2 + y^2}{x + y}\right)$, then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

Ex. 3 If $u = \cos^{-1}\left(\frac{x^3 + y^3 + z^3}{ax + by + cz}\right)$ then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Ex. 4 If $u = 3x^4\cot^{-1}\left(\frac{y}{x}\right) + 16y^4\cos^{-1}\left(\frac{x}{y}\right)$

then prove that $x\, u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 12\, u$

Ex. 5 If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ $= \sin 2u$

Ex. 6 If $V = \sin^{-1}\left(\frac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -3\tan u$

Ex. 7 If $W = \log\left(\frac{x^4 + y^4}{x + y}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$

Ex. 8 If $u = \sin^{-1}\left(\frac{x + y}{\sqrt{x} + \sqrt{y}}\right)$ then find the value of $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$ **Ans. : $\frac{\sin u \cdot \cos 2u}{4\cos^3 u}$**

Ex. 9 If $(x, y) = \sqrt{x^2 - y^2}\,\sin^{-1}\left(\frac{y}{x}\right)$, then prove that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x, y)$

Ex. 10 If $f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$ then prove that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f(x, y) = 0$

Ex. 11 If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ then find the value of $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$

Ex. 12 If $W = x^n f_1\left(\frac{y}{x}\right) + y^{-n} f_2\left(\frac{x}{y}\right)$, then prove that $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} + x\, u_x + y\, u_y = n^2 w$

Ex. 13 If $u = \sin^{-1}\left(\sqrt{x^2 + y^2}\right)$, then find the value of

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ (ii) $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$

Ans. : tan u; $\tan^3 u$

Ex. 14 If $W = \tan^{-1}\left(\frac{\sqrt{x^3 + y^3}}{\sqrt{x} + \sqrt{y}}\right)$ then prove that $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = -\sin^3 u \cdot \cos u$

Ex. 15 If $\sqrt{x} + \sqrt{y}\cos u = x + y$ then find the value of $4x\frac{\partial u}{\partial x} + 4y\frac{\partial u}{\partial y}$ **Ans. : – sin 2u**

Ex. 16 If $u = \sin^{-1}\left(\frac{x^{1/4} + y^{1/4}}{x^{1/6} + y^{1/6}}\right)$, then prove that $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = \frac{1}{144}\tan u\,(\tan^2 u - 1)$

Ex. 17 If $\sqrt{x} + \sqrt{y}\cos u = x + y$ then find the value of $4x\frac{\partial u}{\partial x} + 4y\frac{\partial u}{\partial y}$ **Ans. : – sin 2u**

Ex. 18 If $u = \sin^{-1}\left(\frac{x^{1/4} + y^{1/4}}{x^{1/6} + y^{1/6}}\right)$, then prove that $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = \frac{1}{144}\tan u\,(\tan^2 u - 1)$

Ex. 19 If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, then find the value of

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$= \sin 2u\,(1 - 4\sin^2 u)$

Ex. 20 State and prove Euler's theorem for a function of two variables.

Ex. 21 State and prove Euler's theorem on homogenous function for three variables.

4.19 University Questions and Answers

→ May 18

Q. 1 If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, prove that $x^2 y_{n+2} + (2n+1)xy_{n+1} + 2n^2 y_n = 0$. **(6 Marks)**

Ans. : Here $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$

$$\frac{y}{b} = \cos\left\{n \log\left(\frac{x}{n}\right)\right\}$$

$$y = b\cos\left\{n \log\left(\frac{x}{n}\right)\right\}$$

$$y_1 = -bn \sin\left\{n \log\left(\frac{x}{n}\right)\right\}$$

$$\Rightarrow x^2 y_1^2 = n^2 b^2 \sin^2\left\{n \log\left(\frac{x}{n}\right)\right\} = n^2 b^2\left[1 - \cos^2\left\{n \log\left(\frac{x}{n}\right)\right\}\right]$$

$$= n^2 [b^2 - y^2]$$

$$\Rightarrow x^2 y_2 + x y_1 + n^2 y = 0 \qquad \ldots(1)$$

Differentiating equation (1) n times w.r.t. x by using the Leibnitz's theorem

$$x^2 y_{n+2} + 2nxy_{n+1} + n(n-1) y_n + x y_{n+1} + ny_n + n^2 y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + (2n+1) xy_{n+1} + 2n^2 y_n = 0$$

Q. 2 If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

(Refer Example 4.6.15) **(6 Marks)**

Q. 3 If z is a homogeneous function of degree n in x and y, prove that **(6 Marks)**

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z.$$

Ans. : Since z is a homogeneous function of degree n in x, y hence by Euler's theorem on homogeneous functions

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \qquad \ldots(1)$$

Differentiating equation (1) partially w.r.t x, we obtain

$$\frac{\partial z}{\partial x} + x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x dy} = n\frac{\partial z}{\partial x}$$

$$x^2\frac{\partial^2 z}{\partial x^2} + xy\frac{\partial^2 z}{\partial x dy} = (n-1)\, x\frac{\partial z}{\partial x} \qquad \ldots(2)$$

Simillarly, Differentiating equation (1) w.r.t. y partially

$$y^2\frac{\partial^2 z}{\partial y^2} + xy\frac{\partial^2 z}{\partial x dy} = (n-1)\, y\frac{\partial z}{\partial y} \qquad \ldots(3)$$

Adding equation (2) and (3) we obtain,

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x dy} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

$$= (n-1)\, nz$$

Q. 4 If $r^2 = x^2 + y^2 + z^2$ and $V = r^m$, prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m+1)r^{m-2}$. **(6 Marks)**

Ans. : Given : $r^2 = x^2 + y^2 + z^2$ and $v = r^m$

Differentiating w.r.t. x

$$\frac{\partial v}{\partial x} = \frac{\partial r^m}{\partial x} = \frac{\partial r^m}{\partial \varepsilon} \cdot \frac{\partial r}{\partial x}$$

$$= m\, r^{m-1}\, 1/2\, (x^2 + y^2 + z^2)^{-1/2}\, (2x)$$

$$= \frac{m\, r^{m-1} \cdot x}{\sqrt{x^2 + y^2 + z^2}}$$

Again Differentiating w.r.t. x

$$\frac{\partial^2 v}{\partial x^2} = m\left\{\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\right\} = m\left\{\frac{\partial}{\partial x}\frac{r^{m-1}\, x}{\sqrt{x^2+y^2+z^2}}\right\}$$

$$= m$$

$$\left\{\frac{r^{m-1}}{\sqrt{x^2+y^2+z^2}} + r^{m-1}(x)\frac{-1}{2}(x^2+y^2+z^2)^{-3/2}(2x) + \frac{x}{\sqrt{x^2+y^2+z^2}}\frac{\partial r^{m-1}}{\partial r}\frac{\partial r}{\partial x}\right\}$$

$$= m\left\{\frac{r^{m-1}}{\sqrt{x^2+y^2+z^2}} - \frac{r^{m-1}x^2}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}} + \frac{(m-1)\, x^2 \cdot r^{m-2}}{r^2}\right\}$$

$$= m\left\{r^{m-2} - x^2 r^{m-4} + (m-1)\, x^2 r^{m-4}\right\}$$

$$\frac{\partial^2 v}{\partial x^2} = m\, r^{m-2} - m\, r^{m-4} x^2 + m(m-1)\, x^2 r^{m-4} \qquad \ldots(1)$$

Similarly,

$$\frac{\partial^2 v}{\partial y^2} = m\, r^{m-2} - m\, r^{m-4} y^2 + m(m-1)\, y^2 r^{m-4} \qquad \ldots(2)$$

and

$$\frac{\partial^2 v}{\partial z^2} = m\, r^{m-2} - m\, r^{m-4} z^2 + m(m-1)\, z^2 r^{m-4}$$

Adding equation (1), (2) and (3)

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} = 3m\, r^{m-2} - m\, r^{m-4}(x^2 + y^2 + z^2)$$

$$+ m(m-1)\, r^{m-4}(x^2 + y^2 + z^2)$$

$$= 3\, mr^{m-2} - mr^{m-2} + m(m-1)\, r^{m-2}$$

$$= (m + m^2)\, r^{m-2} = m(m+1)\, r^{m-2}$$

$$= \text{R.H.S.}$$

Q. 5 If $y = x\log(1+x)$, prove that $y_n = \frac{(-1)^{n-2}(n-2)!\,(x+n)}{(x+1)^n}$. **(6 Marks)**

Ans. :

Here, we have $y = x\log(x+1)$

i.e. $y_1 = \log(x+1) + \frac{x}{x+1}$

$= \log(x+1) + 1 - \frac{1}{1+x}$

Differentiating both the sides (n – 1) times,

$$y_n = \frac{(-1)^{n-2}(n-2)!}{(x+1)^n} + 0 - \frac{(-1)^{n-1}(n-1)!}{(x+1)^n}$$

$$= \frac{(-1)^{n-2}(n-2)!}{(x+1)^n}\left[\frac{1}{(x+1)^{-1}} - \frac{(-1)^{-1}(n-1)}{1}\right.$$

$$= \frac{(-1)^{n-2}(n-2)!}{(x+1)^n}\,[(x+1) + (n-1)]$$

$$= \frac{(-1)^{n-2}(n-2)!}{(x+1)^n}\,(x+n)$$

Chapter En

Note

UNIT 2

Partial Differentiation

- Short Questions and Answers
- Fill in the Blanks
- Multiple Choice Questions

UNIT II

Partial Differentiation

Short Questions and Answers

Ex. 1 : Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z = \log(x^2 + y^2)$

Soln. :

Consider $\quad z = \log(x^2 + y^2)$

Step 1 : Differentiate z partially w.r.t. x keeping y as constant

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2}(2x) = \frac{2x}{x^2+y^2}$$

Step 2 : Differentiate z partially w.r.t. y keeping x as constant.

$$\therefore \frac{\partial z}{\partial y} = \frac{1}{x^2+y^2}(2y) = \frac{2y}{x^2+y^2} \checkmark$$

Ex. 2 : Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for $u = x^y + y^x$

Soln. :

Consider $\quad u = x^y + y^x$

Step 1 : Differentiate u partially w. r. t. x, keeping y as constant

$$\therefore \frac{\partial u}{\partial x} = yx^{y-1} + y^x \log y$$

Step 2 : Differentiate u partially w. r. t. y, keeping x constant.

$$\therefore \frac{\partial u}{\partial y} = x^y \log x + xy^{x-1} \checkmark$$

Ex. 3 : If $u = ax + by$; $v = bx - ay$ then show that : $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{a^2}{a^2+b^2}$

Soln. : We have

$$u = ax + by ;$$

Differentiate u partially, w.r.t. x, keeping y as constant

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y = a$$

To get $\left(\frac{\partial x}{\partial u}\right)_v$, first eliminate y.

Consider,
$$u = ax + by$$
$$v = bx - ay$$

Multiply Equation (1) by a and Equation (2) by

$$\therefore \quad au = a^2x + aby$$
$$+ bv = b^2x - aby$$
$$au + bv = (a^2 + b^2)x$$

$$\therefore \quad x = \frac{au + bv}{a^2 + b^2}$$

Differentiate x w.r.t. u, keeping v as constant

$$\therefore \quad \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{a^2+b^2}$$

$$\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right) = a \cdot \frac{a}{a^2+b^2} = \frac{a^2}{a^2+b^2} \checkmark$$

Ex. 4 : If $x = r\cos\theta$, $y = r\sin\theta$ then the v
$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2$

Soln. : We have

$$x = r\cos\theta ; \quad y = r\sin\theta$$
$$r^2 = x^2 + y^2$$

Differentiate r^2

$$\therefore \quad 2r\frac{\partial r}{\partial x} = 2x \quad \text{and} \quad 2r\frac{\partial r}{\partial y} =$$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} \quad ; \quad \frac{\partial r}{\partial y} =$$

$$\therefore \quad \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2+y^2}{r^2} = \frac{r^2}{r^2} = 1$$

$$\therefore \quad \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1 \checkmark$$

... Hence

Ex. 5 : If $u = \cos^{-1}\left[\frac{x^3y^2 + 4y^3x^2}{\sqrt{x^4 + 6y^4}}\right]$, then the
$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Soln. :

Consider $u = \cos^{-1}\left[\dfrac{x^3y^2 + 4y^3x^2}{\sqrt{x^4+6y^4}}\right]$

$$\therefore \cos u = \frac{x^3y^2+4y^3x^2}{\sqrt{x^4+6y^4}} = \frac{x^5\left[\left(\frac{y^2}{x^2}\right)+4\left(\frac{y^3}{x^3}\right)\right]}{\sqrt{x^4\left(1+6\frac{y^4}{x^4}\right)}}$$

$$= \frac{x^5\left[\left(\frac{y}{x}\right)^2+4\left(\frac{y}{x}\right)^3\right]}{x^2\left[\left(1+6\left(\frac{y}{x}\right)^4\right)\right]^{1/2}} = x^3 f\left(\frac{y}{x}\right)$$

$\therefore$ By definition, $z = \cos u$ is homogenous function of degree $n = 3$.

$\therefore$ By deduction of Euler's theorem.

$$\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n\,f(u)}{f'(u)} = 3\frac{\cos u}{-\sin u} = -3\cot u} \checkmark$$

Ex. 6 : If $u = \sin^{-1}\sqrt{x^2+y^2}$ find the value of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$.

Soln. :

Consider, $u = \sin^{-1}\sqrt{x^2+y^2}$

$$\sin u = \sqrt{x^2+y^2} = \sqrt{x^2\left(1+\frac{y^2}{x^2}\right)}$$

$$\sin u = x\left[1+\left(\frac{y}{x}\right)^2\right]^{1/2} = x\,f\left(\frac{y}{x}\right)$$

$\therefore$ By definition ;

$z = \sin u$, is homogenous function of degree $n = 1$

By second deduction of Euler's theorem.

$$\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n\,f(u)}{f'(u)} = (1)\frac{\sin u}{\cos u} = \tan u} \checkmark$$

Ex. 7 : By Euler's theorem on homogenous functions for $u = \dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$ find the value of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$

Soln. :

We know, Euler's theorem on homogenous functions

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu \qquad \ldots(1)$$

where, n is degree of u

$$\textbf{Given, } u = \frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}} = \frac{x\left(1+\frac{y}{x}+\frac{z}{x}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}+\frac{\sqrt{z}}{\sqrt{x}}\right)}$$

$$= x^{1-1/2} f\left(\frac{y}{x},\frac{z}{x}\right) = x^{1/2} f\left(\frac{y}{x},\frac{z}{x}\right)$$

$\therefore$ By definition, u is homogenous function of degree $n = \frac{1}{2}$

$\therefore$ From Equation (1),

$$\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = \frac{1}{2}\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}} \checkmark$$

Ex. 8 : Find $\dfrac{dy}{dx}$ for $x^y + y^x = a^b$, where a and b are constants.

Soln. :

Let $f(x, y) = x^y + y^x - a^b = 0 \qquad \ldots(1)$

Since, $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} \qquad \ldots(2)$

We have to find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$

Differentiate Equation (1) partially w.r.t.x, keeping y constant

$$\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$$

Differentiate Equation (1) partially w.r.t.y keeping x constant.

$$\frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$$

Substitute the value of $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ in Equation (2),

$$\therefore \mathbf{\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{[yx^{y-1} + y^x \log y]}{x^y \log x + xy^{x-1}}} \checkmark$$

Ex. 9 : If $\phi(x, y, z) = 0$, the value of $\left(\dfrac{\partial z}{\partial y}\right)_x\left(\dfrac{\partial x}{\partial z}\right)_y\left(\dfrac{\partial y}{\partial x}\right)_z$

Soln. :

Step 1 : Given, $\phi(x, y, z) = 0$ [Implicit function]

$$\left(\frac{\partial z}{\partial y}\right)_x = -\frac{\partial\phi/\partial y}{\partial\phi/\partial z} \qquad \ldots(1)$$

$$\left(\frac{\partial x}{\partial z}\right)_y = -\frac{\partial\phi/\partial z}{\partial\phi/\partial x} \qquad \ldots(2)$$

$$\left(\frac{\partial y}{\partial x}\right)_z = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y} \qquad \ldots(3)$$

Step 2 :

Consider LHS : $\left(\dfrac{\partial z}{\partial y}\right)_x\left(\dfrac{\partial x}{\partial z}\right)_y\left(\dfrac{\partial y}{\partial x}\right)_z$

Substituting Equations (1), (2) and (3)

$$\textbf{L.H.S.} \mathbf{-\frac{\partial\phi/\partial y}{\partial\phi/\partial z}\times-\frac{\partial\phi/\partial z}{\partial\phi/\partial x}\times-\frac{\partial\phi/\partial x}{\partial\phi/\partial y} = -1} \checkmark$$

Ex. 10 : Find $\dfrac{\partial^2 u}{\partial x\,\partial y}$ for $u = \tan^{-1}\dfrac{y}{x}$

Soln. :

Consider $u = \tan^{-1}\dfrac{y}{x}$

For LHS; differentiate u partially w. r. t. y

Step 1 :

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}$$

$$= \frac{x^2}{x^2+y^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

To find $\dfrac{\partial^2 u}{\partial x\,\partial y}$, differentiate $\dfrac{\partial u}{\partial y}$ partially w. r. t. x

Step 2 : $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{x}{x^2+y^2}\right)$

$$\therefore \text{Using } \frac{\partial}{\partial x}\left(\frac{f}{g}\right) = \frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2} \qquad \text{(Quotient rule)}$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{(x^2+y^2)\frac{\partial}{\partial x}(x) - x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2} = \frac{x^2+y^2(1)-x(2x)}{(x^2+y^2)^2}$$

$$\mathbf{\frac{\partial^2 u}{\partial x\,\partial y} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}} \checkmark$$

Ex. 11 : If $u = e^{x-pt}\cos(x - pt)$, find $\dfrac{\partial u}{\partial t}$.

Soln. : Consider, $u = e^{x-pt}\cos(x-pt)$

Step 1: To prove LHS ; differentiate u partially w. r. t. t .

$$\frac{\partial u}{\partial t} = e^{x-pt}\frac{\partial}{\partial t}\cos(x-pt) + \cos(x-pt)\frac{\partial}{\partial t}(e^{x-pt})$$

$$= e^{x-pt}(+p\sin(x-pt)) + \cos(x-pt)(-p\,e^{x-pt})$$

$$\mathbf{\frac{\partial u}{\partial t} = p\,e^{x-pt}[\sin(x-pt) - \cos(x-pt)]} \checkmark$$

Fill in the Blanks / Multiple Choice Questions

Q. 1 By Euler's Theorem on Homogeneous Function ,if u is a homogeneous function in x and y of degree n then value of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ = ______.

(a) $n\dfrac{f(u)}{f'(u)}$ (b) $n(n-1)u$ (c) nu (d) n^2u **Ans. : (c)**

Q. 2 If u is a homogenous function in x and y of degree n then value of

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \_\_\_\_\_\_.$$

(a) $n(n-1)u$ (b) nu (c) $n\dfrac{f(u)}{f'(u)}$ (d) n^2u

Ans. : (a)

Q. 3 If f(u) is a homogenous function in x and y of degree n then $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$ = ______.

(a) $n\dfrac{f(u)}{f'(u)}$ (b) $n(n-1)u$

(c) nu (d) $G(u)[G'(u) - 1]$,

Where, $G(u) = n\dfrac{f(u)}{f'(u)}$ **Ans. : (d)**

Q. 4 If f(u) is a homogenous function in x and y of degree n then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ = ______.

(a) nu (b) $n\dfrac{f(u)}{f'(u)}$ (c) $n(n-1)u$ (d) $n\,f(u)$

Ans. : (b)

Q. 5 By Euler's Theorem for Homogeneous Function $u = \dfrac{\sqrt{xy}}{\sqrt{x}+\sqrt{y}}$, the value of : $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ is ....

(a) $\dfrac{1}{2}\left(\dfrac{\sqrt{xy}}{\sqrt{x}+\sqrt{y}}\right)$ (b) $\dfrac{\sqrt{xy}}{\sqrt{x}+\sqrt{y}}$

(c) $\sqrt{xy}$ (d) $\sqrt{x}+\sqrt{y}$ **Ans. : (a)**

Q. 6 By Euler's Theorem for Homogeneous Function $u = \dfrac{\sqrt{x}+\sqrt{y}}{xy}$ the value of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ is ______.

(a) $-\dfrac{\sqrt{x}+\sqrt{y}}{xy}$ (b) $\dfrac{\sqrt{x}+\sqrt{y}}{xy}$

(c) $-\dfrac{3}{2}\left(\dfrac{\sqrt{x}+\sqrt{y}}{xy}\right)$ (d) x^2y **Ans. : (c)**

Q. 7 By Euler's Theorem for Homogeneous Function $z = \log x - \log y$, the value of $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}$ is ____.

(a) 1 (b) 0 (c) 3 (d) e^z **Ans. : (b)**

Q. 8 By Euler's Theorem for Homogeneous Function $u = \frac{1}{x^2} + \frac{1}{y^2}$, the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is ______.

(a) $-2\left(\frac{1}{x^2}+\frac{1}{y^2}\right)$ (b) $\frac{x^2}{y^2}$

(c) $\frac{x^2+y^2}{x^2y^2}$ (d) $3\left(\frac{1}{x^2}-\frac{1}{y^2}\right)$ **Ans. : (a)**

Q. 9 If $z = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ then value of $\frac{\partial z}{\partial y} =$ ______.

(a) $x + 2y$ (b) $x - 2y$

(c) $x^2 + 2y\tan^{-1}\left(\frac{x}{y}\right)$ (d) None of the above

Ans. : (d)

Q. 10 If $u = \log(\tan x + \tan y + \tan z)$ then value of $\frac{\partial u}{\partial z} =$ ______.

(a) $\frac{\sec^2 z}{(\tan x + \tan y + \tan z)}$ (b) $\frac{1}{\tan x + \tan y + \tan z}$

(c) $\frac{\sec^2 x}{\tan x + \tan y + \tan z}$ (d) e^u **Ans. : (a)**

Q. 11 If $u = \tan^{-1}\left(\frac{x}{y}\right)$ then value of $\frac{\partial u}{\partial y} =$ ______.

(a) $\frac{x^2}{x^2+y^2}$ (b) $-\frac{x}{x^2+y^2}$

(c) $\frac{xy}{x^2+y^2}$ (d) $\frac{-y}{x^2+y^2}$ **Ans. : (b)**

□□□

UNIT 3

Applications of Partial Differentiation

Syllabus :

Jacobians - properties; Taylor's and Maclaurin's theorems (without proofs) for functions of two variables; Maxima and minima of functions of two variables; Lagrange's method of undetermined multipliers.

- **Chapter 5 : Applications of Partial Differentiation**

CHAPTER 5

UNIT III

Applications of Partial Differentiation

Syllabus

Jacobians - properties; Taylor's and Maclaurin's theorems (without proofs) for functions of two variables; Maxima and minima of functions of two variables; Lagrange's method of undetermined multipliers.

5.1 Introduction

Jacobian is a functional determinant (whose elements are functions). Jacobian is a powerful tool in transformation of Cartesian system to polar, spherical polar and cylindrical co-ordinate system in multiple integrals. Jacobian can also be used to solve system of differential equations at an equilibrium point or approximative solution near an equilibrium point. The area of a small region in the uv-plane is scaled by the Jacobian determinant. Also examine functional dependancy by Jacobian.

The concept of Jacobian was introduced by the German mathematician Carl Gustav Jacob Jacobi (1804-1851). Jacobian determinant is simply called the Jacobian.

5.2 Jacobians

If u (x, y) and v (x, y) are continuous and differentiable functions of two independent variables x and y then the Jacobian of u, v w.r.t. x and y is denoted by $J\left(\frac{u, v}{x, y}\right)$ or $\frac{\partial(u, v)}{\partial(x, y)}$ or simply J and is defined as,

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Therefore, $J = J\left(\frac{u, v}{x, y}\right) = \frac{\partial(u, v)}{\partial(x, y)}$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \text{ or } \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Similarly, if u, v, w are the functions of three independent variables x, y, z then the Jacobian of u, v, w w.r.t. x, y, z is

$$J = J\left(\frac{u, v, w}{x, y, z}\right) = \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$\text{or } \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

In the same manner, we can determine the Jacobian of n functions in n variables.

5.2.1 Properties of Jacobian

(1) If u and v are functions of x and y and $J = \frac{\partial(u, v)}{\partial(x, y)}$ and $J' = \frac{\partial(x, y)}{\partial(u, v)}$ then $JJ' = 1$

i.e. $\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = 1$

(2) If $u = f_1(x, y)$; $v = f_2(x, y)$ and $x = \phi_1(r, \theta)$; $y = \phi_2(r, \theta)$ then $\frac{\partial(u, v)}{\partial(r, \theta)} = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, \theta)}$

it is called "Chain rule" of Jacobian. This chain rule of Jacobian is true for functions of three and more variables.

(3) The functions u, v of two independent variables x, y are functionally dependent if $\frac{\partial(u, v)}{\partial(x, y)} = 0$.

Type I : Examples based on Jacobian

Example 5.2.1

If $x = r\cos\theta$; $y = r\sin\theta$ then find $\frac{\partial(x, y)}{\partial(r, \theta)}$

Solution :

Step I : Given, $x = r\cos\theta$; $y = r\sin\theta$

Here, $x, y \to f(r, \theta)$

Step II : Finding the derivatives of x and y w.r.t r and θ

$$\therefore \left.\begin{array}{ll} \frac{\partial x}{\partial r} = \cos\theta\ ; & \frac{\partial x}{\partial \theta} = -r\sin\theta\ ; \\ \frac{\partial y}{\partial r} = \sin\theta\ ; & \frac{\partial y}{\partial \theta} = r\cos\theta \end{array}\right\} \text{(by standard derivatives)} \quad ...(1)$$

Step III :

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \quad \text{...(by definition of Jacobian)}$$

Substitute values from Equation (1), it gives

$$= \begin{vmatrix} \cos\theta & \boxed{-r\sin\theta} \\ \boxed{\sin\theta} & r\cos\theta \end{vmatrix}$$

$$= \underbrace{(\cos\theta)(r\cos\theta)}_{r\cos^2\theta} - \underbrace{(\sin\theta)(-r\sin\theta)}_{-r\sin^2\theta}$$

$$= r\cos^2\theta + r\sin^2\theta \quad ...(2)$$

→ Using trigonometric formula in eqn. (1)

...[∵ $\cos^2\theta + \sin^2\theta = 1$]

$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$\therefore J = \frac{\partial(x, y)}{\partial(r, \theta)} = r ✓ \quad \text{...Ans.}$$

Example 5.2.2

If $u = x^2 - y^2$; $v = 2x^2 - y^2$ and $x = r\cos\theta$; $y = r\sin\theta$ then show that $\frac{\partial(u, v)}{\partial(x,y)} = 2r^2 \sin 2\theta$

Solution :

Step I : Given, $u = x^2 - y^2$; $v = 2x^2 - y^2$

Here, $u, v \to f(x, y)$ and finding derivatives of u and v w.r.t. x and y.

→ Using standard results of differentiation

Partially differentiate u w.r.t x, keeping y as constant

$$\frac{\partial u}{\partial x} = 2x\ ;$$

Partially differentiate u w.r.t y, keeping x as constant.

$$\frac{\partial u}{\partial y} = -2y\ ;$$

Partially differentiate v w.r.t x, keeping y as constant.

$$\frac{\partial v}{\partial x} = 4x\ ;$$

Partially differentiate v w.r.t y, keeping x as constant.

$$\frac{\partial v}{\partial y} = -2y \quad ...(1)$$

Step II :

$$\therefore J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

...(by definition of Jacobian)

Substitute values from Equation (1), it gives

$$= \begin{vmatrix} 2x & -2y \\ 4x & -2y \end{vmatrix}$$

$$= [(2x) \times (2y)] - [(4x)(-2y)]$$

$$= -4xy + 8xy = 4xy \quad ...(2)$$

Step III : Since, $x = r\cos\theta$; $y = r\sin\theta$

By multiplication of both equations,

$$\therefore xy = (r\cos\theta)(r\sin\theta)$$

$$= r^2 \sin\theta \cdot \cos\theta \quad ...(3)$$

→ Using trigonometric formula in eqn. (2)

...[$2\sin\theta \cdot \cos\theta = \sin 2\theta$]

$$= r^2 \cdot \frac{1}{2}[2\sin\theta\cos\theta] = \frac{r^2}{2}\sin 2\theta$$

Substitute above value in Equation (2), it gives,

$$\therefore J = \frac{\partial(u, v)}{\partial(x,y)} = 4\left(\frac{r^2}{2}\sin 2\theta\right)$$

$$J = \frac{\partial(u, v)}{\partial(x,y)} = 2r^2 \sin 2\theta ✓ \quad \text{...Ans.}$$

Example 5.2.3

If $u = r^2\cos 2\theta$; $v = r^2 \sin 2\theta$, find $\frac{\partial(u, v)}{\partial(r, \theta)}$

Solution :

Step I : Given, $u = r^2\cos 2\theta$; $v = r^2\sin 2\theta$,

Here, $u, v \to f(r, \theta)$ i.e. u, v are functions of r and θ

Differentiate u and v w.r.t r, and θ.

→ Using standard formulae of derivatives

Differentiate u partially w.r.t r, keeping θ as constants.

$$\frac{\partial u}{\partial r} = 2r\cos 2\theta\ ;$$

Differentiate u partially w.r.t θ, keeping r as constants.

$$\frac{\partial u}{\partial \theta} = -2r^2\sin 2\theta \quad ...(1)$$

Differentiate v partially w.r.t r, keeping θ as constants.

$$\frac{\partial v}{\partial r} = 2r\sin 2\theta\ ;$$

Differentiate v partially w.r.t θ, keeping r as constants.

$$\frac{\partial v}{\partial \theta} = 2r^2\cos 2\theta$$

Step II : $J = \frac{\partial(u, v)}{\partial(r, \theta)}$

$$= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} \quad \text{...(by definition of Jacobian)}$$

$$= \begin{vmatrix} 2r\cos 2\theta & \boxed{-2r^2\sin 2\theta} \\ \boxed{2r\sin 2\theta} & 2r^2\cos 2\theta \end{vmatrix}$$

$$= [2r\cos 2\theta \times 2r^2\cos 2\theta] - [2r\sin 2\theta \times (-2r^2\sin 2\theta)]$$

$$= [4r^3\cos^2 2\theta] - [-4r^3\sin^2 2\theta]$$

$$= 4r^3\cos^2 2\theta + 4r^3\sin^2 2\theta$$

$$= 4r^3[\cos^2 2\theta + \sin^2 2\theta] \quad ...(2)$$

→ Using trigonometric formulae in eqn. (2)

...[$\sin^2\theta + \cos^2\theta = 1$]

$$J = \frac{\partial(u, v)}{\partial(r, \theta)} = 4r^3 ✓ \quad \text{...Ans.}$$

Example 5.2.4

If $u = x\sin y$; $v = y\sin x$ then find $\frac{\partial(u, v)}{\partial(x, y)}$

Solution :

Step I : Given, $u = x\sin y$; $v = y\sin x$

Differentiate u and v partially w.r.t x and y

Differentiate u partially w.r.t x, keeping y as constant

$$u_x = \sin y\ ;$$

Differentiate u partially w.r.t y, keeping x as constant.

$$u_y = x\cos y\ ; \quad \text{...(by standard derivatives)} \quad ...(1)$$

Differentiate v partially w.r.t x, keeping y as constant

$$v_x = y\cos x$$

Differentiate v partially w.r.t y, keeping x as constant.

$$v_y = \sin x$$

Step II : We have by definition of Jacobian,

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Substitute values from Equation (1), we get,

$$= \begin{vmatrix} \sin y & \boxed{x\cos y} \\ \boxed{y\cos x} & \sin x \end{vmatrix}$$

$$= [(\sin y)(\sin x)] - [(y\cos x)\cdot(x\cos y)]$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \sin x \sin y - xy\cos x\cos y ✓ \quad \text{...Ans.}$$

Example 5.2.5

If $u = a \cosh x \cos y$; $v = a \sinh x \sin y$, find $\dfrac{\partial (u, v)}{\partial (x, y)}$

Solution :

Step I : **Given ,** $u = a \cosh x \cos y$;

$v = a \sinh x \sin y$

Differentiate u and v partially w.r.t x and y,

$$\left.\begin{array}{l}
\text{Differentiate u partially w.r.t x, keeping y as constant} \\
\dfrac{\partial u}{\partial x} = u_x = a \sinh x \cos y ; \\
\text{Differentiate u partially w.r.t y, keeping x as constant.} \\
\dfrac{\partial u}{\partial y} = u_y = -a \cosh x \sin y ; \\
\text{Differentiate v partially w.r.t x, keeping y as constant.} \\
\dfrac{\partial v}{\partial x} = v_x = a \cosh x \sin y ; \\
\text{Differentiate v partially w.r.t y, keeping x as constant.} \\
\dfrac{\partial v}{\partial y} = v_y = a \sinh x \cos y
\end{array}\right\} \quad \ldots(1)$$

Step II : We know, definition of Jacobian

$$\frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Substitute values from Equation (1), we get,

$$= \begin{vmatrix} a \sinh x \cos y & -a \cosh x \sin y \\ a \cosh x \cos y & a \sinh x \cos y \end{vmatrix}$$

$= [\,(a \sinh x \cos y)\ (a \sinh x \cos y)\,]$

$\quad - [\,(a \cosh x \sin y) \cdot (-a \cosh x \sin y)\,]$

$= [a^2 \sinh^2 x \cos^2 y] - [-a^2 \cosh^2 x \sin^2 y]$

$= a^2 [\sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y] \quad \ldots(2)$

→ Using standard formula in eqn. (2)

$\ldots[\sin^2 \theta + \cos^2 \theta = 1 \Rightarrow \sin^2 \theta = 1 - \cos^2 \theta]$

$= a^2 [\sinh^2 x \cos^2 y + \cosh^2 x (1 - \cos^2 y)]$

$= a^2 [\underline{\sinh^2 x \cos^2 y} + \cosh^2 x - \underline{\cosh^2 x \cos^2 y}]$

$= a^2 [-\cos^2 y (-\sinh^2 x + \cosh^2 x) + \cosh^2 x]$

$= a^2 [-\cos^2 y (\cosh^2 x - \sinh^2 x) + \cosh^2 x] \quad \ldots(3)$

→ Using standard hyperbolic formula for eqn. (3)

$\ldots[\cosh^2 x - \sinh^2 x = 1]$

$$\frac{\partial (u, v)}{\partial (x, y)} = a^2[-\cos^2 y + \cosh^2 x] \checkmark \quad \ldots\text{Ans.}$$

Type II : Examples based on Jacobians

Example 5.2.6

If $u = 1 - x$; $v = x (1 - y)$; $w = xy (1 - z)$ then show that $\dfrac{\partial (u,v,w)}{\partial (x, y, z)} = -x^2 y$

Solution :

Step I : Given, $u = 1 - x$; $v = x(1 - y)$; $w = xy(1 - z)$,

Here, u, v, w → f (x, y, z) i.e. u, v, w are functions of x, y, z.

Differentiate u partially w.r.t x keeping y, z as constant.

$$\therefore \frac{\partial u}{\partial x} = -1;$$

Differentiate u partially w.r.t y, keeping x, z as constants.

$$\frac{\partial u}{\partial y} = 0;$$

Differentiate u partially w.r.t z, keeping x, y as constants.

$$\frac{\partial u}{\partial z} = 0;$$

Differentiate v partially w.r.t x, keeping y, z as constants.

$$\frac{\partial v}{\partial x} = (1 - y);$$

Differentiate v partially w.r.t y, keeping x, z as constant

$$\frac{\partial v}{\partial y} = -x;$$

Differentiate v partially w.r.t z, keeping x, y as constants.

$$\frac{\partial v}{\partial z} = 0$$

Differentiate w partially w.r.t x, keeping y, z as constants.

$$\frac{\partial w}{\partial x} = y(1 - z);$$

Differentiate w partially w.r.t y, keeping x, z as constants.

$$\frac{\partial w}{\partial y} = x(1 - z);$$

Differentiate w partially w.r.t z, keeping x, y as constants

$$\frac{\partial w}{\partial z} = -xy$$

Step II : We know, definition of Jacobian

$$J = \frac{\partial (u,v,w)}{\partial (x, y, z)}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 \\ 1 - y & -x & 0 \\ y(1 - z) & x(1 - z) & -xy \end{vmatrix}$$

$$= (-1) \times [\text{Minor of } (-1)] - \underbrace{0 \times [\text{Minor of } 0]}_{0} + \underbrace{0 \times [\text{Minor of } 0]}_{0}$$

$$= (-1) \times \begin{vmatrix} \cancel{-1} & \cancel{0} & \cancel{0} \\ 1 - y & -x & 0 \\ y(1 - z) & x(1 - z) & -xy \end{vmatrix} = (-1) \times \begin{vmatrix} -x & 0 \\ x(1 - z) & -xy \end{vmatrix}$$

$= (-1) \times [(-x)(-xy) - x(1 - z)(0)]$

$= -1\,[(-x)(-xy) - 0] = -1[x^2 y - 0]$

$$\frac{\partial (u,v,w)}{\partial (x, y, z)} = -x^2 y \checkmark \quad \ldots\text{Ans.}$$

Example 5.2.7

If $u = xyz$; $v = x^2 + y^2 + z^2$; $w = x + y + z$ then find $\dfrac{\partial (u, v, w)}{\partial (x, y, z)}$.

Solution :

Step I : Given , $u = xyz$; $v = x^2 + y^2 + z^2$; $w = x + y + z$

Differentiate u partially w.r.t x,

keeping y, z as constant

$$\frac{\partial u}{\partial x} = yz ;$$

Differentiate u partially w.r.t y,

keeping x, z as constants.

$$\frac{\partial u}{\partial y} = xz ;$$

Differentiate u partially w.r.t z,

keeping x, y as constants

$$\frac{\partial u}{\partial z} = xy$$

Differentiate v partially w.r.t x,

keeping y, z as constants

$$\frac{\partial v}{\partial x} = 2x ;$$

Differentiate v partially w.r.t y,

keeping x, z as constant

$$\frac{\partial v}{\partial y} = 2y ;$$

Differentiate v partially w.r.t z, keeping x, y as constants

$$\frac{\partial v}{\partial z} = 2z$$

Differentiate w partially w.r.t x, keeping y, z as constants

$$\frac{\partial w}{\partial x} = 1 ;$$

Differentiate w partially w.r.t y, keeping x, z as constants.

$$\frac{\partial w}{\partial y} = 1 ;$$

Differentiate w partially w.r.t z, keeping x, y as constants

$$\frac{\partial w}{\partial z} = 1$$

Step II : We know, definition of Jacobian

$$\therefore \frac{\partial (u, v, w)}{\partial (x, y, z)} = J = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

Substitute values from step (1) we get,

$$= \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

For simplicity, to solve determinant use elementary transformations,

Operate R_{13}

$$= - \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ yz & xz & xy \end{vmatrix}$$

Common out 2 from R_2

$$= -2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & xz & xy \end{vmatrix}$$

Operate $C_2 - C_1,\ C_3 - C_1$

$$= -2\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ yz & xz-yz & xy-yz \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ yz & -z(y-x) & -y(z-x) \end{vmatrix}$$

Common out $(y-x)$ from C_2 and $(z-x)$ from C_3, we get,

$$= -2\,(y-x)\,(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ yz & -z & -y \end{vmatrix}$$

$$= -2\,(y-x)\,(z-x)\times$$

$$\{1\times[\text{Minor of }1] - \underbrace{0\times[\text{Minor of }0]}_{0} + \underbrace{0\times[\text{Minor of }0]}_{0}\}$$

$$= 2(y-x)\,(z-x)\times\begin{vmatrix} \cancel{1} & \cancel{0} & \cancel{0} \\ \cancel{x} & 1 & 1 \\ \cancel{yz} & -z & -y \end{vmatrix}$$

$$= -2\,(y-x)\,(z-x)\times\begin{vmatrix} 1 & 1 \\ -z & -y \end{vmatrix}$$

$$= -2\,(y-x)\,(z-x)\cdot[(1)\,(-y)\,(-z)\,(1)]$$

$$= -2\,(y-x)\,(z-x)\,(-y+z)$$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \mathbf{-2\,(x-y)\,(y-z)\,(z-x)}\ ✓$$ ...Ans.

Example 5.2.8

If $xu = vw$; $yv = wu$; $wz = uv$ then find $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$

Solution :

Step I : Given, $x = \dfrac{vw}{u}$; $\quad y = \dfrac{wu}{v}$;

$$z = \frac{uv}{w}$$

$$\text{Using, } d\left(\frac{f_1}{g_1}\right) = \frac{g_1\,d(f_1) - f_1\,d(g_1)}{g_1^2}$$

Differentiate x partially w.r.t. u, keeping v, w as constants

$$\frac{\partial x}{\partial u} = -\frac{vw}{u^2};$$

Differentiate x partially w.r.t. v, keeping u, w as constants

$$\frac{\partial x}{\partial v} = \frac{w}{u};$$

Differentiate x partially w.r.t. w, keeping u, v as constants

$$\frac{\partial x}{\partial w} = \frac{v}{u}$$

Differentiate y partially w.r.t. u, keeping v, w as constants

$$\frac{\partial y}{\partial u} = \frac{w}{v};$$

Differentiate y partially w.r.t. v, keeping u, w as constants

$$\frac{\partial y}{\partial v} = -\frac{wu}{v^2};$$

Differentiate y partially w.r.t. w, keeping u, v as constants

$$\frac{\partial y}{\partial w} = \frac{u}{v}$$

Differentiate z partially w.r.t. u, keeping v, w as constants

and $\quad \dfrac{\partial z}{\partial u} = \dfrac{v}{w};$

Differentiate z partially w.r.t. v, keeping u, w as constants

$$\frac{\partial z}{\partial v} = \frac{u}{w};$$

Differentiate z partially w.r.t. w, keeping u, v as constants

$$\frac{\partial z}{\partial w} = -\frac{uv}{w^2}$$

Step II :

We know, definition of Jacobian,

$$\therefore\ J = \frac{\partial(x, y, z)}{\partial(u, v, w)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Substitute values from step (1), we get

$$= \begin{vmatrix} -\frac{vw}{u^2} & \frac{w}{u} & \frac{v}{u} \\ \frac{w}{v} & -\frac{wu}{v^2} & \frac{u}{v} \\ \frac{v}{w} & \frac{u}{w} & -\frac{uv}{w^2} \end{vmatrix}$$

For simplicity, to solve determinant use elementary transformation and laws of determinant.

Common out $\dfrac{1}{u^2}$ from R_1, $\dfrac{1}{v^2}$ from R_2 and $\dfrac{1}{w^2}$ from R_3, we get,

$$= \frac{1}{u^2 v^2 w^2}\begin{vmatrix} -vw & wu & uv \\ vw & -wu & uv \\ vw & wu & -uv \end{vmatrix}$$

Common out (vw) from C_1, (wu) from C_2, (uv) from C_3, we get,

$$= \frac{((vw)\,(wu)\,(uv))}{u^2 v^2 w^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{\cancel{u^2 v^2 w^2}}{\cancel{u^2 v^2 w^2}}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= (-1)\times[\text{Minor of }(-1)] - 1\text{x}\ [\text{minor of }1]$

$\quad + 1\times[\text{minor of }1]$

$$= (-1)\times\begin{vmatrix} \cancel{-1} & \cancel{1} & \cancel{1} \\ \cancel{1} & -1 & 1 \\ \cancel{1} & 1 & -1 \end{vmatrix} - 1\times\begin{vmatrix} \cancel{-1} & \cancel{1} & \cancel{1} \\ 1 & \cancel{-1} & 1 \\ 1 & \cancel{1} & -1 \end{vmatrix} + 1\times\begin{vmatrix} \cancel{-1} & \cancel{1} & \cancel{1} \\ 1 & -1 & \cancel{1} \\ 1 & 1 & \cancel{-1} \end{vmatrix}$$

$$= (-1)\times\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - 1\times\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 1\times\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$= (-1)\,[(-1)\times(-1) - (1)\times(1)] - 1$

$\quad [(-1)\times(1) - (1)\times(1)] + 1\,[(1)\times(1) - (1)\,(-1)]$

$= -[1-1] - 1\,[-1-1] + [1+1]$

$= 0 - (-2) + 2 = 2 + 2 = 4$ ...Ans.

Example 5.2.9

If $x + y + z = u$; $y + z = uv$; $z = uvw$ then find $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$

Solution :

Given,

Step I : $\quad x + y + z = u; \qquad y + z = uv;$

$$z = uvw$$

First find x, y, z interms of u, v, w

$\Rightarrow\ x = u - (y + z); \qquad y = uv - z;$

$\quad\ \ \mathbf{y = uv - uvw}; \qquad \mathbf{x = u - uv}$

$\quad\ \ \mathbf{z = uvw}$

Differentiate x, y, z partially w.r.t. u, v, w

Differentiate x partially w.r.to u keeping v, w as constants

$$\therefore\ \frac{\partial x}{\partial u} = 1 - v$$

Differentiate x partially w.r.to v, keeping u, w as constants

$$\frac{\partial x}{\partial v} = -u$$

Differentiate x partially w.r.to w, keeping u, v as constants

$$\frac{\partial x}{\partial w} = 0$$

Differentiate y partially w.r.to u, keeping v, w as constants

$$\frac{\partial y}{\partial u} = v - vw$$

Differentiate y partially w.r.to v, keeping u, w as constants

$$\frac{\partial y}{\partial v} = u - uw$$

Differentiate y partially w.r.to w, keeping u, v as constants

$$\frac{\partial y}{\partial w} = -uv$$

Differentiate z partially w.r.to u, keeping v, w as constants

$$\frac{\partial z}{\partial u} = vw$$

Differentiate z partially w.r.to v, keeping u, w as constants

$$\frac{\partial z}{\partial v} = uw$$

Differentiate z partially w.r.to w, keeping u, v as constants

$$\frac{\partial z}{\partial w} = uv$$

Step II : We know, definition of Jacobian

$$\therefore\ \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

For simplicity, to evaluate determinant use elementary terms formations.

Operate $R_1 + R_2 + R_3$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$= 1 \times [\text{Minor of } 1] - \underbrace{0 \times [\text{Minor of } 0]}_{0} + \underbrace{0 \times [\text{Minor of } 0]}_{0}$

$$= 1 \times \begin{vmatrix} 1 & 0 & 0 \\ -vw & u-uw & -uv \\ w & uw & uv \end{vmatrix} = 1 \times \begin{vmatrix} u-uw & -uv \\ uw & uv \end{vmatrix}$$

$= 1 \times [(u - uw)(uv) - (uw)(-uv)]$

$= (u - uw)\, uv + uv\, uw$

$= u^2 v - \cancel{uwuv} + \cancel{uvuw} = u^2 v$

$$\therefore \frac{\partial (x, y, z)}{\partial (u, v, w)} = u^2 v \checkmark \qquad \text{...Ans.}$$

Example 5.2.10

If $ux = yz$; $vy = zx$, $wz = xy$ then find $\dfrac{\partial (u, v, w)}{\partial (x, y, z)}$

Solution :

Step I : Given , $ux = yz$; $vy = zx$; $wz = xy$

$$\therefore u = \frac{yz}{x}; \qquad v = \frac{zx}{y};$$

$$w = \frac{xy}{z}$$

Here u, v, w $\rightarrow$ F (x, y, z) i.e. u, v, w are functions of x, y, z

Differentiate u, v, w partially w.r.t x, y, z by

Partially differentiate u w.r. to x keeping y, z as constant

$$\frac{\partial u}{\partial x} = -\frac{yz}{x^2}$$

Differentiate u partially w.r.to y, keeping x, z as constants.

$$\frac{\partial u}{\partial y} = \frac{z}{x}$$

Differentiate u partially w.r. to z, keeping x, y as constants

$$\frac{\partial u}{\partial z} = \frac{y}{x}$$

Differentiate v partially w.r.to x, keeping y, z as constants

$$\frac{\partial v}{\partial x} = \frac{z}{y}$$

Differentiate v partially w.r.to y, keeping x, z as constant

$$\frac{\partial v}{\partial y} = -\frac{zx}{y^2}$$

Differentiate v partially w.r.to z, keeping x, y as constants

$$\frac{\partial v}{\partial z} = \frac{x}{y}$$

Differentiate w partially w.r.to x, keeping y, z as constants.

$$\frac{\partial w}{\partial x} = \frac{y}{z}$$

Differentiate w partially w.r.to y, keeping x, z as constants.

$$\frac{\partial w}{\partial y} = \frac{x}{z}$$

Differentiate w partially w.r.to z, keeping x, y as constants

$$\frac{\partial w}{\partial z} = -\frac{xy}{z^2}$$

Step II : We know, definition of Jacobian

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Substitute values from step (1), we get

$$= \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

Common out $\frac{1}{x^2}$ from R_1, $\frac{1}{y^2}$ form R_2 and $\frac{1}{z^2}$ form R_3, we get

$$= \frac{1}{x^2 y^2 z^2} \begin{vmatrix} -yz & xz & xy \\ yz & -zx & xy \\ yz & zx & -xy \end{vmatrix}.$$

Common out yz from C_1, xz from C_2 and xy from C_3, we get

$$= \frac{(yz)(xz)(xy)}{x^2 y^2 z^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{\cancel{x^2 y^2 z^2}}{\cancel{x^2 y^2 z^2}} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= (-1) \times [\text{Minor of } (-1) - 1 \times [\text{minor of } 1]]$

$+ 1 \times [\text{minor of } 1]$

$$= (-1) \times \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} - 1 \times \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} + 1 \times \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= (-1) \times \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - 1 \times \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$= -1(1 - 1) - 1(-1 - 1) + 1(1 + 1)$

$$= 0 + 2 + 2 = 4 \checkmark \qquad \text{...Ans.}$$

Example 5.2.11

If $u = \dfrac{x}{y - z}$; $v = \dfrac{y}{z - x}$; $w = \dfrac{z}{x - y}$

then find : $\dfrac{\partial (u, v, w)}{\partial (x, y, z)}$.

Solution :

Step I : Given , $u = \dfrac{x}{y - z}$; $v = \dfrac{y}{z - x}$

$$w = \frac{z}{x - y}$$

u, v, w $\rightarrow$ f (x, y, z) i.e. u, v, w are functions of x, y, z

Differentiate u, v, w partially w.r.t x, y, z

→ Using standard, rules of differentiation

Partially differentiate u w.r. to x keeping y, z as constant

$$\frac{\partial u}{\partial x} = \frac{1}{y - z}$$

Differentiate u partially w.r.to y, keeping x, z as constants

$$\frac{\partial u}{\partial y} = -\frac{x}{(y - z)^2}$$

Differentiate u partially w.r. to z, keeping x, y as constants

$$\frac{\partial u}{\partial z} = \frac{x}{(y - z)^2}$$

Differentiate v partially w.r.to x, keeping y, z as constants

$$\frac{\partial v}{\partial x} = \frac{y}{(z - x)^2}$$

Differentiate v partially w.r.to y, keeping x, z as constant

$$\frac{\partial v}{\partial y} = \frac{1}{z - x}$$

Differentiate v partially w.r.to z, keeping x, y as constants

$$\frac{\partial v}{\partial z} = -\frac{y}{(z - x)^2}$$

Differentiate w partially w.r.to x, keeping y, z as constants

$$\frac{\partial w}{\partial x} = -\frac{z}{(x - y)^2}$$

Differentiate w partially w.r.to y, keeping x, z as constants

$$\frac{\partial w}{\partial y} = \frac{z}{(x - y)^2}$$

Differentiate w partially w.r.to z, keeping x, y as constants

$$\frac{\partial w}{\partial z} = \frac{1}{(x - y)}$$

Step II : We know, by definition of Jacobian

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Substitute values from step (1), we get

$$= \begin{vmatrix} \frac{1}{(y-z)} & \frac{-x}{(y-z)^2} & \frac{x}{(y-z)^2} \\ \frac{y}{(z-x)^2} & \frac{1}{(z-x)} & \frac{-y}{(z-x)^2} \\ \frac{-z}{(x-y)^2} & \frac{z}{(x-y)^2} & \frac{1}{(x-y)} \end{vmatrix}$$

Common out $\frac{1}{(y-z)^2}$ from R_1, $\frac{1}{(z-x)^2}$ from R_2 and $\frac{1}{(z-x)^2}$ from R_3, we get

$$= \frac{1}{(y-z)^2} \cdot \frac{1}{(z-x)^2 (x-y)^2} \begin{vmatrix} y-z & -x & x \\ y & (z-x) & -y \\ -z & z & (z-y) \end{vmatrix}$$

By solving determinant,

$$= \frac{1}{(y-z)^2 (z-x)^2 (x-y)^2}$$

$\{(y - z) \times [\text{Minor of } (y - z)] - (-x) \times [\text{Minor of } (-x)] + x \times [\text{Minor of } x]\}$

$$= \frac{1}{(y-z)^2 (z-x)^2 (x-y)^2}$$

$$(y-z) \times \begin{vmatrix} y-z & -x & x \\ y & z-x & -y \\ -z & z & x-y \end{vmatrix} + x \times \begin{vmatrix} y-z & -x & x \\ y & z-x & -y \\ -z & z & x-y \end{vmatrix} + x \times \begin{vmatrix} y-z & -x & x \\ y & z-x & -y \\ -z & z & x-y \end{vmatrix}$$

$$= \frac{1}{(y-z)^2 (z-x)^2 (x-y)^2} \left\{ (y-z) \times \begin{vmatrix} z-x & -y \\ z & x-y \end{vmatrix} + x \times \begin{vmatrix} y & -y \\ -z & x-y \end{vmatrix} + x \times \begin{vmatrix} y & z-x \\ -z & z \end{vmatrix} \right\}$$

$$= \frac{1}{(y-z)^2 (z-x)^2 (x-y)^2} \left\{ \begin{array}{l} (y-z)\,[(z-x)(x-y) + yz] \\ + x\,[y(x-y) - yz] + x\,[yz + z(z-x)] \end{array} \right\}$$

$$= \frac{1}{(x-y)^2 (y-z)^2 (z-x)^2} \left\{ \begin{array}{l} (y-z)\,[xz - x^2 + xy] \\ + x\,[xy - y^2 - yz] \\ + x\,[yz + z^2 - xz] \end{array} \right\}$$

$$= \frac{1}{(x-y)^2 (y-z)^2 (z-x)^2} \left[\begin{array}{l} \cancel{xzy} - \cancel{x^2y} + \cancel{xy^2} - \cancel{xz^2} + \cancel{zx^2} - \cancel{xyz} \\ + \cancel{x^2y} - \cancel{xy^2} - \cancel{xyz} + \cancel{xyz} + \cancel{xz^2} - \cancel{x^2z} \end{array} \right]$$

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = 0 \checkmark \qquad \text{...Ans.}$$

5.3 Type : JJ′ = 1

If $u = f_1(x, y)$; $v = f_2(x, y)$ and $x = \phi_1(u, v)$; $y = \phi_2(u, v)$

then $\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = 1$ i.e. $JJ' = 1$

If $u = f_1(x, y)$; and $v = f_2(x, y)$ are functions of x and y are given then (i) fist find directly $J = \frac{\partial(x, y)}{\partial(u, v)}$ and (ii) for $J' = \frac{\partial(x, y)}{\partial(u, v)}$, first, we have to find x, y are functions of u and v.

This can be extended for three functions of three variables and so on.

Note : If we have to verify JJ′ = 1 ; at that time do not write directly $J' = \frac{1}{J}$ or $J = \frac{1}{J'}$ term.

Type I : Examples Based on JJ′ = 1

Example 5.3.1

If x = u cos v; y = u sin v, prove that $\frac{\partial(x, y)}{\partial(u,v)} \cdot \frac{\partial(u, v)}{\partial(x,y)} = 1$

Solution :

Step I : Given, $x = u\cos v$; $y = u \sin v$ ...(1)

Here $x, y \to f(u, v)$, i.e. x and y are functions of u, v

We know, definition of Jacobian,

$$\therefore J = \frac{\partial(x, y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad ...(2)$$

Differentiate x, y partially w.r.t of u, v

Differentiate x partially w.r.t u, keeping v as constant

$$\therefore \frac{\partial x}{\partial u} = \cos v$$

Differentiate x partially w.r.t. to v, keeping u as constant

$$\frac{\partial x}{\partial v} = u(-\sin v) = -u \sin v$$

Differentiate y partially w.r.t. to u, keeping v as constant.

$$\frac{\partial y}{\partial u} = \sin v$$

Differentiate y partially w.r.t. to v, keeping u as constant.

$$\frac{\partial y}{\partial v} = u \cos v$$

Substitute values in Equation (2), we get

$$J = \begin{vmatrix} \cos v & -u \sin v \\ \sin v & u\cos v \end{vmatrix}$$

$= [(\cos v)(u \cos v)] - [(\sin v)(-u \sin v)]$

$= [u \cos^2 v] - [-u \sin^2 v]$

$= u\cos^2 v + u \sin^2 v$

$= u(\cos^2 v + \sin^2 v) = u$ ...($\because \sin^2\theta + \cos^2\theta = 1$)

$$\therefore J = \frac{\partial(x, y)}{\partial(u, v)} = u \quad ...(3)$$

Step II : Now, we have to find $J' = \frac{\partial(u, v)}{\partial(x, y)}$

i.e. we have to find u, v as functions of x and y

From Equation (1), eliminate v

$$\therefore x^2 + y^2 = u^2(\cos^2 v + \sin^2 v) = u^2$$

...($\because \sin^2\theta + \cos^2\theta = 1$)

$$\therefore u^2 = x^2 + y^2$$

Taking square root of both sides,

$$u = \sqrt{x^2+y^2}$$

Again from Equation (1) eliminate u as,

and $\frac{y}{x} = \frac{u \sin v}{u \cos v} = \tan v$

$$\therefore \tan v = \frac{y}{x} \Rightarrow v = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore u = \sqrt{x^2+y^2} \quad \text{and} \quad v = \tan^{-1}\left(\frac{y}{x}\right) \quad ...(4)$$

Differentiate u and v w.r.t. x and y

Step III :

Partially differentiate u w.r.t. to x, keeping y as constant

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} 2x = \frac{x}{\sqrt{x^2+y^2}} \quad ...(5)$$

Partially differentiate u w.r.t. to y, keeping x as constant

$$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

Partially differentiate v w.r. to x keeping y as constant.

$$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{(x^2+y^2)}$$

Partially differentiate v w.r. to y keeping x as constant.

and $\frac{\partial v}{\partial y} = \frac{x}{(x^2+y^2)}$

Step IV : We know, definition of Jacobian,

$$\therefore J' = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Substitute values from step (3), it gives

$$= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \boxed{\frac{y}{\sqrt{x^2+y^2}}} \\ \boxed{\frac{-y}{(x^2+y^2)}} & \frac{x}{(x^2+y^2)} \end{vmatrix}$$

$$= \left[\frac{x}{\sqrt{x^2+y^2}} \cdot \frac{x}{(x^2+y^2)}\right] - \left[\frac{-y}{(x^2+y^2)} \cdot \frac{y}{(x^2+y^2)}\right]$$

$$= \frac{x^2}{\sqrt{x^2+y^2}(x^2+y^2)} + \frac{y^2}{\sqrt{x^2+y^2}(x^2+y^2)}$$

$$\therefore \frac{\partial(u, v)}{\partial(x, y)} = \frac{x^2+y^2}{\sqrt{x^2+y^2}(x^2+y^2)} = \frac{1}{\sqrt{x^2+y^2}} \quad ...(6)$$

From Equations (3) and (6,)

Hence, $JJ' = \frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)} = u \cdot \frac{1}{\sqrt{x^2+y^2}}$

$$= \left(\cancel{\sqrt{x^2+y^2}}\right) \cdot \left(\frac{1}{\cancel{\sqrt{x^2+y^2}}}\right)$$

...($\because u = \sqrt{x^2+y^2}$)

JJ′ = 1✓ **...Hence Proved.**

Example 5.3.2

Verify JJ′ = 1 for the transformation $x = uv$, $y = \frac{u}{v}$.

Solution :

Step I : Given, $x = uv$ and $y = \frac{u}{v}$ ...(1)

Differentiate x and y partially w.r.t. u and v

Differentiate x partially w.r. to u, keeping v as constant

$$\therefore \frac{\partial x}{\partial u} = v$$

Differentiate x partially w.r. to v, keeping u as constant.

$$\frac{\partial x}{\partial v} = u$$

Differentiate y partially w.r. to u, keeping v as constant.

$$\frac{\partial y}{\partial u} = \frac{1}{v}$$

Differentiate y partially w.r. to v, keeping u as constant.

$$\frac{\partial y}{\partial v} = -\frac{u}{v^2}$$

...(2)

Step II : We know definition of Jacobian,

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Substitute values from Equation (2), we get,

$$= \begin{vmatrix} v & \boxed{u} \\ \boxed{\frac{1}{v}} & \frac{-u}{v^2} \end{vmatrix}$$

$$= \left[(v)\left(\frac{-u}{v^2}\right)\right] - \left[\left(\frac{1}{v}\right)(u)\right] = \frac{-u}{v} - \frac{u}{v}$$

$$\therefore J = \frac{-2u}{v} \quad ...(3)$$

Step III : Now, for $J' = \frac{\partial(u,v)}{\partial(x, y)}$, Express u and v in terms of x, y

From Equation (1), Eliminate v,

$$xy = uv \cdot \frac{u}{v} = u^2$$

From Equation (1), eliminate u,

and $\frac{x}{y} = \frac{uv}{\frac{u}{v}} = uv \cdot \frac{v}{u} = v^2$

$\therefore\ u = \sqrt{xy}$ and $v = \sqrt{\dfrac{x}{y}} = \dfrac{\sqrt{x}}{\sqrt{y}}$ ...(4)

Differentiate u, v partially w.r. to x, y, we get,

Partially differentiate u w.r. to x, keeping y as constant

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{xy}} \cdot y$$

Partially differentiate u w.r. to y, keeping x as constant.

$$\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{xy}} \cdot x$$

Partially differentiate v w.r. to x keeping y as constant.

$$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{y}}\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{xy}}$$

Partially differentiate v w.r. to y keeping x as constant.

$$\frac{\partial v}{\partial y} = \sqrt{x} \cdot \left(-\frac{1}{2} y^{-3/2}\right) = -\frac{1}{2}\frac{\sqrt{x}}{y\sqrt{y}}$$

Step IV : We know, definition of Jacobian,

$$\therefore \frac{\partial (u, v)}{\partial (x, y)} = J'$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Substitute values from Equation (4), we get,

$$= \begin{vmatrix} \dfrac{1 \cdot y}{2\sqrt{xy}} & \dfrac{1 \cdot x}{2\sqrt{xy}} \\ \dfrac{1}{2\sqrt{xy}} & \dfrac{-\sqrt{x}}{2y\sqrt{y}} \end{vmatrix}$$

Common out $\dfrac{1}{2\sqrt{xy}}$ from R_1 and $\dfrac{1}{2\sqrt{y}}$ from R_2

$$= \frac{1}{2\sqrt{xy}} \cdot \frac{1}{2\sqrt{y}} \begin{vmatrix} y & x \\ \dfrac{1}{\sqrt{x}} & \dfrac{-\sqrt{x}}{y} \end{vmatrix}$$

$$= \frac{1}{4\sqrt{xy}\sqrt{y}}\left\{\left[(y)\left(\frac{-\sqrt{x}}{y}\right)\right] - \left[\left(\frac{1}{\sqrt{x}}\right) \cdot (x)\right]\right\}$$

$$= \frac{1}{4\sqrt{xy}\sqrt{y}}\{-\sqrt{x} - \sqrt{x}\} = \frac{1}{4\sqrt{xy}\sqrt{y}}(-2\sqrt{x})$$

$$= \frac{1}{2y} = -\frac{1}{2} \cdot \frac{1}{\frac{u}{v}} \quad \text{...[From Equation (1)]}$$

$$J' = \frac{\partial (u,v)}{\partial (x, y)} = -\frac{v}{2u} \quad \text{...(5)}$$

Step V : From Equations (2) and (5),

$$JJ' = \frac{\partial (x,y)}{\partial (u, v)} \cdot \frac{\partial (u, v)}{\partial (x, y)}$$

$$= \left(\frac{-2\cancel{u}}{\cancel{v}}\right)\left(\frac{-\cancel{v}}{2\cancel{u}}\right) = 1$$

JJ′ = 1✓ **...Hence verified**

Example 5.3.3

If $x = u(1 - v)$; $y = uv$ then prove that $JJ' = 1$.

Solution :

Step I : Given , $x = u(1 - v)$; $y = uv$ ...(1)

Step II : We know, definition of Jacobian,

$$J = \frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

Differentiate x partially w.r. to u, keeping v as constant

$$\frac{\partial x}{\partial u} = (1 - v)$$

Differentiate x partially w.r. to v, keeping u as constant.

$$\frac{\partial x}{\partial v} = u(-1) = -u$$

Differentiate y partially w.r. to u, keeping v as constant.

$$\frac{\partial y}{\partial u} = v$$

Differentiate y partially w.r. to v, keeping u as constant.

$$\frac{\partial y}{\partial v} = u$$

...(2)

Substitute values from Equation (2), we get

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= [(1 - v)(u)] - [(v)(-u)]$$

$$J = u(1 - v) + uv = u - \cancel{uv} + \cancel{uv} = u \quad \text{...(3)}$$

Step III : Now, for $J' = \dfrac{\partial (u, v)}{\partial (x, y)}$, we have to find u and v in terms of x and y.

From Equation (1),

$x = u(1 - v) = u - uv$ and $y = uv$

$\Rightarrow \quad x = u - y \ \Rightarrow u = x + y$

$$\text{and} \quad v = \frac{y}{u} = \frac{y}{x + y}$$

$$\therefore \quad u = x + y \text{ and } v = \frac{y}{x + y} \quad \text{...(4)}$$

Partially differentiate u w.r. to x, keeping y as constant

$$\frac{\partial u}{\partial x} = 1$$

Partially differentiate u w.r.t y, keeping x as constant.

$$\frac{\partial u}{\partial y} = 1$$

Partially differentiate v w.r.t x keeping y as constant.

$$\frac{\partial v}{\partial x} = y\left[\frac{-1}{(x + y)^2}\right] = \frac{-y}{(x + y)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x + y)(1) - y(1)}{(x + y)^2} \quad \text{(by Quatient rule of derivative)}$$

$$= \frac{x + \cancel{y} - \cancel{y}}{(x + y)^2} = \frac{x}{(x + y)^2}$$

Step IV : We know, by definition of Jacobian,

$$J' = \frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Substitute values from Step (3), we get,

$$= \begin{vmatrix} 1 & 1 \\ \dfrac{-y}{(x + y)^2} & \dfrac{(x + y)1 - y(1)}{(x + y)^2} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ \dfrac{-y}{(x + y)^2} & \dfrac{x}{(x + y)^2} \end{vmatrix}$$

$$= \left[(1)\left(\frac{x}{(x + y^2)}\right)\right] - \left[\frac{-y}{(x + y^2)} \cdot 1\right]$$

$$= \frac{x}{(x + y)^2} + \frac{y}{(x + y)^2} = \frac{x + y}{(x + y)^2}$$

$$J' = \frac{1}{(x + y)} = \frac{1}{u} \text{[From Equation (4)]} \quad \text{...(5)}$$

Step V : $\therefore$ From Equations (3) and (5), we get,

$$JJ' = \frac{\partial (x, y)}{\partial (u, v)} \cdot \frac{\partial (u, v)}{\partial (x, y)} = \cancel{u} \cdot \frac{1}{\cancel{u}} = 1$$

JJ′ = 1 ✓ **...Hence Proved.**

Example 5.3.4

If $x = e^u \cos v$; $y = e^u \sin v$ then prove that $JJ' = 1$.

Solution :

Step I : Given , $x = e^u \cos v$; $y = e^u \sin v$ ...(1)

Step II :

$$J = \frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \quad \text{...(2)}$$

➔ **Using product rule of derivative for Eqn. (1)**

...[d (uv) = udv + vdu]

Partially differentiate x w.r. to u, keeping v as constant

$$\frac{\partial x}{\partial u} = e^u \cos v$$

Partially differentiate x w.r. to v, keeping u as constant

$$\frac{\partial x}{\partial v} = e^u(-\sin) v = -e^u \sin v$$

Partially differentiate y w.r. to u keeping v as constant

$$\frac{\partial y}{\partial u} = e^u \sin v$$

Partially differentiate y w.r. to v keeping u as constant.

$$\frac{\partial y}{\partial v} = e^u \cos v$$

Using these values in Equation (2), we get

$$= \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix}$$

$= [(e^u \cos v)(e^u \cos v) - (e^u \sin v)(-e^u \sin v)]$

$= e^{2u} \cos^2 v + e^{2u} \sin^2 v$

$J = e^{2u} [\cos^2 v + \sin^2 u] = e^{2u}$ ($\because \sin^2\theta + \cos^2\theta = 1$)

$J = e^{2u}$ ...(3)

Step III : Now $\quad J' = \dfrac{\partial (u, v)}{\partial (x, y)}$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Express u and v in terms of x and y.

From Equation (1)

$$\therefore \quad x^2 + y^2 = (e^u \cos v)^2 + (e^u \sin v)^2$$
$$= e^{2u} \cos^2 v + e^{2u} \sin^2 v = e^{2u} (\cos^2 v + \sin^2 v)$$
$$= e^{2u} \quad \ldots(\because \sin^2\theta + \cos^2\theta = 1)$$
$$\therefore e^{2u} = x^2 + y^2$$
$$\Rightarrow 2u = \log (x^2 + y^2)$$
$$u = \frac{1}{2} \log (x^2 + y^2)$$

and $\quad \dfrac{y}{x} = \dfrac{\cancel{e^u} \sin v}{\cancel{e^u} \cos v} = \tan v$

and $\quad v = \tan^{-1}\left(\dfrac{y}{x}\right) \quad \ldots(4)$

➔ Using standard rules of derivatives

Step IV : We know definition Jacobian

$$J' = \frac{\partial (u, v)}{\partial (x, y)} = \frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

From Equation (5),

$$\frac{\partial u}{\partial x} = \frac{1}{2} \frac{1}{(x^2 + y^2)} \cdot (2x)$$
$$= \frac{x}{x^2 + y^2}; \frac{\partial u}{\partial y} = \frac{1 \cdot 2y}{2 (x^2 + y^2)} = \frac{y}{x^2 + y^2}$$
$$\frac{\partial v}{\partial x} = \frac{1\left(-\frac{y}{x^2}\right)}{1 + \left(\frac{y}{x}\right)^2} \frac{-y}{x^2 + y^2}$$
$$\frac{\partial v}{\partial y} = \frac{\left(\frac{1}{x}\right)}{1 + \left(\frac{y}{x}\right)^2} \frac{x}{x^2 + y^2}$$
$$= \begin{vmatrix} \frac{1 \cdot 2x}{2 (x^2 + y^2)} & \frac{1 \cdot 2y}{2 (x^2 + y^2)} \\ \frac{1\left(-\frac{y}{x^2}\right)}{1 + \left(\frac{y}{x}\right)^2} & \frac{\left(\frac{1}{x}\right)}{1 + \left(\frac{y}{x}\right)^2} \end{vmatrix}$$
$$= \begin{vmatrix} \frac{x}{(x^2 + y^2)} & \frac{y}{(x^2 + y^2)} \\ \frac{-y}{(x^2 + y^2)} & \frac{x}{(x^2 + y^2)} \end{vmatrix}$$
$$= \frac{1}{(x^2 + y^2)^2} = \begin{vmatrix} x & \boxed{y} \\ \boxed{-y} & x \end{vmatrix}$$
$$= \frac{1}{(x^2 + y^2)^2} (x^2 + y^2)$$
$$= \frac{1}{(x^2 + y^2)} = \frac{1}{e^{2u}} \quad \ldots\text{from (2)}$$
$$J' = \frac{1}{e^{2u}} \quad \ldots(5)$$

Step V : From Equations (3) and (5), we get,

$$J J' = \frac{\partial (x, y)}{\partial (u, v)} \cdot \frac{\partial (u, v)}{\partial (x, y)}$$
$$= (\cancel{e^{2u}}) \frac{1}{(\cancel{e^{2u}})} = 1 \checkmark$$ **...Hence Proved.**

Example 5.3.5

For the transformations $x = e^v \sec u$; $y = e^v \tan u$ then prove that $\dfrac{\partial(u, v)}{\partial(x, y)} \cdot \dfrac{\partial(x, y)}{\partial(u, v)} = 1$ I.e. $J J' = 1$

Solution :

Step I : Given : $x = e^v \sec u$; $y = e^v \tan u \quad \ldots(1)$

➔ Using standard rules of derivatives

Step II : $\quad J = \dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$\begin{vmatrix} e^v \sec u . \tan u & \boxed{e^v . \sec u} \\ \boxed{e^v \sec^2 u} & e^v \tan u \end{vmatrix}$$

$= [(e^v \sec u \tan u) (e^v \tan u) - (e^v \sec^2 u) (e^v \sec u)]$

$= [e^{2v} \sec u \tan^2 u - e^{2v} \sec^3 u]$

Common out $(e^{2v} \sec u)$

$$= e^{2v} \sec u [\tan^2 u - \sec^2 u] \quad \ldots(2)$$

➔ Using standard formula for eqn. (2)

$$\ldots[\tan^2 x - \sec^2 x = -1]$$
$$J = e^{2v} \sec u (-1) = -e^{2v} \cdot \sec u \quad \ldots(3)$$

Step III : Now, $J' = \dfrac{\partial(u, v)}{\partial(x, y)}$; express u and v are in x and y.

Form Equation (1),

$$\therefore \frac{y}{x} = \frac{e^v \tan u}{e^v \sec u} = \tan u \cdot \frac{1}{\sec u} = \frac{\sin u}{\cancel{\cos u}} \cancel{\cos u} = \sin u$$
$$\Rightarrow u = \sin^{-1}\left(\frac{y}{x}\right)$$

Again from Equation (1)

$$x^2 - y^2 = e^{2v} \sec^2 u - e^{2v} \tan^2 v$$
$$= e^{2v} [\sec^2 u - \tan^2 u]$$

➔ Using standard formula

$$\ldots[1 + \tan^2 x = \sec^2 x \Rightarrow \sec^2 x - \tan^2 x = 1]$$
$$x^2 - y^2 = e^{2v}$$
$$e^{2v} = x^2 - y^2 \Rightarrow 2v = \log (x^2 - y^2)$$

(taking log of both sides)

$$v = \frac{1}{2} \log (x^2 - y^2)$$

Step IV : We know definition of Jacobinan,

$$\therefore J' = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad \ldots(4)$$

Since $u = \sin^{-1}\left(\frac{y}{x}\right)$ and $v = \frac{1}{2} \log (x^2 - y^2)$

Differentiate partially u w.r.t x, keeping y as constant

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \left(\frac{y}{x}\right)^2}} \cdot \frac{d}{dx}\left(\frac{y}{x}\right)$$

...(Using Composite Rule of derivative)

$$= \frac{1}{\sqrt{1 - \left(\frac{y}{x}\right)^2}} \left(\frac{-y}{x^2}\right) = \frac{-y}{x(\sqrt{x^2 + y^2})}$$

Differentiate u partially w.r.t y, keeping x as constant.

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \left(\frac{y}{x}\right)^2}} \cdot \left(\frac{1}{x}\right) (1) = \frac{1}{\sqrt{x^2 - y^2}}$$

Differentiate v partially w.r.t x keeping y as constant.

$$\frac{\partial v}{\partial x} = \frac{1}{2} \frac{1}{x^2 - y^2} \cdot (2x) = \frac{x}{x^2 - y^2}$$

Differentiate u partially w.r.t x keeping y, z as constant.

$$\frac{\partial v}{\partial y} = \frac{1}{2} \frac{1}{x^2 - y^2} \cdot (-2y) = \frac{-y}{x^2 + y^2}$$

➔ Substitute above value in Equation (4), it gives,

$$= \begin{vmatrix} \frac{-y}{x\left(\sqrt{x^2 - y^2}\right)} & \left(\frac{1}{\sqrt{x^2 + y^2}}\right) \\ \left(\frac{x}{x^2 - y^2}\right) & \left(\frac{-y}{x^2 + y^2}\right) \end{vmatrix}$$

Common out $\dfrac{1}{\sqrt{x^2 - y^2}}$ from R_1 and $\dfrac{1}{x^2 + y^2}$ from R_2

$$= \frac{1}{\sqrt{x^2 - y^2} (x^2 - y^2)} \begin{vmatrix} -\frac{y}{x} & \boxed{1} \\ \boxed{x} & y \end{vmatrix}$$
$$= \frac{1}{(x^2 - y^2)^{3/2}} \left[\left(\frac{-y}{x}\right) (-y) - (x) (1)\right]$$
$$= \frac{1}{(x^2 - y^2)^{3/2}} \left(\frac{y^2}{x} - x\right) = \frac{1}{(x^2 - y^2)^{3/2}} \left(\frac{y^2 - x^2}{x}\right)$$
$$= \frac{-(x^2 - y^2)}{(x^2 - y^2)^{3/2} \cdot x} = \frac{-1}{x (x^2 - y^2)^{1/2}}$$

Since $\quad e^{2v} = x^2 - y^2 \Rightarrow e^v = \sqrt{x^2 - y^2}$

$$\therefore J' = \frac{-1}{x \cdot e^v} = -\frac{1}{e^{2v} \cdot \sec u} \quad \ldots(\because x = e^v \sec u)$$
$$\ldots(5)$$

Step V : From Equations (3) and (5), we get,

$$JJ' = \frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)}$$
$$= (-\cancel{e^{2v}} \sec u) \left(\frac{-1}{\cancel{e^{2v} \sec u}}\right) = 1$$

$JJ' = 1 \checkmark$ **...Hence Proved.**

Example 5.3.6

If $x = u, y = u \tan v , z = w$ prove that $JJ^1 = 1$.

Solution :

Step I : Given , $\quad x = u, \quad y = u \tan v, \quad z = w \quad \ldots(1)$

Step II : $\quad \therefore J = \dfrac{\partial(x, y, z)}{\partial(u, v, w)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \quad \ldots(2)$$

Differentiate x partially w.r.to u, keeping v, w as constants

$$\frac{\partial x}{\partial u} = 1$$

Differentiate x partially w.r.to v, keeping u, w as constants

$$\frac{\partial x}{\partial v} = 0$$

Differentiate x partially w.r.to w, keeping u, v as constants

$$\frac{\partial x}{\partial w} = 0$$

Differentiate y partially w.r.t u, keeping v, w as constants

$$\frac{\partial y}{\partial u} = \tan v$$

Differentiate y partially w.r.t v, keeping u, w as constants

$$\frac{\partial y}{\partial v} = u \sec^2 v$$

Differentiate y partially w.r.t w, keeping u, v as constants

$$\frac{\partial y}{\partial w} = 0$$

Differentiate z partially w.r.t u, keeping v, w as constants

$$\frac{\partial z}{\partial u} = 0$$

Differentiate z partially w.r.t v, keeping u, w as constants

$$\frac{\partial z}{\partial v} = 0$$

Differentiate z partially w.r.t w, keeping u, v as constants

$$\frac{\partial z}{\partial w} = 1$$

Substitute above values in Equation (2), we get

$$J = \begin{vmatrix} 1 & 0 & 0 \\ \tan v & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1 \times [\text{Minor of } 0] - \underbrace{0 \times [\text{Minor of } 0]}_{0} + \underbrace{0 \times [\text{Minor of } 0]}_{0}$$

$$J = 1 \times = \begin{vmatrix} \cancel{1} & \cancel{0} & \cancel{0} \\ \cancel{\tan v} & u\sec^2 v & 0 \\ \cancel{0} & 0 & 1 \end{vmatrix}$$

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = 1\,[u \sec^2 v - 0] = u \sec^2 v$$

$$J = u \sec^2 v \qquad \ldots(3)$$

Step III : Now, for $J' = \frac{\partial(u, v, w)}{\partial(x, y, z)}$, express u, v, w as functions of x, y, z.

From Equation (1),

$$u = x, \quad \tan v = \frac{y}{u} = \frac{y}{x},$$

$$\therefore u = x; \; v = \tan^{-1}\left(\frac{y}{x}\right) \text{ and } w = z$$

Step IV : Now, $J' = \frac{\partial(u, v, w)}{\partial(x, y, z)}$

By definition of Jocobian

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \qquad \ldots(4)$$

Since $u = x, \; v = \tan^{-1}\left(\frac{y}{x}\right), \; w = z$

Differentiate partially u w.r.t x keeping y, z as constant

$$\frac{\partial u}{\partial x} = 1$$

Differentiate u partially w.r.t y, keeping x, z as constants.

$$\frac{\partial u}{\partial y} = 0$$

Differentiate u partially w.r.t z, keeping x, y as constants

$$\frac{\partial u}{\partial z} = 0$$

Differentiate v partially w.r.t x, keeping y, z as constants

$$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

➔ **Using standard Derivative**

$$\ldots\left[\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

Differentiate v partially w.r.to y, keeping x, z as constant

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\frac{\partial v}{\partial z} = 0; \qquad \frac{\partial w}{\partial x} = 0$$

$$\frac{\partial w}{\partial y} = 0; \qquad \frac{\partial w}{\partial z} = 1$$

Substitute above values from Equation (4) we get,

$$= \begin{vmatrix} 1 & 0 & 0 \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1 \times [\text{Minor of } 1] - \underbrace{0 \times [\text{Minor of } 0]}_{0} + \underbrace{0 \times [\text{Minor of } 0]}_{0}$$

$$= 1 \times \begin{vmatrix} \cancel{1} & \cancel{0} & \cancel{0} \\ \cancel{\frac{-y}{x^2+y^2}} & \frac{x}{x^2+y^2} & 0 \\ \cancel{0} & 0 & 1 \end{vmatrix}$$

$$= 1 \times \left[\left(\frac{x}{x^2+y^2}\right)(1) - 0\right]$$

$$= \frac{x}{x^2+y^2} = \frac{u}{u^2+u^2\tan^2 v} \qquad (\because x = u, y = u\tan v)$$

$$J' = \frac{u}{u^2(1+\tan^2 v)}$$

➔ **Using standard trigonometric formula**

$$\ldots[1+\tan^2 x = \sec^2 x]$$

$$J' = \frac{1}{u\sec^2 v} \qquad \ldots(5)$$

Step V : From Equations (3) and (5),

$$\therefore JJ' = \frac{\partial(x, y, z)}{\partial(u, v, w)} \cdot \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

$$= \cancel{(u\sec^2 v)}\,\frac{1}{\cancel{u\sec^2 v}} = 1$$

$$JJ' = 1 \checkmark$$

...Hence Proved.

5.4 Jacobians of Composite Function (Chain Rule of Jacobian)

If u, v are functions of x, y and x, y are functions of r, θ such that

$$u = f_1(x, y) \;;\quad v = f_2(x, y)$$

$$x = \phi_1(r, \theta) \;;\quad y = \phi_2(r, \theta)$$

$$\text{then } \frac{\partial(u, v)}{\partial(r, \theta)} = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, \theta)}$$

Similarly, If u, v, w are functions of x, y, z and x, y, z are functions of r, θ, ϕ such that

$$u = f_1(x, y, z)\;;$$
$$v = f_2(x, y, z)\;;$$
$$w = f_3(x, y, z)$$
$$\text{and} \quad x = \psi_1(r, \theta, \phi)\;;$$
$$y = \psi_2(r, \theta, \phi)\;;$$
$$z = \psi_3(r, \theta, \phi)$$

$$\text{then } \frac{\partial(u, v, w)}{\partial(r, \theta, \phi)} = \frac{\partial(u, v, w)}{\partial(x, y, z)} \cdot \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} \text{ and so on.}$$

Type I : Examples based on Jacobians of Composite Function

Example 5.4.1

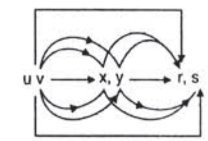
If $u = e^x \cos y$; $v = e^x \sin y$; where $x = l r + s m$; $y = mr - l s$.

Find the value of $\frac{\partial(u, v)}{\partial(r, s)}$.

Solution

Step I : Given, $u = e^x \cos y$; $v = e^x \sin y$;

$x = lr + sm$; $y = mr - ls$



l, m are constants

Step II : We knows Jacobian of composite functions,

$$\frac{\partial(u, v)}{\partial(r, s)} = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, s)}$$

➔ **Using definition of Jacobian**

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix} \qquad \ldots(1)$$

Differentiate u partially w.r.to x, keeping y as constant

$$\frac{\partial u}{\partial x} = e^x \cos y$$

Differentiate u partially w.r.to y, keeping x as constant.

$$\frac{\partial u}{\partial y} = e^x(-\sin y) = -e^x \sin y$$

Differentiate v partially w.r.to x keeping y as constant.

$$\frac{\partial v}{\partial x} = e^x \sin y$$

Differentiate v partially w.r.t y keeping x as constant.

$$\frac{\partial v}{\partial y} = e^x \cos y$$

Differentiate x partially w.r.to r keeping s as constant.

$$\frac{\partial x}{\partial r} = l$$

Differentiate x partially w.r.to keeping r as constant.

$$\frac{\partial x}{\partial s} = m$$

Differentiate x partially w.r.to r keeping r as constant.

$$\frac{\partial y}{\partial r} = m$$

Differentiate y partially w.r.to s keeping r as constant.

$$\frac{\partial y}{\partial s} = -l$$

Substitute above values in Equation (1), we get

$$= \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix} \cdot \begin{vmatrix} l & m \\ m & -l \end{vmatrix}$$

$$= [(e^x \cos y)(e^x \cos y) - (e^x \sin y)(-e^x \sin y)]$$

$$[(l)(-l) - (m)(m)]]$$

$$= [(e^{2x} \cos^2 y) + (e^{2x} \sin^2 y)][(-l)^2 - m^2]$$

$$= e^{2x} \underbrace{[\cos^2 y + \sin^2 y]}_{1} \cdot (-l^2 - m^2)$$

$$\therefore \frac{\partial (u, v)}{\partial (r, s)} = -e^{2x} (l^2 + m^2) \checkmark \qquad \text{...Ans.}$$

5.5 Jacobians of an Implicit Functions

If u, v are implicit functions of x, y mapped by f_1, f_2 such that $u(x, y) = 0$; $v(x, y) = 0$;

$\Rightarrow \quad f_1(u, v, x, y) = 0; \quad f_2(u, v, x, y) = 0$ then

$$\frac{\partial (u, v)}{\partial (x, y)} = (-1)^2 \left[\frac{\dfrac{\partial (f_1, f_2)}{\partial (x, y)}}{\dfrac{\partial (f_1, f_2)}{\partial (u, v)}}\right] = \frac{N_r}{D_r}$$

Similarly, if u, v, w are implicit functions of x, y, z mapped by f_1, f_2, f_3

Such that $u(x, y, z) = 0$;

$v(x, y, z) = 0$;

$w(x, y, z) = 0$

$\Rightarrow \quad f_1(u, v, w, x, y, z) = 0,$

$f_2(u, v, w, x, y, z) = 0,$

$f_3(u, v, w, x, y, z) = 0$

Then $$\frac{\partial (u, v, w)}{\partial (x, y, z)} = (-1)^3 \left[\frac{\dfrac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)}}{\dfrac{\partial (f_1, f_2, f_3)}{\partial (u, v, w)}}\right] = -\frac{N_r}{D_r}$$

It is true for functions of several variable.

Type I : Examples Based on Jacobians of an Implicit Functions

Example 5.5.1

It $u^3 + v^3 = x + y$; $u^2 + v^2 = x^3 + y^3$ then show that $\dfrac{\partial (u, v)}{\partial (x, y)} = \dfrac{1}{2} \dfrac{y^2 - x^2}{uv(u - v)}$.

Solution

Step I : Let, the implicit functions

$$f_1 \equiv u^3 + v^3 - x - y = 0;$$
$$f_2 \equiv u^2 + v^2 - x^3 - y^3 = 0$$

Step II : Differentiate f_1 and f_2 partially,

$$\left.\begin{aligned} \frac{\partial f_1}{\partial x} &= -1; & \frac{\partial f_1}{\partial y} &= -1; \\ \frac{\partial f_2}{\partial x} &= -3x^2; & \frac{\partial f_2}{\partial y} &= -3y^2 \\ \frac{\partial f_1}{\partial u} &= 3u^2; & \frac{\partial f_1}{\partial v} &= 3v^2; \\ \frac{\partial f_2}{\partial u} &= 2u; & \frac{\partial f_2}{\partial v} &= 2v \end{aligned}\right\} \qquad \text{...(1)}$$

Step III : We know, Jacobian implicit functions,

$$\frac{\partial (u, v)}{\partial (x, y)} = (-1)^2 \left[\frac{\dfrac{\partial (f_1, f_2)}{\partial (x, y)}}{\dfrac{\partial (f_1, f_2)}{\partial (u, v)}}\right]$$

$$= \frac{N_r}{D_r} \qquad \text{...(2)}$$

Step IV : Where, $N_r = \dfrac{\partial (f_1, f_2)}{\partial (x, y)}$

$$= \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix}$$

Using values from Equation (1), we get

$$= \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix}$$

$$N_r = 3y^2 - 3x^2 = 3(y^2 - x^2) \qquad \text{...(3)}$$

Step V : Also, $D_r = \dfrac{\partial (f_1, f_2)}{\partial (u, v)}$

$$= \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}$$

Using values from Equation (1), we get

$$D_r = 6(u^2 v - v^2 u) = 6uv(u - v) \qquad \text{...(4)}$$

Step VI : From Equations (2), (3) and (4) we get,

$$\frac{\partial (u, v)}{\partial (x, y)} = \frac{3(y^2 - x^2)}{6uv(u - v)}$$

$$\frac{\partial (u, v)}{\partial (x, y)} = \frac{1}{2} \frac{(y^2 - x^2)}{uv(u - v)} \checkmark \qquad \text{...Hence proved.}$$

Example 5.5.2

If $x^2 + y^2 + u^2 - v^2 = 0$ and $uv + xy = 0$ then prove that $\dfrac{\partial (u, v)}{\partial (x, y)} = \dfrac{x^2 - y^2}{u^2 + v^2}$.

Solution :

Step I : Let, the implicit functions,

$$f_1 \equiv x^2 + y^2 + u^2 - v^2 = 0,$$
$$f_2 \equiv uv + xy = 0$$

Step II : We know, Jacobian of implicit functions,

$$\frac{\partial (u, v)}{\partial (x, y)} = (-1)^2 \left[\frac{\dfrac{\partial (f_1, f_2)}{\partial (x, y)}}{\dfrac{\partial (f_1, f_2)}{\partial (u, v)}}\right]$$

$$= \frac{N_r}{D_r} \qquad \text{...(1)}$$

Step III : Here, $N_r = \dfrac{\partial (f_1, f_2)}{\partial (x, y)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix}$

Differentiate f_1, f_2 w.r.t. x and y

$$= \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix}$$

$$= 2(x^2 - y^2) \qquad \text{...(2)}$$

Step IV : $D_r = \dfrac{\partial (f_1, f_2)}{\partial (u, v)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}$

Differentiate f_1, f_2 w.r.t. u and v

$$= \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix}$$

$$= 2(u^2 + v^2) \qquad \text{...}$$

Step V : From Equation (1), (2) and (3), we get

$$\frac{\partial (u, v)}{\partial (x, y)} = \frac{2(x^2 - y^2)}{2(u^2 + v^2)}$$

$$= \frac{x^2 - y^2}{u^2 + v^2} \qquad \text{...Hence prov}$$

Step VI : $\therefore \dfrac{\partial (x, y)}{\partial (u, v)} = \dfrac{1}{x(1 - y) + xy}$

$$= \frac{1}{u + v} = \frac{1}{x} \checkmark \qquad \text{...A}$$

Type II : Examples Based on Jacobians of an Implicit Functions

Example 5.5.3

If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$, $u + v + w = x^2 + y^2 + z^2$ then show that $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} = \dfrac{(x - y)(y - z)(z - x)}{(u - v)(v - w)(w - u)}$

Solution :

Step I : Let,
$$\left.\begin{aligned} f_1 &\equiv u^3 + v^3 + w^3 - x - y - z = 0 \\ f_2 &\equiv u^2 + v^2 + w^2 - x^3 - y^3 - z^3 = 0 \\ f_3 &\equiv u + v + w - x^2 - y^2 - z^2 = 0 \end{aligned}\right\} \text{...}$$

Step II : $$\frac{\partial (u, v, w)}{\partial (x, y, z)} = (-1)^3 \left[\frac{\dfrac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)}}{\dfrac{\partial (f_1, f_2, f_3)}{\partial (u, v, w)}}\right]$$

$$= -\frac{N_r}{D_r} \qquad \text{...}$$

Step III : We know, Jacobian of implicit functions,

Where, $N_r = \dfrac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)}$

$$= \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z} \end{vmatrix} \qquad \text{...}$$

Differentiate f_1, f_2, f_3 partially w.r.t. x, y, z using standard rules of derivatives,

Differentiate f_1 partially w.r.t x, keeping remaining variables constants

$$\frac{\partial f_1}{\partial x} = -1$$

Differentiate f_1 partially w.r.t y, keeping remaining variables constants

$$\frac{\partial f_1}{\partial y} = -1$$

Differentiate f_1 partially w.r.t z, keeping remaining variables constants

$$\frac{\partial f_1}{\partial z} = -1$$

Differentiate f_2 partially w.r.t x, keeping remaining variables constants

$$\frac{\partial f_2}{\partial x} = -3x^2$$

Differentiate f_2 partially w.r.t y, keeping remaining variables constants

$$\frac{\partial f_2}{\partial y} = -3y^2$$

Differentiate f_2 partially w.r.t z, keeping remaining variables constants

$$\frac{\partial f_2}{\partial z} = -3z^2$$

Differentiate f_3 partially w.r.t x, keeping remaining variables constants

$$\frac{\partial f_3}{\partial x} = -2x$$

Differentiate f_3 partially w.r.t y, keeping remaining variables constants

$$\frac{\partial f_3}{\partial y} = -2y$$

Differentiate f_3 partially w.r.t z, keeping remaining variables constants

$$\frac{\partial f_3}{\partial z} = -2z$$

Substitute above values in Equation (3), we get

$$N_r = \begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix}$$

Common out (– 1) from R_1, (– 3) from R_2, and (– 2) from R_3, we get,

$$= -6\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}$$

To solve determinant for simplicity use elementary transformations,

By $C_2 - C_1$, $C_3 - C_1$

$$\sim -6\begin{vmatrix} 1 & 0 & 0 \\ x^2 & y^2 - x^2 & z^2 - x^2 \\ x & y - x & z - x \end{vmatrix}$$

$$\ldots[\because a^2 - b^2 = (a-b)(a+b)]$$

$$= -6\begin{vmatrix} 1 & 0 & 0 \\ x^2 & (y-x)(y+x) & (z-x)(z+x) \\ x & (y-x) & (z-x) \end{vmatrix}$$

$$= (-6) \times 1x \begin{vmatrix} \cancel{1} & \cancel{0} & \cancel{0} \\ \cancel{x^2} & (y-x)(y+x) & (z-x)(z+x) \\ \cancel{x} & (y-x) & (z-x) \end{vmatrix}$$

$$= -6x \begin{vmatrix} (y-x)(y+x) & (z-x)(z+x) \\ y-x & z-x \end{vmatrix}$$

$$= -6.1\,[\,(z-x)(y-x)(y+x) - (z-x)(z+x)(y-x)\,]$$

Taking $(z-x)(y-x)$ common we get,

$$= -6\,[\,(z-x)(y-x)\,[(y+x)-(z+x)]\,]$$

$$= -6\,[\,(z-x)(y-x)(y-z)\,]$$

$$N_r = 6(x-y)(y-z)(z-x) \qquad \ldots(4)$$

Step IV :

Differentiate f_1, f_2, f_3 partially w.r.t. u, v, w

→ Using standard rules of derivatives

and, $D_r = \dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} \qquad \ldots(5)$$

Differentiate f_1 partially w.r.t u, keeping remaining variables constants

$$\frac{\partial f_1}{\partial u} = 3u^2$$

Differentiate f partially w.r.t v, keeping remaining variables constants

$$\frac{\partial f_1}{\partial v} = 3v^2$$

Differentiate f_1 partially w.r.t w, keeping remaining variables constants

$$\frac{\partial f_1}{\partial w} = 3w^2$$

Differentiate f_2 partially w.r.t u, keeping remaining variables constants

$$\frac{\partial f_2}{\partial u} = 2u$$

Differentiate f_2 partially w.r.t v, keeping remaining variables constants

$$\frac{\partial f_2}{\partial v} = 2v$$

Differentiate f_2 partially w.r.t w, keeping remaining variables constants

$$\frac{\partial f_2}{\partial w} = 2w$$

Differentiate f_3 partially w.r.t u, keeping remaining variables constants

$$\frac{\partial f_3}{\partial u} = 1$$

Differentiate f_3 partially w.r.t v, keeping remaining variables constants

$$\frac{\partial f_3}{\partial v} = 1$$

Differentiate f_3 partially w.r.t w, keeping remaining variables constants

$$\frac{\partial f_3}{\partial w} = 1$$

Substituting above values in Equation (5) we get,

$$D_r = \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2v & 2w \\ 1 & 1 & 1 \end{vmatrix}$$

Common out 3 from R_1 and 2 from R_2, we get,

$$= (3)(2) \times \begin{vmatrix} u^2 & v^2 & w^2 \\ u & v & w \\ 1 & 1 & 1 \end{vmatrix}$$

To solve determinant for simplicity use elementary transformations,

Operate $C_2 - C_1$, $C_3 - C_1$

$$= 6\begin{vmatrix} u^2 & v^2 - u^2 & w^2 - u^2 \\ u & v - u & w - u \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 6\begin{vmatrix} u^2 & (v-u)(v+u) & (w-u)(w+u) \\ u & (v-u) & (w-u) \\ 1 & 0 & 0 \end{vmatrix}$$

$$\ldots[\because a^2 - b^2 = (a-b)(a+b)]$$

Common out $(v-u)$ from C_2 and $(w-u)$ from C_3

$$= 6(v-u)(w-u)\begin{vmatrix} u^2 & v+u & w+u \\ u & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

Operator $R_1 \leftrightarrow R_3 = -6(v-u)(w-u) \times 1 \times \begin{vmatrix} \cancel{1} & \cancel{0} & \cancel{0} \\ \cancel{u} & 1 & 1 \\ \cancel{u^2} & v+u & w+u \end{vmatrix}$

$$= -6(v-u)(w-u) = 1 \times [\text{Minor of } 1] \underbrace{-\,0 \times [\text{Minor of } 0]}_{0} + \underbrace{0 \times [\text{Minor of } 0]}_{0}$$

$$= -6(v-u)(w-u) \times \begin{bmatrix} 1 & 0 & 0 \\ u & 1 & 1 \\ u^2 & v+u & w+u \end{bmatrix}$$

[Then value of determinate change by (– 1)]

$$= -6(v-u)(w-u)(w+u) - (v+u)$$

$$= 6(v-u)(w-u)(v-w)$$

$$D_r = -6(u-v)(v-w)(w-u) \qquad \ldots(6)$$

Step V : Using values from Equation (4) and Equation (6) in Equation (2) we get,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{6(x-y)(y-z)(z-x)}{6(u-v)(v-w)(w-u)}$$

$$\therefore \quad \frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)} ✓$$

...Hence proved

Example 5.5.4

If u + v + w = x + y + z ; uv + vw + wu = $x^2 + y^2 + z^2$ and uvw = $\frac{1}{3}(x^3 + y^3 + z^3)$ then find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$.

Solution :

Step I : Let,

$$\left.\begin{aligned} f_1 &\equiv u + v + w - x - y - z = 0, \\ f_2 &\equiv uv + vw + wu - x^2 - y^2 - z^2 = 0 \\ f_3 &\equiv uvw - \frac{1}{3}(x^3 + y^3 + z^3) = 0 \end{aligned}\right\} \quad \ldots(1)$$

Step II : We know, Jacobian of implicit functions,

$$\therefore \frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \left[\frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}}\right]$$

$$= -\frac{N_r}{D_r} \quad \ldots(2)$$

Step III : where, $N_r = \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}$

We know definition of Jacobian,

$$N_r = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} \quad \ldots(3)$$

Differentiate f_1, f_2, f_3 partially w.r.t. x, y, z using standard rules of derivatives

Differentiate f_1 partially w.r.t x, keeping remaining variables constants

$$\frac{\partial f_1}{\partial x} = -1$$

Differentiate f_1 partially w.r.t y, keeping remaining variables constants

$$\frac{\partial f_1}{\partial y} = -1$$

Differentiate f_1 partially w.r.t z, keeping remaining variables constants

$$\frac{\partial f_1}{\partial z} = -1$$

Differentiate f_2 partially w.r.t x, keeping remaining variables constants

$$\frac{\partial f_2}{\partial x} = -2x$$

Differentiate f_2 partially w.r.t y, keeping remaining variables constants

$$\frac{\partial f_2}{\partial y} = -2y$$

Differentiate f_2 partially w.r.t z, keeping remaining variables constants

$$\frac{\partial f_2}{\partial z} = -2z$$

Differentiate f_3 partially w.r.t x, keeping remaining variables constants

$$\frac{\partial f_3}{\partial x} = -\frac{1}{3}(3x^2) = -x^2$$

Differentiate f_3 partially w.r.t y, keeping remaining variables constants

$$\frac{\partial f_3}{\partial} = -\frac{1}{3}(3y^2) = -y^2$$

Differentiate f_3 partially w.r.t z, keeping remaining variables constants

$$\frac{\partial f_3}{\partial z} = -\frac{1}{3}(3z^2) = -z^2$$

Using above values in Equation (3), we get,

$$N_r = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

$$= (-1)(-2)(-1)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

By $C_2 - C_1$, $C_3 - C_1$ -

$$= -2\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & (y-x)(y+x) & (z-x)(z+x) \end{vmatrix}$$

$\ldots[\because a^2 - b^2 = (a-b)(a+b)]$

Common out (y – x) from C_2 and (z – x) from C_3, it gives,

$$= 2(y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & (y+x) & (z+x) \end{vmatrix}$$

$$= -2(y-x)(z-x)\cdot\{1\times[\text{Minor of }1]\} - \underbrace{0\times[\text{Minor of }0]}_{0}$$

$$\underbrace{+0\times[\text{minor of }0]}_{0}$$

$$= 2(y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & (y+x) & (z+x) \end{vmatrix}$$

$$= -2(y-x)(z-x) \times \begin{vmatrix} 1 & 1 \\ y+x & z+x \end{vmatrix}$$

$$= -2(y-x)(z-x)\cdot 1\,[z + x - (y + x)]$$

$$= -2(y-x)(z-x)(z-y)$$

$$N_r = -2(x-y)(y-z)(z-x) \quad \ldots(4)$$

Step IV :

Differentiate f_1, f_2, f_3 partially w.r.t. u, v, w using standard rules of derivatives

and, $D_r = \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}$

We know definition of Jacobian,

$$D_r = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} \quad \ldots(5)$$

Differentiate f_1 partially w.r.t u, keeping remaining variables constants

$$\frac{\partial f_1}{\partial x} = 1$$

Differentiate f_1 partially w.r.t v, keeping remaining variables constants

$$\frac{\partial f_1}{\partial v} = 1$$

Differentiate f_1 partially w.r.t w, keeping remaining variables constants

$$\frac{\partial f_1}{\partial w} = 1$$

Differentiate f_2 partially w.r.t u, keeping r
variables constants

$$\frac{\partial f_2}{\partial u} = v + w$$

Differentiate f_2 partially w.r.t w, keeping r
variables constants

$$\frac{\partial f_2}{\partial w} = u + v$$

Differentiate f_2 partially w.r.t v, keeping r
variables constants

$$\frac{\partial f_2}{\partial v} = u + w$$

Differentiate f_3 partially w.r.t u, keeping r
variables constants

$$\frac{\partial f_3}{\partial u} = vw$$

Differentiate f_3 partially w.r.t v, keeping r
variables constants

$$\frac{\partial f_3}{\partial v} = uw$$

Differentiate f_3 partially w.r.t w, keeping r
variables constants

$$\frac{\partial f_3}{\partial w} = uv$$

$$D_r = \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix}$$

To solve determinant for simplicity use el
transformations.

$C_2 - C_1$, $C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & \\ v+w & u-v & u- \\ vw & w(u-v) & v(u \end{vmatrix}$$

Common out (u – v) from C_2, and (u – w) from C_3 i

$$= (u-v)(u-w)\begin{vmatrix} 1 & 0 \\ v+w & 1 \\ vw & w \end{vmatrix}$$

$$= (u-v)(u-w)\cdot\{1\times[\text{Minor of }1]\} - \underbrace{0\times[\text{Minor c}}_{0}$$

$$\underbrace{+0\times[\text{minor of }0]}_{0}$$

$$= (u-v)(u-w) \times 1 \times \begin{vmatrix} 1 & 0 & 0 \\ v+w & 1 & 1 \\ vw & w & v \end{vmatrix}$$

$$= (u-v)(u-w) \times \begin{vmatrix} 1 & 1 \\ w & v \end{vmatrix}$$

$$= (u-v)(u-w)[(v-w)]$$

$$D_r = -(u-v)(v-w)(w-u) \qquad \ldots(6)$$

Step V : From Equations (4) and (6), Equation (2) becomes,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -\left[\frac{-2(x-y)(y-z)(z-x)}{-(u-v)(v-w)(w-u)}\right]$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{-2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)} \checkmark$$

...Ans.

Example 5.5.5

If u , v, w are the roots of the equation $(\lambda-x)^3+(\lambda-y)^3+(\lambda+z)^3 = 0$, in terms of λ then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Solution :

Step I : Given equation is,

$$(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3 = 0$$

➔ **Using standard formula**

$$\ldots[(a-b)^3 = a^3-3a^2b+3ab^2-b^3]$$

$$\lambda^3-3\lambda^2x+3\lambda x^2-x^3+\lambda^3-3\lambda^2y+3\lambda y^2-y^3+\lambda^3-3\lambda^2z+3\lambda z^2-z^3=0$$

$$3\lambda^3-(x+y+z)3\lambda^2+3(x^2+y^2+z^2)\lambda-(x^3+y^3+z^3)=0 \qquad \ldots(1)$$

It is similar, $\quad ax^3+bx^2+cx+d = 0$

Check now,

> If α, β, γ be the roots of $ax^3+bx^2+cx+d = 0$
>
> Then, sum of roots $(\alpha+\beta+\gamma) = -\frac{b}{a}$
>
> Sum of product of roots $(\alpha\beta+\beta\gamma+\gamma\alpha) = \frac{c}{a}$
>
> and product of roots $(\alpha\cdot\beta\cdot\gamma) = -\frac{d}{a}$

Here, $a = 3$, $\quad b = -3(x+y+z)$;

$c = 3(x^2+y^2+z^2)$; $d = -(x^3+y^3+z^3)$

Given, u, v, w are the roots of Equation (1),

$$u+v+w = \frac{-b}{a} = -\left[\frac{-3(x+y+z)}{3}\right]$$

$$= x+y+z$$

$$uv+vw+wu = \frac{c}{a} = \left[\frac{3(x^2+y^2+z^2)}{3}\right]$$

$$= x^2+y^2+z^2$$

$$uvw = \frac{-d}{a} = \left[\frac{(x^3+y^3+z^3)}{3}\right]$$

$$= \frac{1}{3}(x^3+y^3+z^3)$$

Step II : Let,

$$f_1 \equiv u+v+w-x-y-z=0 ;$$
$$f_2 \equiv uv+vw+wu-x^2-y^2-z^2=0;$$
$$f_3 \equiv uvw-\frac{1}{3}(x^3+y^3+z^3)=0$$

For remaining solution, refer Example 5.5.6.

Example 5.5.6

If $x = v^2+w^2$; $y = w^2+u^2$; $z = u^2+v^2$ then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution :

Step I : Let

$$\left.\begin{aligned} f_1 &\equiv v^2+w^2-x=0 ; \\ f_2 &\equiv w^2+u^2-y=0 ; \\ f_3 &\equiv u^2+v^2-z=0 \end{aligned}\right\} \qquad \ldots(1)$$

Step II : We know, Jacobian of implicit functions,

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} = -\frac{N_r}{D_r} \qquad \ldots(2)$$

Step III : Where, $N_r = \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$

We know definition of Jacobian,

$$N_r = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} \qquad \ldots(3)$$

Differentiate f_1, f_2, f_3 partially w.r.t. x, y, z using standard rules of derivatives.

➔ **Using standard rules of derivatives**

Differentiate f_1 partially w.r.t x, keeping remaining variables constants

$$\frac{\partial f_1}{\partial x} = -1$$

Differentiate f_1 partially w.r.t y, keeping remaining variables constants

$$\frac{\partial f_1}{\partial y} = 0$$

Differentiate f_1 partially w.r.t z, keeping remaining variables constants

$$\frac{\partial f_1}{\partial z} = 0$$

Differentiate f_2 partially w.r.t x, keeping remaining variables constants

$$\frac{\partial f_2}{\partial x} = 0$$

Differentiate f_2 partially w.r.t y, keeping remaining variables constants

$$\frac{\partial f_2}{\partial y} = -1$$

Differentiate f_2 partially w.r.t z, keeping remaining variables constants

$$\frac{\partial f_2}{\partial z} = 0$$

Differentiate f_3 partially w.r.t x, keeping remaining variables constants

$$\frac{\partial f_3}{\partial x} = 0$$

Differentiate f_3 partially w.r.t y, keeping remaining variables constants

$$\frac{\partial f_3}{\partial y} = 0$$

Differentiate f_3 partially w.r.t z, keeping remaining variables constants

$$\frac{\partial f_3}{\partial z} = -1$$

Using above values in Equation (3) we get,

$$N_r = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

We know, value of upper or lower triangular matrix is product of diagonal elements.

$$N_r = (-1)(-1)(-1)$$

$$N_r = -1 \qquad \ldots(4)$$

Step IV : and $D_r = \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$

$$D_r = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} \qquad \ldots(5)$$

We know definition of Jacobian

Differentiate f_1 partially w.r.t u, keeping remaining variables constants

$$\frac{\partial f_1}{\partial u} = 0$$

Differentiate f partially w.r.t v, keeping remaining variables constants

$$\frac{\partial f_1}{\partial v} = 2v$$

Differentiate f_1 partially w.r.t w, keeping remaining variables constants

$$\frac{\partial f_1}{\partial w} = 2w$$

Differentiate f_2 partially w.r.t u, keeping remaining variables constants

$$\frac{\partial f_2}{\partial u} = 2u$$

Differentiate f_2 partially w.r.t v, keeping remaining variables constants

$$\frac{\partial f_2}{\partial v} = 0$$

Differentiate f_2 partially w.r.t w, keeping remaining variables constants

$$\frac{\partial f_2}{\partial w} = 2w$$

Differentiate f_3 partially w.r.t u, keeping remaining variables constants

$$\frac{\partial f_3}{\partial u} = 2u$$

Differentiate f_3 partially w.r.t v, keeping remaining variables constants

$$\frac{\partial f_3}{\partial v} = 2v$$

Differentiate f_3 partially w.r.t w, keeping remaining variables constants

$$\frac{\partial f_3}{\partial w} = 0$$

Using above values in Equation (5) we get,

$$D_r = \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix}$$

Common out (2u) from R_1, (2v) from R_2 and (2w) from R_3 it gives.

$$\therefore \quad D_r = (2u)(2v)(2w)\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 8\,uvw \times \{\underbrace{0 \times [\text{Minor of } 0]}_{0} - (1) \times [\text{Minor of } 1] + 1 \times$$

[Minor of 1]}

$$= 8uvw \times \left\{(-1) \times \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} + 1 \times \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}\right\}$$

$$= 8uvw \times \left\{(-1) \times \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + (1) \times \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}\right\}$$

$$= 8\,uvw\,[\,0 - 1\,(0-1) + 1\,(1-0)\,]$$

$$D_r = 16\,uvw$$

Step V : From Equations (4), (6) and (2), we get,

$$\therefore \frac{\partial (u, v, w)}{\partial (x, y, z)} = -\left[\frac{-1}{16\,uvw}\right] = \mathbf{\frac{1}{16\,uvw}} \checkmark \qquad \textbf{...Ans.}$$

5.6 Functional Dependence or Independence

Let $u = f_1(x, y)$; $v = f_2(x, y)$ are functions of independent variables x and y.

(i) If $J = \frac{\partial (u, v)}{\partial (x, y)} = 0$ then u, v are functionally dependent on each other.

Otherwise it is called functionally independent.

(ii) If the given functions are functionally dependent then, there is a relation between them.

(iii) If number of functions are less than number of independent variables. i.e. if $u = f(x, y, z)$; $v = f(x, y, z)$ and $\frac{\partial (u,v)}{\partial (x,y)} = 0$, $\frac{\partial (u,v)}{\partial (y,z)} = 0$, $\frac{\partial (u,v)}{\partial (z,x)} = 0$

then given functions are functionally dependent.

(iv) m functions of n variables are **always** functionally dependent if $m > n$.

(v) If the given functions are functionally dependent, we can find the relation between them.

(vi) Functionally dependent means one function is a function of other functions.

Example 5.6.1

Examine for functionally dependent, for $u = e^x \sin y$; $v = e^x \cos y$

Solution :

Step I : **Given ,** $u = e^x \sin y, \quad v = e^x \cos y$ ...(1)

Differentiate u and v partially w.r.t. x and y using standard rules of derivatives.

Step II : We know definition of Jacobian,

$$\frac{\partial (u,v)}{\partial (x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \qquad ...(2)$$

Differentiate u partially w.r.t x, keeping y as constant

$$\frac{\partial u}{\partial x} = e^x \sin y$$

Differentiate u partially w.r.t y, keeping x as constant

$$\frac{\partial u}{\partial y} = e^x \cos y$$

Differentiate v partially w.r.t x keeping y as constant

$$\frac{\partial v}{\partial x} = e^x \cos y$$

Differentiate v partially w.r.t y keeping x as constant.

$$\frac{\partial v}{\partial y} = e^x(-\sin y) = -e^x \sin y$$

Using above values in Equation (2) we get,

$$\frac{\partial (u,v)}{\partial (x,y)} = \begin{vmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{vmatrix}$$

$$= [(e^x \sin y)(-e^x \sin y)] - [(e^x \cos y)(e^x \cos y)]$$

$$= -e^{2x} \sin^2 y - e^{2x} \cos^2 y$$

$$y = -[e^{2x} \sin^2 y + \cos^2 y] = -e^{2x}$$

$$...(\because \sin^2 y + \cos^2 y = 1)$$

$$\frac{\partial (u,v)}{\partial (x,y)} \neq 0$$

Step III : This shows that the **given functions are functionally independent.**

$\therefore$ There is no relation between the given functions.

Example 5.6.2

Determine whether the following functions are functionally dependent. If functionally dependent, find the relation between them :

$u = \sin x + \sin y$; $v = \sin (x + y)$

Solution :

Step I : Given,

$$u = \sin x + \sin y \text{ and } v = \sin (x + y) \qquad ...(1)$$

➔ Using standard rules of derivatives

Step II : We know definition of Jacobian,

$$\frac{\partial (u\,v)}{\partial (x\,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

For functionally dependent, we must have $\frac{\partial (u\,v)}{\partial (x\,y)} = 0$

Differentiate u, v partially w.r.t. x and y

From Equation (1)

$$\frac{\partial u}{\partial x} = \cos x, \frac{\partial u}{\partial y} = \cos y, \frac{\partial v}{\partial x} = \cos (x + y),$$

$$\frac{\partial v}{\partial y} = \cos (x + y)$$

$$\frac{\partial (u\,v)}{\partial (x\,y)} = \begin{vmatrix} \cos x & \cos y \\ \cos(x+y) & \cos(x+y) \end{vmatrix}$$

$$= \cos x \cdot \cos (x + y) - \cos y \cdot \cos(x + y)$$

$$\neq 0$$

Step III :

This shows **that the given functions are functionally** independent. So, there is **no** any relation between them.

Example 5.6.3

Examine for functional dependence of

$u = \frac{x}{y - z}$; $v = \frac{y}{z - x}$; $w = \frac{z}{x - y}$

Solution :

Step I : Here u, v, w are functions of x, y, z. Therefore, for functional dependence, we have to find $\frac{\partial (u, v, w)}{\partial (x, y, z)}$

Given, $u = \frac{x}{y - z}$; $\quad v = \frac{y}{z - x}$;

$$w = \frac{z}{x - y}$$

Differentiate u, v, w partially w.r.t. x, y, z using standard rules of derivatives

$$...\left[d\left(\frac{u}{v}\right) = \frac{u\,du - v\,dv}{v^2}\right]$$

Differentiate u partially w.r.t x keeping y, z as constant,

$$\frac{\partial u}{\partial x} = \frac{1}{y - z}$$

Differentiate u partially w.r.t y, keeping x, z as constants

$$\frac{\partial u}{\partial y} = -\frac{x}{(y - z)^2}$$

Differentiate u partially w.r.t z, keeping x, y as constants

$$\frac{\partial u}{\partial z} = \frac{x}{(y - z)^2}$$

Differentiate v partially w.r.t x, keeping y, z a constants

$$\frac{\partial v}{\partial x} = \frac{y}{(z - x)^2}$$

Differentiate v partially w.r.t y, keeping x, z a constant

$$\frac{\partial v}{\partial y} = \frac{1}{z - x}$$

Differentiate v partially w.r.t z, keeping x, y a constants

$$\frac{\partial v}{\partial z} = -\frac{y}{(z - x)^2}$$

Differentiate w partially w.r.t x, keeping y, z a constants.

$$\frac{\partial w}{\partial x} = -\frac{z}{(x - y)^2}$$

Differentiate w partially w.r.t y, keeping x, z as constants.

$$\frac{\partial w}{\partial y} = -\frac{z}{(x-y)^2}$$

Differentiate w partially w.r.t z, keeping x, y as constants

$$\frac{\partial w}{\partial z} = \frac{1}{(x-y)}$$

Step II : We know definition of Jacobian,

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{(y-z)} & \frac{-x}{(y-z)^2} & \frac{x}{(y-z)^2} \\ \frac{y}{(z-x)^2} & \frac{1}{(z-x)} & \frac{-y}{(z-x)^2} \\ \frac{-z}{(x-y)^2} & \frac{z}{(x-y)^2} & \frac{1}{(x-y)} \end{vmatrix}$$

Common out $\frac{1}{(y-z)^2}$ from R_1, $\frac{1}{(z-x)^2}$ from R_2, and $\frac{1}{(x-y)^2}$ from R_3, it gives

$$= \frac{1}{(y-z)^2} \cdot \frac{1}{(z-x)^2(x-y)^2} \begin{vmatrix} (y-z) & -x & x \\ y & (z-x) & -y \\ -z & z & (x-y) \end{vmatrix}$$

To solve determinant for simplicity use elementary transformations.

By solving determinant

$$= \frac{1}{(y-z)^2(z-x)^2(x-y)^2} \left\{ \begin{array}{l} (y-z)[(z-x)(x-y)+yz] \\ +x[y(x-y)-yz]+x[yz+z(z-x)] \end{array} \right\}$$

$$= \frac{1}{(x-y)^2(y-z)^2(z-x)^2} \left\{ \begin{array}{l} (y-z)[xz-x^2+xy] \\ +x[xy-y^2-yz] \\ +x[yz+z^2-xz] \end{array} \right\}$$

$$= \frac{1}{(x-y)^2(y-z)^2(z-x)^2}$$

$$\left[\begin{array}{l} \cancel{xzy}-\cancel{x^2y}+\cancel{xy^2}-\cancel{xz^2}+\cancel{zx^2}-\cancel{xyz} \\ +\cancel{x^2y}-\cancel{xy^2}-\cancel{xyz}+\cancel{xyz}+\cancel{xz^2}-\cancel{x^2z} \end{array} \right] = 0$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

Hence, u, v, w functions are functionally dependent. ✓ **...Ans.**

Example 5.6.4

Examine the functions are functionally dependent or not, if so find the relation between them $u = \frac{x-y}{x+y}$; $v = \frac{x+y}{x}$

Solution :

Step I : Given , $u = \frac{x-y}{x+y}$; $v = \frac{x+y}{x}$

Step II : We know definition of Jacobian,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{(x+y)\cdot 1-(x-y)\cdot 1}{(x+y)^2} & \frac{(x+y)(-1)-(x-y)}{(x+y)^2} \\ \frac{x(1)-(x+y)1}{x^2} & \frac{1}{x} \end{vmatrix}$$

$$= \frac{1}{(x+y)^2} \begin{vmatrix} (x+y)-(x-y) & -x-y-x+y \\ \frac{-y}{x^2} & \frac{1}{x} \end{vmatrix}$$

$$= \frac{1}{(x+y)^2} = \begin{vmatrix} 2y & \boxed{-2x} \\ \boxed{\frac{-y}{x^2}} & \frac{1}{x} \end{vmatrix} = \frac{1}{(x+y)^2}\left[\frac{2y}{x} - \frac{2yx}{x^2}\right]$$

$$= \frac{1}{(x+y)^2} \times 0$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

Step III : This shows that the given functions are functionally dependant.

Relation

$$\therefore u = \frac{x-y}{x+y} \quad ; \quad v = \frac{x+y}{x}$$

$$\Rightarrow v = \frac{\cancel{x}}{\cancel{x}} + \frac{y}{x} = 1 + \frac{y}{x}$$

$$\Rightarrow \frac{y}{x} = v - 1$$

$$u \cdot v = \left(\frac{x-y}{x+y}\right) \cdot \left(\frac{x+y}{x}\right)$$

$$= \frac{x-y}{x} = 1 - \frac{y}{x} = 1 - (v-1)$$

$$\mathbf{u \cdot v = 2 - v}$$

It is required relation.

Example 5.6.5

Examine the following for functionally dependent or independent. If dependent, find relation between them for

$u = \sin^{-1} x + \sin^{-1} y$; $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

Solution :

Step I : Given , $u = \sin^{-1} x + \sin^{-1} y$;

$$v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

Step II : Consider,

$$\frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2}+\frac{y(-2x)}{2\sqrt{1-x^2}} & \frac{x(-2y)}{2\sqrt{1-y^2}}+\sqrt{1-x^2} \end{vmatrix}$$

$$= \left[\frac{-2xy}{2\sqrt{1-y^2}} \cancel{\frac{1}{\sqrt{1-x^2}}} + 1\right] - \left[1 - \frac{2xy}{2\sqrt{1-x^2}} \cancel{\frac{1}{\sqrt{1-y^2}}}\right]$$

$$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = 0$$

This shows that the given Functions are functionally dependant.

Step III : For relation :

$\therefore \sin^{-1} x = \alpha$

and $\sin^{-1} y = \beta$

$x = \sin\alpha$

and $y = \sin\beta$

$$v = \sin\alpha \cdot \sqrt{1-\sin^2\beta} + \sin\beta\sqrt{1-\sin^2\alpha}$$

$$= \sin\alpha\cos\beta + \sin\beta \cdot \cos\alpha$$

$$= \sin(\alpha+\beta) = \sin(\sin^{-1} x + \sin^{-1} y)$$

$\mathbf{v = \sin u}$ is the required relation. ✓ **...Ans.**

Example 5.6.6

Examine for functionally dependent, if so find the relation between them $u = \frac{x-y}{x+z}$; $v = \frac{x+z}{y+z}$

Solution :

Step I : **Given ,** $u = \frac{x-y}{x+z}$; $v = \frac{x+z}{y+z}$

Step II : We know definition of Jacobian,

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad \text{...(1)}$$

Differentiate u, v w.r.t. x and y using standard rules of derivatives

$$\ldots\left[d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}\right]$$

$$\frac{\partial u}{\partial x} = \frac{(x+z)1-(x-y)1}{(x+z)^2}$$

$$\frac{\partial u}{\partial y} = -\left(-\frac{1}{x+z}\right)$$

$$\frac{\partial v}{\partial x} = \left(\frac{1}{y+z}\right)$$

$$\frac{\partial v}{\partial y} = \frac{-(x+z)}{(y+z)^2}$$

Using above values in Equation (1)

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{z+y}{(x+z)^2} & \boxed{\frac{-1}{(x+z)}} \\ \boxed{\frac{1}{(y+z)}} & \frac{-(x+z)}{(y+z)^2} \end{vmatrix}$$

$$= \left(\frac{(z+y)}{(x+z)^2}\right) \cdot \left(\frac{-(x+z)}{(y+z)^2}\right) - \left(\frac{1}{y+z}\right)\left(\frac{-1}{x+z}\right)$$

$$= \frac{-1}{(x+z)(y+z)} + \frac{1}{(x+z)(y+z)}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

Step III : We know definition of Jacobian,

$$\text{Also, } \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix}$$

Differentiate u, v w.r.t. x and y

→ Using standard rules of derivatives

$$\ldots\left[d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}\right]$$

Since, $u = \frac{x-y}{x+z}$; $v = \frac{x+z}{y+z}$

$u_y = \frac{-1}{x+z}$; $u_z = -\frac{(x-y)}{(x+z)^2}$

$$v_z = \frac{(y+z)-(x+z)}{(y+z)^2}$$

$$v_y = \frac{-(x+z)}{(y+z)^2} \; ; \quad v_z = \frac{y-x}{(y+z)^2}$$

Step IV : $\therefore \dfrac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} -\dfrac{1}{(x+z)} & -\dfrac{(x-y)}{(x+z)^2} \\ \dfrac{-(x+z)}{(y+z)^2} & \dfrac{(y-x)}{(y+z)^2} \end{vmatrix}$

$$= \frac{1}{(x+z)^2} \cdot \frac{1}{(y+z)^2} = \begin{vmatrix} -(x+z) & -(x-y) \\ -(x+z) & y-x \end{vmatrix}$$

$$= \frac{1}{(x+z)^2 (y+z)^2} \cdot \left[-(x+z)(y-x) - (x-y)(x+z)\right] = 0$$

Differentiate u, v w.r.t. x and z

→ **Using standard rules of derivatives**

$$\ldots\left[d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2} \right]$$

Step V : and $\dfrac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} u_z & u_x \\ v_z & v_x \end{vmatrix}$

$$= \begin{vmatrix} \dfrac{-(x-y)}{(x+z)^2} & \dfrac{z+y}{(x+z)^2} \\ \dfrac{y-x}{(y+z)^2} & \dfrac{1}{y+z} \end{vmatrix}$$

$$= \frac{-(x-y)}{(x+z)^2 (y+z)} - \frac{(z+y)(y-x)}{(y+z)^2 (x+z)^2} = 0$$

All the Jacobins are zero.

Hence, given functions are functionally dependant.

Step VI : For relation :

Since, $\quad u = \dfrac{x-y}{x+z}$; $\quad v = \dfrac{x+z}{y+z}$

$$u + \frac{1}{v} = \frac{x-y}{x+z} + \frac{y+z}{x+z} = \frac{x+z}{x+z}$$

$\dfrac{u+1}{v} = 1$ is the required relations. ✓ **...Ans.**

Example 5.6.7

Examine whether the given functions are functionally dependent, if so, find the relation between them $u = xy + yz + zx$; $v = x^2 + y^2 + z^2$; $w = x + y + z$

Solution :

Step I : Given, $\quad u = xy + yz + zx$;

$$v = x^2 + y^2 + z^2 \; ;$$

$$w = x + y + z$$

Differentiate u, v, w w.r.t. x, y, z

→ **Using standard rules of derivatives**

Step II : $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$

$$= \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} y+z & x+z & y+x \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & y+x \end{vmatrix}$$

By $C_2 - C_1$, $C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ y+z & x-y & x-z \end{vmatrix}$$

$$= -2\left[(y-x)\cancel{(x-z)} - (x-y)\cancel{(z-x)} \right]$$

$$= 0$$

Hence, the given functions are functionally dependant.

Step III : For relation

Since, $\quad u = xy + yz + zx$

$$v = x^2 + y^2 + z^2$$

$$w = x + y + z$$

$$\therefore \quad 2u + v = 2(xy + yz + zx) + x^2 + y^2 + z^2$$

$$= (x + y + z)^2 = w^2$$

$\Rightarrow \quad$ **$2u + v = w^2$** is required relation.

Differentiate u, v, w partially w.r.t. x and y

→ **Using standard rules of derivatives**

$$\ldots\left[d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2} \right]$$

Since, $\quad u = \dfrac{x-y}{x+z}$; $\quad v = \dfrac{x+z}{y+z}$

Differentiate u partially w.r.t y, keeping x, z as constants.

$$\frac{\partial u}{\partial y} = \frac{-1}{x+z}$$

Differentiate u partially w.r. t z, keeping x, y as constants

$$\frac{\partial u}{\partial z} = -\frac{(x-y)}{(x+z)^2}$$

Differentiate v partially w.r.t y, keeping x, z as constants

$$\therefore \frac{\partial v}{\partial y} = -\frac{(x+z)}{(y+z)^2}$$

Differentiate v partially w.r.t z, keeping x, y as constants

$$\frac{\partial v}{\partial z} = \frac{(y+z)-(x+z)}{(y+z)^2} = \frac{y-x}{(y+z)^2}$$

Differentiate v partially w.r.t y, keeping x, z as constant

Step IV :

By definition of Jacobian,

$$\therefore \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} -\dfrac{1}{(x+z)} & -\dfrac{(x-y)}{(x+z)^2} \\ \dfrac{-(x+z)}{(y+z)^2} & \dfrac{(y-x)}{(y+z)^2} \end{vmatrix}$$

Common out $\dfrac{1}{(x+z)^2}$ from R_1 and $\dfrac{1}{(y+z)^2}$ from R_2, we get

$$= \frac{1}{(x+z)^2} \cdot \frac{1}{(y+z)^2} \begin{vmatrix} -(x+z) & -(x-y) \\ -(x+z) & (y-x) \end{vmatrix}$$

$$= \frac{1}{(x+z)^2 (y+z)^2} \cdot \left[-(x+z)(y-x) - (x-y)(x+z)\right] = 0$$

$$\therefore \frac{\partial(u,v)}{\partial(y,z)} = 0$$

Step V :

We know definition of Jacobian,

Now, $\dfrac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \dfrac{\partial u}{\partial z} & \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial z} & \dfrac{\partial v}{\partial x} \end{vmatrix}$

Differentiate u, v partially w.r.t. x and z

$$\frac{\partial u}{\partial x} = \frac{(x+z)\cdot 1 - (x-y)\cdot 1}{(x+z)^2} = \frac{z+y}{(x+z)^2}$$

$$\frac{\partial v}{\partial x} = \left(\frac{1}{y+z}\right)$$

→ **Using standard rules of derivatives**

$$\ldots\left[d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2} \right]$$

$$= \begin{vmatrix} \dfrac{-(x-y)}{(x+z)^2} & \dfrac{z+y}{(x+z)^2} \\ \dfrac{y-x}{(y+z)^2} & \dfrac{1}{y+z} \end{vmatrix}$$

...(using values from above steps)

$$= \left(\frac{-(x-y)}{(x+z)^2}\right)\left(\frac{1}{y+z}\right) - \left(\frac{y-x}{(y+z)^2}\right)\left(\frac{z+y}{(x+z)^2}\right)$$

$$= \frac{-(x-y)}{(x+z)^2 (y+z)} - \frac{(z+y)(y-x)}{(y+z)^2 (x+z)^2}$$

$$= \frac{(y-x)}{(x+z)^2 (y+z)} - \frac{(y-x)}{(x+z)^2 (y+z)}$$

$$= 0$$

All the Jacobins are zero.

Hence, **given functions are functionally dependant.**

Step VI : For relation :

Since, $\quad u = \dfrac{x-y}{x+z}$; $\quad v = \dfrac{x+z}{y+z}$

$$u + \frac{1}{v} = \frac{x-y}{x+z} + \frac{y+z}{x+z} = \frac{x-y+y+z}{x+z} = \frac{x+z}{x+z}$$

$\dfrac{u+1}{v} = 1$ is the required relation. ✓ **...Ans.**

Example 5.6.8

Examine whether the given functions are functionally dependent, if so, find the relation between them $u = xy + yz + zx$; $v = x^2 + y^2 + z^2$; $w = x + y + z$

Solution :

Step I : Given, $\quad u = xy + yz + zx$;

$$v = x^2 + y^2 + z^2 \; ;$$

$$w = x + y + z$$

Differentiate u, v, w partially w.r.t. x, y, z

→ Using standard rules of derivatives

$$\textbf{Step II :} \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \quad \ldots(1)$$

Differentiate u partially w.r.t x, keeping y, z as constants.

$$\frac{\partial u}{\partial x} = y+z$$

Differentiate u partially w.r.t y, keeping x, z as constants.

$$\frac{\partial u}{\partial y} = x+z$$

Differentiate u partially w.r. t z, keeping x, y as constants

$$\frac{\partial u}{\partial z} = y+x$$

Differentiate v partially w.r.t x, keeping y, z as constants

$$\frac{\partial v}{\partial x} = 2x$$

Differentiate v partially w.r.t y, keeping x, z as constant

$$\frac{\partial v}{\partial y} = 2y$$

Differentiate v partially w.r.t z, keeping x, y as constants

$$\frac{\partial v}{\partial z} = 2z$$

Differentiate w partially w.r.t x, keeping y, z as constants.

$$\frac{\partial w}{\partial x} = 1$$

Differentiate w partially w.r.t y, keeping x, z as constants.

$$\frac{\partial w}{\partial y} = 1$$

Differentiate w partially w.r.t z, keeping x, y as constants

$$\frac{\partial w}{\partial z} = 1$$

By putting above values, Equation (1) becomes,

$$= \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

Common out 2 from R_2, it gives,

$$= 2\begin{vmatrix} y+z & x+z & y+x \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

Operate $R_1 \leftrightarrow R_3$ [value of determinant change by (–)]

$$= -2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & y+x \end{vmatrix}$$

To solve determinant, for simplicity use elementary transformation.

Operate $C_2 - C_1, C_3 - C_1$

$$= -2\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ y+z & x-y & x-z \end{vmatrix}$$

Adding $R_2 + R_3$

$$= -2\begin{vmatrix} 1 & 0 & 0 \\ x+y+z & 0 & 0 \\ x+y+z & x-y & x-z \end{vmatrix}$$

We know, value of upper or lower triangular matrix is product of diagonal elements

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = (1)(0)(x-z) = 0$$

Hence, the given functions are functionally dependant.

Step III : For relation

$$\begin{aligned} \text{Since,} \quad u &= xy+yz+zx \\ v &= x^2+y^2+z^2 \\ w &= x+y+z \\ \therefore \quad 2u+v &= 2(xy+yz+zx)+x^2+y^2+z^2 \\ &= x^2+y^2+z^2+2xy+2yz+2xz \\ &= (x+y+z)^2 = w^2 \\ \Rightarrow \quad \mathbf{2u+v} &= \mathbf{w^2} \quad \textbf{is required relation.} \checkmark \end{aligned}$$

...Ans.

Example 5.6.9

Examine for functionally dependent, if so find the relation between them $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$.

Solution :

Step I :

Given , $u = \frac{x+y}{1-xy}$; $v = \tan^{-1}x + \tan^{-1}y$ ...(1)

Differentiate u, v partially w.r.t. x and y

→ Using standard rule of derivative

Differentiate u partially w.r.t x, keeping y as constant

$$\frac{\partial u}{\partial x} = \frac{(1-xy)-(x+y)\cdot(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

Differentiate u partially w.r.t y, keeping x as constant.

$$\frac{\partial u}{\partial y} = \frac{(1-xy)\cdot 1-(x+y)\cdot(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

Differentiate v partially w.r.t x, keeping y as constant. ...(2)

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} \quad \ldots\left[\because \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}\right]$$

Differentiate v partially w.r.t y, keeping x as constant.

$$\frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

Step II :

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Substitute above values from Equation (2), it gives,

$$= \begin{vmatrix} \frac{(1+y^2)}{(1-xy)^2} & \frac{(1+x^2)}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1+y^2}{(1-xy)^2}\left(\frac{1}{1+y^2}\right) - \left(\frac{1}{1+x^2}\right)\left(\frac{1+x^2}{(1-xy^2)}\right)$$

$$= \frac{\cancel{1}}{(\cancel{1}-xy)^2} - \frac{\cancel{1}}{(\cancel{1}-xy)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

This shows that, the given functions are functionally dependant.

Step III : For relation

$$\begin{aligned} \text{Since,} \quad u &= \frac{x+y}{1-xy}, \\ v &= \tan^{-1}x + \tan^{-1}y \\ \Rightarrow \quad v &= \tan^{-1}x + \tan^{-1}y \\ y &= \tan^{-1}\left(\frac{x+y}{1-xy}\right) \quad \text{...(by standard formula)} \\ v &= \tan^{-1}(u) \quad \text{...[from Equation (1)]} \\ \Rightarrow \quad \mathbf{u} &= \mathbf{\tan v} \quad \textbf{is the required relation.} \end{aligned}$$

✓ ...Ans.

Example 5.6.10

Examine for functionally dependent, if so find the relation between them $u = \frac{x-y}{1+xy}$; $v = \tan^{-1}x - \tan^{-1}y$

Solution :

Step I : Given , $u = \frac{x-y}{1+xy}$;

$v = \tan^{-1}x - \tan^{-1}y$

Differentiate u, v w.r.t. x and y

→ Using standard rule of derivative

Partially differentiate u w.r.t x, keeping y as constant

$$\frac{\partial u}{\partial x} = \frac{(1+xy)\,1-(x-y)\cdot y}{(1+xy)^2} = \frac{1+y^2}{(1+xy)^2}$$

Partially differentiate u w.r.t y, keeping x as constant.

$$\frac{\partial u}{\partial y} = \frac{(1+xy)(-1)-(x-y)\,x}{(1+xy)^2} = \frac{-1-x^2}{(1+xy)^2}$$

$$\frac{\partial u}{\partial y} = -\left(\frac{1+x^2}{(1+xy)^2}\right)$$

Partially differentiate v w.r.t x keeping y as constant.

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} ;$$

Partially differentiate v w.r.t y keeping x as constant.

$$\frac{\partial v}{\partial y} = -\frac{1}{1+y^2}$$

$$\textbf{Step II :} \quad \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{(1+y^2)}{(1+xy)^2} & \frac{-(1+x^2)}{(1+xy)^2} \\ \frac{1}{(1+x^2)} & \frac{-1}{(1+y^2)} \end{vmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{-1}{(1+xy)^2} + \frac{1}{(1+xy)^2} = 0$$

This shows that the given factions are functionally dependent.

Step III : For relation

Since, $v = \tan^{-1} x - \tan^{-1} t$

$$= \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$v = \tan^{-1}(u)$$

$\Rightarrow$ **$\tan v = u$ ✓ is the required relation.** **...Ans.**

Example 5.6.11

Examine whether functions :

$u = x + y + z$, $v = x^2 + y^2 + z^2$; $w = x^3 + y^3 + z^3 - 3xyz$ are functionally dependant. If so find the relation between them.

Solution :

Step I : Given ,

$$\left.\begin{aligned} u &= x+y+z \ ; \\ v &= x^2+y^2+z^2 \\ w &= x^3+y^3+z^3-3xyz \end{aligned}\right\} \quad ...(1)$$

Differentiate u, v, w partially w.r.t. x, y, z

→ **Using standard rules of derivatives**

Step II :

We know definition of Jacobian,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \quad ...(2)$$

Differentiate u partially w.r.t x, keeping y, z as constant

$$\frac{\partial u}{\partial x} = 1$$

Differentiate u partially w.r.t y, keeping x, z as constants.

$$\frac{\partial u}{\partial y} = 1$$

Differentiate u partially w.r.t z, keeping x, y as constants

$$\frac{\partial u}{\partial z} = 1$$

Differentiate v partially w.r.t x, keeping y, z as constants

$$\frac{\partial v}{\partial x} = 2x$$

Differentiate v partially w.r.t y, keeping x, z as constant

$$\frac{\partial v}{\partial y} = 2y$$

Differentiate v partially w.r.t z, keeping x, y as constants

$$\frac{\partial v}{\partial z} = 2z$$

Differentiate w partially w.r.t x, keeping y, z as constants.

$$\frac{\partial w}{\partial k} = 3x^2 - 3yz$$

Differentiate w partially w.r.t y, keeping x, z as constants.

$$\frac{\partial w}{\partial y} = 3y^2 - 3xz$$

Differentiate w partially w.r.t z, keeping x, y as constants

$$\frac{\partial w}{\partial z} = 3z^2 - 3xy$$

Substituting above values in Equation (2), we get

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

Common out 2 from R_2 and 3 from R_3, it gives,

$$= 2 \times 3 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

To solve determinant, for simplicity use elementary transformations.

By $C_2 - C_1$, $C_3 - C_1$

$$= 6 \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2-yz & (y-x)(x+y+z) & (z-x)(x+y+z) \end{vmatrix}$$

$$\ldots\left\{\begin{aligned} y^2-xz-x^2+yz &= (y^2-x^2)-xz+yz \\ &= (y-x)(y+x)+z(y-x) \\ &= (y-x)(x+y+z) \end{aligned}\right\}$$

Common out $(y-x)$ from C_2 and $(z-x)$ from C_3

$$= 6(y-x)(z-x) \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2-yz & x+y+z & x+y+z \end{vmatrix}$$

Since two columns (C_2, C_3) are identical, so value of determinant is zero.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

This show that u, v, w are functionally dependent.

Step III : For relation

Since,

$$\begin{aligned} u &= x+y+z \ ; \\ v &= x^2+y^2+z^2 \ ; \\ w &= x^3+y^3+z^3-3xyz \end{aligned}$$

We can write

$$\begin{aligned} w &= (x^3+y^3+z^3-3xyz) \\ &= (x+y+z)\cdot(x^2+y^2+z^2-xy-yz-zx) \end{aligned}$$

$$\therefore w = u(v - xy - yz - zx)$$

Since,

$$u^2 = (x+y+z)^2 = x^2+y^2+z^2+2xy+2xz+2yz$$

Now, $u^2 = v + 2(xy + yz + zx)$

$$\therefore xy + yz + zx = \frac{u^2 - v}{2}$$

$$\therefore w = u\left[v - \frac{u^2-v}{2}\right]$$

$$= u\left[\frac{2v - u^2 + v}{2}\right]$$

$$w = u\left[\frac{3v - u^2}{2}\right]$$

$\Rightarrow$ **$2w = u(3v - u^2)$ is the required relation. ✓ ...Ans.**

Example 5.6.12

Examine whether functions $u = x + y + z$; $v = x - y + z$; $w = x^2 + y^2 + z^2 + 2xz$ are functionally dependant. If so find the relation between them.

Solution :

Step I : Given ,

$$\left.\begin{aligned} u &= x+y+z \ ; \quad v = x-y+z \\ w &= x^2+y^2+z^2+2xz \end{aligned}\right\} \quad ...(1)$$

Differentiate u, v, w partially w.r.t. x, y, z

→ **Using standard rules of derivatives**

Differentiate u partially w.r.t x keeping y, z as constants.

$$\frac{\partial u}{\partial x} = 1$$

Differentiate u partially w.r.t y, keeping x, z as constants.

$$\frac{\partial u}{\partial y} = 1$$

Differentiate u partially w.r. t z, keeping x, y as constants

$$\frac{\partial u}{\partial z} = 1$$

Differentiate v partially w.r.t x, keeping y, z as constants

$$\frac{\partial v}{\partial x} = 1$$

Differentiate v partially w.r.t y, keeping x, z as constant

$$\frac{\partial v}{\partial y} = 1$$

Differentiate v partially w.r.t z, keeping x, y as constants

$$\frac{\partial v}{\partial z} = 1$$

Differentiate w partially w.r.t x, keeping y, z as constants.

$$\frac{\partial w}{\partial x} = 2x + 2z$$

Differentiate w partially w.r.t y, keeping x, z as constants.

$$\frac{\partial w}{\partial y} = 2y$$

Differentiate w partially w.r.t z, keeping x, y as constants

$$\frac{\partial w}{\partial z} = 2z + 2x$$

Step II : We know definition of Jacobian,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Substituting above values, we get

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2x+2z & 2y & 2z+2x \end{vmatrix}$$

To solve determinant, for simplicity use elementary transformations.

By $C_2 - C_1$, $C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 2x+2z & 2y-2x-2z & 0 \end{vmatrix}$$

We know value of upper or lower triangular matrix is product of diagonal elements.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (1)(-2)(0) = 0$$

This show that u, v, w functions are functionally dependent.

Step III : For relation

From Equation (1),

$$u^2 = (x+y+z)^2$$

$$= x^2+y^2+z^2+2xy+2xz+2yz$$

$$v^2 = (x-y+z)^2$$

$$= x^2+y^2+z^2-2xy+2xz-2yz$$

$$u^2+v^2 = 2(x^2+y^2+z^2+2xz)$$

$u^2+v^2 = 2w$ **is the required relation.** ✓ **...Ans.**

5.7 Partial Derivatives Using Jacobians

If u, v are implicit functions of x, y mapped by f_1 and f_2 such that

$$u(x,y) = 0; \qquad v(x,y) = 0$$

$$\Rightarrow f_1(u,v,x,y) = 0; \quad f_2(u,v,x,y) = 0$$

then,
$$\frac{\partial u}{\partial x} = -\left[\frac{\dfrac{\partial(f_1,f_2)}{\partial(x,v)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}}\right];$$

$$\frac{\partial x}{\partial v} = -\left[\frac{\dfrac{\partial(f_1,f_2)}{\partial(v,y)}}{\dfrac{\partial(f_1,f_2)}{\partial(x,y)}}\right]$$

Similarly, if u, v, w are implicit functions of x, y, z connected by f_1, f_2, f_3 such that $f_1(u,v,w,x,y,z)=0$; $f_2(u,v,w,x,y,z)=0$; $f_3(u,v,w,x,y,z)=0$ then

$$\frac{\partial u}{\partial z} = -\left[\frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(z,v,w)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}\right];$$

$$\frac{\partial y}{\partial w} = -\left[\frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,w,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}\right]$$

Note : (i) For partial derivative $\frac{\partial x}{\partial u}$ (say) first write Jacobian in the denominator and then to write the Jacobin in numerator by replacing x by u.

(ii) Partial derivatives exists only when Jacobian of dependent variables are non zero.

Type I : Examples based on Partial Derivatives using Jacobian

Example 5.7.1

If $u^2+xv^2-uxy=0$; $v^2-xy^2+2uv+u^2=0$; then find $\frac{\partial u}{\partial x}$ by choosing u, v as dependent and x, y as independent variables.

Solution :

Step I :

Let,
$$f_1 \equiv u^2+xv^2-uxy=0;$$
$$f_2 \equiv v^2-xy^2+2uv+u^2=0$$

We know partial derivative by using Jacobian is,

$$\frac{\partial u}{\partial x} = -\left[\frac{\dfrac{\partial(f_1,f_2)}{\partial(x,v)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}}\right] = -\frac{N_r}{D_r} \qquad \text{...(1)}$$

Step II :

Now, We know definition of Jacobian,

$$N_r = \frac{\partial(f_1,f_2)}{\partial(x,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix}$$

Differentiate f_1, f_2, partially w w.r.t. x, v

➔ **Using standard rules of derivatives**

$$\frac{\partial f_1}{\partial x} = v^2 - uy \quad \text{... (Differentiate } f_1 \text{ w.r.t x, keeping others constants)}$$

$$\frac{\partial f_1}{\partial v} = 2xv \quad \text{... (Differentiate } f_1 \text{ w.r.t v, keeping others constants)}$$

$$\frac{\partial f_2}{\partial v} = 2v+2u \quad \text{... (Differentiate } f_2 \text{ w.r. t v, keeping others constants)}$$

$$\therefore N_r = \begin{vmatrix} v^2-uy & \boxed{2xv} \\ \boxed{-y^2} & 2v+2u \end{vmatrix}$$

$$= [(v^2-uy)(2v+2u)] - [(-y^2)(2xv)]$$

$$= (v^2-uy)(2v+2u)+2xy^2v \qquad \text{...(2)}$$

Also, We know definition of Jacobian,

$$D_r = \frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2u-xy & 2xv \\ 2v+2u & 2v+2u \end{vmatrix}$$

$$= (2u-xy)(2u+2v)-2xv(2u+2v)$$

$$= (2u+2v)[2u-xy-2xv] \qquad \text{...(3)}$$

Step III :

By using Equations (2) and (3), Equation (1) becomes,

$$\frac{\partial u}{\partial x} = -\frac{N_r}{D_r}$$

$$= -\left[\frac{(v^2-uy)(2u+2v)+2xy^2v}{2(u+v)(2u-xy-2xv)}\right] \checkmark \quad \text{...Ans.}$$

Example 5.7.2

If $x=\cos\theta - r\sin\theta$; $y=\sin\theta + r\cos\theta$, then find $\frac{\partial r}{\partial x}$.

Solution :

Step I : Given, $x=\cos\theta - r\sin\theta$; and $y=\sin\theta + r\cos\theta$

Let,
$$f_1 \equiv x-\cos\theta + r\sin\theta = 0$$
$$f_2 \equiv y-\sin\theta - r\cos\theta = 0$$

We know partial derivative by using Jacobian is,

$$\frac{\partial r}{\partial x} = -\left[\frac{\dfrac{\partial(f_1,f_2)}{\partial(x,\theta)}}{\dfrac{\partial(f_1,f_2)}{\partial(r,\theta)}}\right] = -\frac{N_r}{D_r} \qquad \text{...(1)}$$

Differentiate f_1, f_2, w w.r.t. x, θ

➔ **Using standard rules of derivatives**

Step II :
$$N_r = \frac{\partial(f_1,f_2)}{\partial(x,\theta)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \sin\theta + r\cos\theta \\ 0 & -\cos\theta + r\sin\theta \end{vmatrix}$$

$$N_r = r\sin\theta - \cos\theta$$

Differentiate f_1, f_2, w.r.t. r, θ

➔ **Using standard rules of derivatives**

and $D_r = \frac{\partial(f_1,f_2)}{\partial(r,\theta)}$

$$= \begin{vmatrix} \sin\theta & \sin\theta + r\cos\theta \\ -\cos\theta & -\cos\theta + r\sin\theta \end{vmatrix}$$

$$= (\cancel{-\sin\theta\cdot\cos\theta} + r\sin^2\theta) + (\cancel{\sin\theta\cdot\cos\theta} + r\cos^2\theta)$$

$$D_r = r \qquad \text{...}(\because \sin^2\theta + \cos^2\theta = 1)$$

Step III : From Equation (1)

$$\frac{\partial r}{\partial x} = -\frac{(r\sin\theta - \cos\theta)}{r} \checkmark \quad \text{...Ans.}$$

Example 5.7.3

If $u = xyz$; $v = x^2+y^2+z^2$, $w = x+y+z$ then prove that $\frac{\partial x}{\partial u} = \frac{1}{(x-y)(x-z)}$

Solution :

Step I : Given : $u = xyz$; $v = x^2+y^2+z^2$;

$$w = x+y+z$$

Let,
$$f_1 \equiv xyz - u = 0;$$
$$f_2 \equiv x^2+y^2+z^2-v=0;$$
$$f_3 \equiv x+y+z-w=0$$

be the implicit functions

Step II : We know partial derivative by using Jacobian is,

$$\frac{\partial x}{\partial u} = -\left[\frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}\right] = -\frac{N_r}{D_r} \qquad \text{...(1)}$$

Where, $N_r = \frac{\partial(f_1,f_2,f_3)}{\partial(u,y,z)}$

We know, definition of Jacobian,

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & xz & xy \\ 0 & 2y & \boxed{2z} \\ 0 & \boxed{1} & 1 \end{vmatrix} \quad \left[\begin{array}{c}\text{Differentiate } f_1, f_2, f_3, \\ \text{partially w.r.t u, y, z} \\ \text{using standard rules} \\ \text{of derivatives}\end{array}\right]$$

By solving determinant

$$= -1(2y-2z)$$

$$N_r = -2(y-z) \qquad \text{...(2)}$$

Step III :

$$D_r = \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$$

We know definition of Jacobian,

$$\frac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

Differentiate f_1, f_2, f_3, partially w.r.t. x, y, z, it gives

$$D_r = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} \quad \begin{bmatrix} \text{Differentiate } f_1, f_2, f_3, \\ \text{partially w.r.t u, y, z} \\ \text{using standard rules} \\ \text{of derivatives} \end{bmatrix}$$

common out 2 from R_2, it gives

$$= 2 \begin{vmatrix} yz & xz & xy \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

To solve determinant for simplicity use elementary transformation.

By $C_2 - C_1$, $C_3 - C_1$:

$$= 2 \begin{vmatrix} yz & (x-y)z & (x-z)y \\ x & (y-x) & (z-x) \\ 1 & 0 & 0 \end{vmatrix}$$

Common out $(x - y)$ from C_2 and $(x - z)$ from C_3, we get

$$= 2\,(1)\,(x-y)\,(x-z) \begin{vmatrix} yz & z & y \\ x & \boxed{-1} & -1 \\ 1 & 0 & 0 \end{vmatrix}$$

By solving determinant and using 3rd row,

$$D_r = 2(x-y)\,(x-z)\cdot 1\,(-z+y)$$

$$D_r = 2\,(x-y)\,(x-z)\,(y-z) \qquad \ldots(3)$$

Step IV : Hence, From Equations (1) (2) and (3),

$$\frac{\partial x}{\partial u} = -\left[\frac{N_r}{D_r}\right]$$

$$= -\left[\frac{-2\,(y-z)}{2(x-y)\,(x-z)\,(y-z)}\right]$$

$$\therefore \quad \frac{\partial x}{\partial u} = \frac{1}{(x-y)\,(x-z)} \checkmark \quad \text{...Hence proved.}$$

Example 5.7.4

If $u = xyz$, $v = x^2 + y^2 + z^2$; $w = x + y + z$ then find (i) $\frac{\partial x}{\partial u}$ (ii) $\frac{\partial y}{\partial u}$.

Solution :

Step I :

Given ,

$$u = xyz \; ;$$
$$v = x^2 + y^2 + z^2 \; ;$$
$$w = x + y + z$$

Let,

$$f_1 \equiv u - xyz = 0 \; ;$$
$$f_2 \equiv v - x^2 - y^2 - z^2 = 0 \; ;$$
$$f_3 \equiv w - x - y - z = 0$$

be the implicit functions.

Step II : (i) We know partial derivative using Jacobian is,

$$\frac{\partial x}{\partial u} = (-1)\frac{\dfrac{\partial (f_1, f_2, f_3)}{\partial (u, y, z)}}{\dfrac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)}}$$

$$= -\frac{N_r}{D_r} \qquad \ldots(1)$$

Step III : Where, $N_r = \dfrac{\partial (f_1, f_2, f_3)}{\partial (u, y, z)}$

We know definition of Jacobian

$$\frac{\partial (f_1, f_2, f_3)}{\partial (u, y, z)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -xz & -xy \\ 0 & -2y & \boxed{-2z} \\ 0 & \boxed{-1} & -1 \end{vmatrix} \quad \begin{bmatrix} \text{Differentiate } f_1, f_2, f_3, \\ \text{partially w.r.t x, y, z} \\ \text{u using standard rules} \\ \text{of derivatives} \end{bmatrix}$$

By solving determinant

$$N_r = 1\,[2y - 2z] = 2\,(y-z) \qquad \ldots(2)$$

Step IV : Also, $D_r = \dfrac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)}$

We know definition of Jacobian,

$$\frac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -yz & -xz & -xy \\ -2x & -2y & -2z \\ -1 & -1 & -1 \end{vmatrix} \quad \begin{bmatrix} \text{Differentiate } f_1, f_2, f_3, \\ \text{partially w.r.t x, y, z} \\ \text{u using standard rules} \\ \text{of derivatives} \end{bmatrix}$$

Common out (-1) from R_1, (-2) from R_2 and (-1) from R_3, it gives

$$= (-1)\,(-2)\,(-1) \begin{vmatrix} yz & xz & xy \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

To solve determinant for simplicity use elementary transformation

By $C_2 - C_1$, $C_3 - C_1$

$$= -2 \begin{vmatrix} yz & z\,(x-y) & y\,(x-z) \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

Common out $(x - y)$ from C_2 and $(z - x)$ from C_3, we get

$$= -2\,(x-y)\,(z-x) = \begin{vmatrix} yz & z & \boxed{-y} \\ x & \boxed{-1} & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

By solving determinant and using 3rd row

$$= -2\,(x-y)\,(z-x)\,[\,1\,(z-y)\,]$$

$$D_r = 2\,(x-y)\,(y-z)\,(z-x) \qquad \ldots(3)$$

Step V : By using Equations (2) and (3), Equation (1) becomes

$$\frac{\partial x}{\partial u} = -\frac{2\,(y-z)}{2\,(x-y)\,(y-z)\,(z-x)}$$

$$\frac{\partial x}{\partial u} = -\frac{1}{(x-y)\,(z-x)} = \frac{1}{(x-y)\,(x-z)} \checkmark \quad \text{...Ans.}$$

Step VI : (ii) We know partial derivative by using Jacobian is,

$$\frac{\partial y}{\partial u} = (-1)\frac{\dfrac{\partial (f_1, f_2, f_3)}{\partial (x, u, z)}}{\dfrac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)}} = -\frac{N_r}{D_r} \qquad \ldots(4)$$

Step VII : Where,

$$N_r = \frac{\partial (f_1, f_2, f_3)}{\partial (x, u, z)}$$

We know definition of Jacobian,

$$\frac{\partial (f_1, f_2, f_3)}{\partial (x, u, z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -yz & 1 & -xy \\ -2x & 0 & \boxed{-2z} \\ \boxed{-1} & 0 & -1 \end{vmatrix}$$

By solving determinant

$$= (z-x)$$

Step VIII : And $D_r = \dfrac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)} = 2\,(x-y)\,(y-z)\,(z-x)$

$\because$ (From Equation (3))

$$\therefore \quad \frac{\partial y}{\partial u} = -\frac{2\,\cancel{(z-x)}}{2\,(x-y)\,(y-z)\,\cancel{(z-x)}}$$

$$\frac{\partial y}{\partial u} = \frac{-1}{(x-y)\,(y-z)} \checkmark \quad \text{...Ans.}$$

Example 5.7.5

If $x = u + v + w$; $y = u^2 + v^2 + w^2$, $z = u^3 + v^3 + w^3$, then find $\frac{\partial u}{\partial x}$.

Soln. :

Step I : Given ,

$$x = u + v + w \; ;$$
$$y = u^2 + v^2 + w^2 \; ;$$
$$z = u^3 + v^3 + w^3$$

Let,

$$f_1 \equiv u + v + w - x = 0 \; ;$$
$$f_2 \equiv u^2 + v^2 + w^2 - y = 0 \; ;$$
$$f_3 \equiv u^3 + v^3 + w^3 - z = 0$$

Step II : We know partial derivative by using Jacobian is,

$$\frac{\partial u}{\partial x} = (-1)\frac{\dfrac{\partial (f_1, f_2, f_3)}{\partial (x, v, w)}}{\dfrac{\partial (f_1, f_2, f_3)}{\partial (u, v, w)}}$$

$$= -\frac{N_r}{D_r} \qquad \ldots(1)$$

Step III :

Now, $\quad N_r = \dfrac{\partial (f_1, f_2, f_3)}{\partial (x, v, w)}$

We know definition of Jacobian,

$$= \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$$

Differentiate f_1, f_2, f_3, partially w.r.t. u, v, w, x

→ **Using standard rules of derivatives**

$$\therefore N_r = \begin{vmatrix} -1 & 1 & 1 \\ 0 & 2v & 2w \\ 0 & 3v^2 & 3w^2 \end{vmatrix}$$

To solve determinant for simplicity use elementary transformation

By $C_2 + C_1$, $C_3 + C_1$

$$= \begin{vmatrix} -1 & 0 & 0 \\ 0 & 2v & 2w \\ 0 & 3v^2 & 3w^2 \end{vmatrix}$$

By solving determinant.

$$= -1\,[6vw^2 - 6wv^2]$$

$$= -6\,vw\,(w-v)$$

$$N_r = 6\,vw\,(v-w) \qquad \ldots(2)$$

Step IV : Also,

$$D_r = \frac{\partial (f_1, f_2, f_3)}{\partial (u, v, w)}$$

We know definition of Jacobian,

$$D_r = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2u & 2v & 2w \\ 3u^2 & 3v^2 & 3w^2 \end{vmatrix}$$

Common out (2) from R_2 and (2) from R_3, we get,

$$= 6 \begin{vmatrix} 1 & 1 & 1 \\ u & v & w \\ u^2 & v^2 & w^2 \end{vmatrix}$$

To solve determinant for simplicity use elementary transformation

By $C_2 - C_1$, $C_3 - C_1$

$$D_r = 6 \begin{vmatrix} 1 & 0 & 0 \\ u & v-u & w-u \\ u^2 & v^2-u^2 & w^2-u^2 \end{vmatrix}$$

$$= 6 \begin{vmatrix} 1 & 0 & 0 \\ u & v-u & w-u \\ u^2 & (v-u)(v+u) & (w-u)(w+u) \end{vmatrix}$$

Common out $(v-u)$ from C_2 and $(w-u)$ from C_3, it gives,

$$= 6\,(v-u)\,(w-u) \begin{vmatrix} 1 & 0 & 0 \\ u & 1 & 1 \\ u^2 & v+u & w+u \end{vmatrix}$$

$$= 6\,(v-u)\,(w-u)\,(w+u-v-u)$$

....(By solving determinant)

$$D_r = 6\,(v-u)\,(w-u)\,(w-v) \qquad \ldots(3)$$

Step VII : From Equations (2) and (3), Equation (1) becomes,

$$\frac{\partial u}{\partial x} = -\left[\frac{6\,vw\,(v-w)}{6\,(v-u)\,(w-u)\,(w-v)}\right]$$

$$= \frac{\mathbf{vw}}{\mathbf{(u-v)\,(u-w)}} \checkmark \qquad \textbf{...Ans.}$$

Example 5.7.6

Use Jacobian to show that $\frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v + \frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u = 0$; where $ux + vy = a$; $\frac{u}{x} + \frac{v}{y} = 1$

Solution :

Step I : Let $f_1 \equiv ux + vy - a = 0$;

$$f_2 \equiv \frac{u}{x} + \frac{v}{y} - 1 = 0$$

Step II : We know partial derivative by using Jacobian is,

$$\left(\frac{\partial x}{\partial u}\right)_v = -\left(\frac{\frac{\partial (f_1, f_2)}{\partial (u, y)}}{\frac{\partial (f_1, f_2)}{\partial (x, y)}}\right)_v$$

We know definition of Jacobian,

$$\left(\frac{\partial x}{\partial u}\right)_v = -\frac{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial y} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix}}$$

Differentiate f_1, f_2, partially w.r.t. x, y, u

→ **Using standard rules of derivatives**

$$= -\frac{\begin{vmatrix} x & v \\ \frac{1}{x} & \frac{-v}{y^2} \end{vmatrix}}{\begin{vmatrix} u & v \\ \frac{-u}{x^2} & \frac{-v}{y^2} \end{vmatrix}}$$

By solving determinants,

$$= -\frac{\left[\frac{-vx}{y^2} - \frac{v}{x}\right]}{\left[\frac{-uv}{y^2} + \frac{uv}{x^2}\right]} = \frac{v\left(\frac{x}{y^2} + \frac{1}{x}\right)}{uv\left(\frac{-x^2+y^2}{x^2y^2}\right)}$$

$$= \frac{v\left(\frac{x^2+y^2}{xy^2}\right)}{uv} \times \frac{x^2y^2}{(y^2-x^2)}$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{x}{u}\left(\frac{x^2+y^2}{y^2-x^2}\right)$$

$$\therefore \frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v = \left(\frac{x^2+y^2}{y^2-x^2}\right) \qquad \ldots(1)$$

Step III : Also, We know partial derivative by using Jacobian is,

$$\left(\frac{\partial y}{\partial v}\right)_u = -\left(\frac{\frac{\partial (f_1, f_2)}{\partial (x, v)}}{\frac{\partial (f_1, f_2)}{\partial (x, y)}}\right)_u$$

$$= -\frac{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix}} = \frac{\begin{vmatrix} u & y \\ \frac{-u}{x^2} & \frac{1}{y} \end{vmatrix}}{uv\left(\frac{y^2-x^2}{x^2y^2}\right)}$$

...(from previous denominator)

$$= -\left[\frac{\frac{u}{y} + \frac{uy}{x^2}}{uv\left(\frac{y^2-x^2}{x^2y^2}\right)}\right] = \frac{-u\left(\frac{1}{y} + \frac{y}{x^2}\right)}{uv\left(\frac{y^2-x^2}{x^2y^2}\right)}$$

$$= -\frac{u}{x^2y}\left(\frac{x^2+y^2}{1}\right) \times \frac{x^2y^2}{uv(y^2-x^2)}$$

$$\left(\frac{\partial y}{\partial v}\right)_u = -\frac{y}{v}\left(\frac{x^2+y^2}{y^2-x^2}\right);$$

$$\frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u = -\left(\frac{x^2+y^2}{y^2-x^2}\right) \qquad \ldots(2)$$

Step IV : Adding Equations (1) and (2), we get,

$$\frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v + \frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u = \mathbf{0} \checkmark \qquad \textbf{...Hence proved.}$$

Example 5.7.7

If $x = u + v$; $y = v^2 + w^2$; $z = w^3 + u^3$ then show that $\frac{\partial u}{\partial x} = \frac{vw}{vw + u^2}$

Solution :

Step I : Let
$f_1 \equiv x - u - v = 0$;
$f_2 \equiv y - v^2 - w^2 = 0$;
$f_3 \equiv z - w^3 - u^3 = 0$

Step II : We know partial derivative by using Jacobian is,

$$\frac{\partial u}{\partial x} = (-)\left[\frac{\frac{\partial (f_1, f_2, f_3)}{\partial (x, v, w)}}{\frac{\partial (f_1, f_2, f_3)}{\partial (u, v, w)}}\right]$$

$$= -\frac{N_r}{D_r} \qquad \ldots(1)$$

Step III : Now, $N_r = \frac{\partial (f_1, f_2, f_3)}{\partial (x, v, w)}$

We know definition of Jacobian,

$$= \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$$

Differentiate f_1, f_2, f_3 partially w.r.t. u, v, w and x

→ Using standard rules of derivatives

$$= \begin{vmatrix} 1 & -1 & 0 \\ 0 & -2v & -2w \\ 0 & 0 & -3w^2 \end{vmatrix}$$

By solving determinant,

$$N_r = 6vw^2 \qquad ...(2)$$

Step IV : Also, $D_r = \frac{\partial (f_1, f_2, f_3)}{\partial (u, v, w)}$

We know definition of Jacobian,

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 & 0 \\ 0 & -2v & -2w \\ -3u^2 & 0 & -3w^2 \end{vmatrix}$$

By solving determinant,

$$= -[6vw^2] + 1\,[0 - 6u^2w]$$

$$= -6vw^2 - 6u^2w$$

$$D_r = -6w(vw + u^2) \qquad ...(3)$$

Step V : Substitute values from Equation (2) and (3) in Equation (1), it gives

$$\frac{\partial u}{\partial x} = -\left[\frac{6vw^2}{-6w(vw + u^2)}\right]$$

$$\frac{\partial u}{\partial x} = \frac{vw}{vw + u^2} \checkmark \qquad \text{...Hence proved.}$$

Example 5.7.8

If $u = x + y$; $v = y + z$; $w = z + x$. Find the values of $\frac{\partial^2 x}{\partial u \, \partial v}$

Solution :

Step I : Let

$$f_1 \equiv x + y - u = 0 ;$$

$$f_2 \equiv y + z - v = 0 ;$$

$$f_3 \equiv z + x - w = 0$$

Step II : $\frac{\partial^2 x}{\partial u \, \partial v} = \frac{\partial}{\partial u}\left[\frac{\partial x}{\partial v}\right]$ ...(1)

Now, We know partial derivative by using Jacobian is,

$$\frac{\partial x}{\partial v} = -\left[\frac{\frac{\partial (f_1, f_2, f_3)}{\partial (v, y, z)}}{\frac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)}}\right] = -\frac{N_r}{D_r} \qquad ...(2)$$

Step III : Where, $N_r = \frac{\partial (f_1, f_2, f_3)}{\partial (v, y, z)}$

We know definition of Jacobian,

$$= \begin{vmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

Differentiate f_1, f_2, f_3 partially w.r.t. x, y, z, v

→ Using standard rules of derivatives

$$= \begin{vmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

By solving determinant,

$$N_r = 1(1) = 1 \qquad ...(3)$$

Step IV : Also, $D_r = \frac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)}$

We know definition of Jacobian,

$$= \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

By solving determinant,

$$D_r = 1[1 - 0] - 1[0 - 0] = 1 \qquad ...(4)$$

From Equations (2), (3) and (4)

$$\therefore \frac{\partial x}{\partial v} = -\left[\frac{1}{1}\right] = 1$$

$$\therefore \text{From Equation (1), } \frac{\partial^2 x}{\partial u \, \partial v} = \frac{\partial}{\partial u}(1) = 0$$

$$\frac{\partial^2 x}{\partial u \, \partial v} = 0 \checkmark \qquad \text{...Ans.}$$

Example 5.7.9

If $x = u^2 - v^2$; $y = uv$, find $\frac{\partial u}{\partial x}$

Solution :

Step I : Let,

$$f_1 \equiv u^2 - v^2 - x = 0 ;$$

$$f_2 \equiv uv - y = 0$$

Step II : We know partial derivative by using Jacobian is,

$$\frac{\partial u}{\partial x} = -\left[\frac{\frac{\partial (f_1, f_2)}{\partial (x, v)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}}\right] = -\frac{N_r}{D_r} \qquad ...(1)$$

Step III : Where, $N_r = \frac{\partial (f_1, f_2)}{\partial (x, v)}$

We know definition of Jacobian,

$$= \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix}$$

Differentiate f_1, f_2 partially w.r.t. u, v and x

→ Using standard rules of derivatives

$$= \begin{vmatrix} -1 & -2v \\ 0 & u \end{vmatrix}$$

...By solving determinant,

$$\mathbf{N_r = -u} \qquad \mathbf{...(2)}$$

Step IV : Also, $D_r = \frac{\partial (f_1, f_2)}{\partial (u, v)}$

We know definition of Jacobian,

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix}$$

By solving determinant,

$$\therefore \quad \mathbf{D_r = 2u^2 + 2v^2 = 2(u^2 + v^2)} \qquad \mathbf{...(3)}$$

Hence, From Equations (1), (2) and (3)

$$\frac{\partial u}{\partial x} = -\left[\frac{-u}{2(u^2 + v^2)}\right]$$

$$\frac{\partial u}{\partial x} = \frac{u}{2(u^2 + v^2)} \checkmark \qquad \text{...Ans.}$$

Example 5.7.10

If $u^2 + xv^2 = x + y$ and $v^2 + yu^2 = x - y$ then, find $\frac{\partial v}{\partial y}$.

Solution :

Step I : Let,

$$f_1 \equiv u^2 + xv^2 - (x + y) = 0 ;$$

$$f_2 \equiv v^2 + yu^2 - x + y = 0$$

Step II : We know partial derivative by using Jacobian is,

$$\frac{\partial v}{\partial y} = -\left[\frac{\frac{\partial (f_1, f_2)}{\partial (u, y)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}}\right] = -\frac{N_r}{D_r} \qquad ...(1)$$

Step III : Where, $N_r = \frac{\partial (f_1, f_2)}{\partial (u, y)}$

We know definition of Jacobian,

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial y} \end{vmatrix}$$

Differentiate f_1, f_2 partially w.r.t. u, v and y

→ Using standard rules of derivatives

$$= \begin{vmatrix} 2u & -1 \\ 2yu & u^2+1 \end{vmatrix}$$

By solving determinants

$$N_r = 2u(u^2 + 1) + 2yu$$

$$N_r = 2u[u^2 + 1 + y] \qquad ...(2)$$

Step IV : Also, $D_r = \frac{\partial (f_1, f_2)}{\partial (u, v)}$

We know definition of Jacobian,

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2u & 2xv \\ 2yu & 2v \end{vmatrix}$$

By solving determinant,

$$\therefore \quad D_r = 4\,uv - 4\,xyuv = 4\,uv\,(1 - xy) \quad \ldots(3)$$

From Equations (1) (2) and (3),

$$\frac{\partial v}{\partial y} = \frac{2u\,(u^2 + 1 + y)}{4uv\,(1 - xy)}$$

$$\therefore \quad \frac{\partial v}{\partial y} = -\frac{1}{2}\left(\frac{1 + u^2 + y}{v\,(1 - xy)}\right) \checkmark \quad \ldots\textbf{Ans.}$$

> **Note :** If $\frac{\partial^2 x}{\partial v^2}$ is any function of x, y, z ; first find the partial derivative of $\frac{\partial x}{\partial v}$ w.r.t u by jacobian and then again find the partial derivatives such as $\frac{\partial^2 x}{\partial v^2}$ by using Jacobian methods. We get second order partial derivatives.

Exercise 1

Ex. 1 Find the jacobian $\frac{\partial(u, v)}{\partial(x, y)}$ of

(i) $u = x^2 - y^2$; $v = 2xy$

(ii) $u = x + \frac{y^2}{x}$; $v = \frac{y^2}{x}$

(iii) $u = \frac{x + y}{1 - xy}$;

$v = \tan^{-1}(x) + \tan^{-1} y$

Ans. : (i) $4\,(x^2 + y^2)$; (ii) $\frac{2y}{x}$ (iii) 0

Ex. 2 If $x = r\cos\theta$; $y = r\sin\theta$ then find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$

Ans. : r and $\frac{1}{r}$

Ex. 3 If $u = x^2 - y^2$; $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$ then find $\frac{\partial(u,v)}{\partial(r,\theta)}$

Ans. : $4r^3$

Ex. 4 If $x = uv$ and $y = \frac{u + v}{u - v}$ then find $\frac{\partial(u,v)}{\partial(x,y)}$

Ans. : $\frac{(u - v)^2}{4uv}$

Ex. 5 If $u = \frac{2yz}{x}$, $v = \frac{2zx}{y}$, $w = \frac{2xy}{z}$ then prove that $JJ' = 1$

Ex. 6 If $u = e^x \cos y$; $v = e^x \sin y$ and $x = \ell r + sm$; $y = mr - \ell s$ then find $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)}$

Ans. : $e^{2x}(\ell^2 + m^2)$

Ex. 7 If $u = x + y + z$; $uv = y + z$, $uvw = z$; then find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Ans. : $u^2 v$

Ex. 8 If $u = x + y + z$, $u^2 v = y + z$; $u^3 w = z$, then find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Ans. : u^5

Ex. 9 If $u^3 + v + w = x + y^2 + z^2$; $u + v^3 + w = x^2 + y + z^2$; $u + v + w^3 = x^2 + y + z$; then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Ans. : $\frac{1 - 4(xy + yz + zx) + 16xyz}{27u^2v^2w^2 - 3\,(u^2 + v^2 + w^2) + 2}$

Ex. 10 If $u^3 + v^3 = x + y$; $u^2 + v^2 = x^3 + y^3$, then show that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}\left[\frac{y^2 - x^2}{uv(u - v)}\right]$

Ex. 11 If $u + v^2 = x$; $v + w^2 = y$; $w + u^2 = z$ find $\frac{\partial u}{\partial x}$

Ans. : $\frac{-1}{1 + 8\,uvw}$

Ex. 12 If $x = u + v + w$; $y = u^2 + v^2 + w^2$, $z = u^3 + v^3 + w^3$; then show that

$$\frac{\partial u}{\partial x} = \frac{vw}{(u - v)(u - w)}$$

5.8 Taylor's and Maclaurin's Theorem For Functions of two Variables

Since, we know the Taylor's theorem for functions of one variable.

If,

(i) f(x) and its first (n − 1) derivatives are continuous in the interval [a, a + h] and

(ii) n^{th} derivatives $f_{(n)}^{(n)}$ exists for every x in (a, a + h) then there is at least one number k (0 < k < 1), such that,

$$f(a + h) = f(a) + h\,f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{2!} f^{(n)}(a + kh) \quad \ldots(1)$$

Now, let f(x, y) be a function of two variables.

By using Equation (1),

By considering y as constant,

$$f(x + h, y + k) = f(x, y + k) + h\,\frac{\partial f(xy + k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x, y + k)}{\partial x^2} \quad \ldots(2)$$

And then expanding f(x, y + k) by considering x as constant,

$$f(x, y + k) = f(x, y) + k\,\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} \quad \ldots(3)$$

By using equation (3), Equation (2) becomes,

$$f(x + h, y + k) = f(x, y) + k\,\frac{\partial f}{\partial y}(x, y) + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots$$

$$+ h\,\frac{\partial}{\partial x}\left[f(x, y) + k\,\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots\right]$$

$$+ \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left[f(x, y) + k\,\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots\right] + \ldots$$

$$= f(x, y) + k\,\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots$$

$$+ h\,\frac{\partial f(x, y)}{\partial x} + h\,k\,\frac{\partial^2 f(x, y)}{\partial x\,\partial y} + h \cdot \frac{k^2}{2!}\,\frac{\partial^2 f(x, y)}{\partial x\,\partial y^2} + \ldots$$

$$+ \frac{h^2}{2!}\frac{\partial^2 f(x, y)}{\partial x^2} + \frac{h^2}{2!} k\,\frac{\partial^3 f(x, y)}{\partial x^2\,\partial y} + \frac{h^2}{2!}\frac{k^2}{2!}\frac{\partial^4 f(x, y)}{\partial x^2\,\partial y^2} + \ldots$$

$$= f(x, y) + h\,\frac{\partial f(x, y)}{\partial x} + k\,\frac{\partial f(x, y)}{\partial y}$$

$$+ \frac{1}{2!}\left[h^2 \cdot \frac{\partial^2 f(x, y)}{\partial x^2} + 2\,hk\,\frac{\partial^2 f(x, y)}{\partial x\,\partial y} + k^2\,\frac{\partial^2 f(x, y)}{\partial y^2}\right] + \ldots(4)$$

Hence,

$$f(x + h, y + k) = f(x, y) + \left(h\,\frac{\partial}{\partial x} + k\,\frac{\partial}{\partial y}\right) f(x, y)$$

$$+ \frac{1}{2!}\left(h\,\frac{\partial}{\partial x} + k\,\frac{\partial}{\partial y}\right)^2 + f(x, y) + \ldots(5)$$

Let x = a and y = b, we get

$$f(a + h, b + k) = f(a, b) + [h\,f_x(a, b) + k\,f_y(a, b)]$$

$$+ \frac{1}{2!}[h^2 f_{xx}(a, b) + 2\,hk\,f_{xy}(a, b) + k^2 f_{yy}(a, b)]$$

Now, Taylor's theorem in powers of (x − a) and (y − b) is,

Let a + h = x and b + k = y

∴ h = x − a and k = y − b,

The above equation becomes,

$$f(x, y) = f(a, b) + [(x - a) \cdot f_x(a, b) + (y - b)\,f_y(a, b)]$$

$$+ \frac{1}{2!}[(x - a)^2 f_{xx}(a, b) + 2\,(x - a)(y - b)\,f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] + \ldots \quad \ldots(6)$$

Equation (6) is called Taylor's series expansion of f(x,y) in two variables.

Also, if we put a = 0, b = 0 in above equation, we have

$$f(x, y) = f(0, 0) + [x\,f_x(0, 0) + y\,f_y(0, 0)]$$

$$+ \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy\,f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \quad \ldots(7)$$

This is called Maclaurin's theorem for two variables.

> **Note :** Equation (7) is used to expand f(x, y) in power of x and y or in the neighborhood of (0, 0) which is Maclaurin's series of f(x,y).

5.8.1 Examples on Taylor's Theorem for two Variables

Example 5.8.1

Expand $x^2y + 3y - 2$ in powers of (x − 1) and (y + 2)

Solution :

Step I : We know, Taylor's expansion of f(x, y) about (a, b) is,

$$f(x, y) = f(a, b) + [\,(x - a)\,f_x(a, b) + (y - b)\,f_y(a, b)]$$

$$+ \frac{1}{2!}[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)$$

$$f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)\,] + \ldots \quad \ldots(1)$$

Given : $f(x, y) = x^2y + 3y - 2$ and $(a, b) = (1, -2)$

Step II : $f(1, -2) = (1)^2(-2) + 3(-2) - 2 = -2 - 6 - 2 = -10$

$f_x(x, y) = 2xy$	$f_x(1, -2) = 2(1)(-2) = -4$
$f_y(x, y) = x^2 + 3$	$f_y(1, -2) = (1)^2 + 3 = 4$
$f_{xy}(x, y) = 2x$	$f_{xy}(1, -2) = 2(1) = 2$
$f_{xx}(x, y) = 2y$	$f_{xx}(1, -2) = 2(-2) = -4$
$f_{yy}(x, y) = 0$	$f_{yy}(1, -2) = 0$
$f_{xxx}(x, y) = 0$	$f_{xxx}(1, -2) = 0$
$f_{xxy}(x, y) = 2$	$f_{xxy}(1, -2) = 2$
$f_{xyy}(x, y) = 0$	$f_{xyy}(1, -2) = 0$
$f_{yyy}(x, y) = 0$	$f_{yyy}(1, -2) = 0$

Step III : Substitute these values in Equation (1)

$$f(x, y) = -10 + [\,(x - 1)(-4) + (y + 2)(4)\,] + \frac{1}{2!}$$

$$[\,(x - 1)^2(-4) + (x - 1)(y + 2)(2)$$

$$+ (y + 2)^2(0)\,] + \frac{1}{3!}[\,(x - 1)^3(0)$$

$$+ 3\,(x - 1)^2(y + 2)(2) + 3(x - 1)(y + 2)^2 (0) + (y + 2)^3(0)\,] + \ldots$$

$$\mathbf{f(x, y) = -10 - 4(x - 1) + 4(y + 2) - 2\,(x - 1)^2 + (x - 1)(y + 2) + (x - 1)^2(y + 2) + \ldots} \checkmark \ldots\textbf{Ans.}$$

Example 5.8.2

Using Taylor's theorem, verify the result

$$\mathbf{\sin(x + y) = x + y - \frac{(x + y)^3}{3!} + \ldots}$$

Solution :

Step I : We know, Taylor's expansion of f(x, y) about (a, b) is,

$$f(x, y) = f(a, b) + [(x-a) f_x(a, b) + (y-b) f_y(a, b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)] + \frac{1}{3!}[(x-a)^3 f_{xxx}(a, b) + 3(x-a)^2 (y-b) f_{xxy}(a, b) + 3(x-a)(y-b)^2 f_{xyy}(a, b) + (y-b)^3 f_{yyy}(a, b)] + \ldots \quad \ldots(1)$$

Step II : Given, $f(x, y) = \sin(x+y)$

and $a = 0, \quad b = 0$

$f(0, 0) = \sin(0+0) = 0$

$f_x(x, y) = \cos(x+y)$	$f_x(0, 0) = \cos(0) = 1$
$f_y(x, y) = \cos(x+y)$	$f_y(0, 0) = \cos 0 = 1$
$f_{xx}(x, y) = -\sin(x+y)$	$f_{xx}(0, 0) = -\sin 0 = 0$
$f_{xy}(x, y) = -\sin(x+y)$	$f_{xy}(0, 0) = -\sin 0 = 0$
$f_{yy}(x, y) = -\sin(x+y)$	$f_{yy}(0, 0) = -\sin(0) = 0$
$f_{xxx}(x, y) = -\cos(x+y)$	$f_{xxx}(0, 0) = -\cos 0 = -1$
$f_{xxy}(x, y) = -\cos(x+y)$	$f_{xxy}(0, 0) = -\cos 0 = -1$
$f_{xyy}(x, y) = -\cos(x+y)$	$f_{xyy}(0, 0) = -\cos 0 = -1$
$f_{yyy}(x, y) = -\cos(x+y)$	$f_{yyy}(0, 0) = -\cos 0 = -1$

Step III : Substitute these values in Equation (1),

$$f(x, y) = 0 + [x(1) + y(1)] + \frac{1}{2!}[x^2(0) + 2xy(0) + y^2(0)] + \frac{1}{3!}[x^3(-1) + 3x^2 y(-1) + 3xy^2(-1) + y^3(-1)] + \ldots$$

$$\sin(x+y) = x + y - \frac{1}{3!}(x+y)^3 + \ldots \checkmark \qquad \ldots\text{Ans.}$$

Example 5.8.3

Expand $f(x, y) = y^x$ in the neighbourhood of (1, 1) upto the terms of second degree.

Solution :

Step I : We know, Taylor's expansion of f(x, y) about (a, b) is,

$$f(x, y) = f(a, b) + [(x-a) f_x(a, b) + (y-b) f_y(a, b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)] + \ldots \quad \ldots(1)$$

Given : $f(x, y) = y^x$ and $(a, b) = (1, 1)$,

$f(1, 1) = (1)^{(1)} = 1$

$f_x(x, y) = y^x \log y$

$f_x(1, 1) = (1)^1 \log 1 = 0$

$f_y(x, y) = x y^{x-1}$

$f_y(1, 1) = (1)(1)^0 = 1$

$f_{xx} = \log y \cdot y^x \cdot \log y = y^x \log y)^2$

$f_{xx}(1, 1) = (1)^1 (\log 1)^2 = 0$

$f_{xy} = y^x \cdot \frac{1}{y} + \log y \cdot x y^{x-1}$

$f_{xy}(0, 0) = -\sin 0 = 0$

$= y^{x-1}[1 + x \log y]$

$f_{xy}(1, 1) = (1)^0 [1 + (1) \log 1] = 1 + 0 = 1$

$f_{yy} = x(x-1) y^{x-2}$

$f_{yy}(1, 1) = 1(1, 1)(1)^{1-2} = 0$

Step II : Substitute these values in Equation (1),

$$\therefore f(x, y) = 1 + [(x-1)(0) + (y-1)(1)] + \frac{1}{2!}[(x-1)^2(0) + 2(x-1)(y-1)(1) + (y-1)^2(0)] + \ldots$$

$$y^x = 1 + (y-1) + (x-1)(y-1) + \ldots \checkmark \ldots\text{Ans.}$$

Example 5.8.4

Expand $f(x, y) = \sin xy$ in powers of $(x-1)$ and $\left(y - \frac{\pi}{2}\right)$ upto the second degree terms.

Solution :

Step I : We know, Taylor's expansion of f(x, y) about (a, b) is,

$$f(x, y) = f(a, b) + [(x-a) f_x(a, b) + (y-b) f_y(a, b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)] + \ldots \quad \ldots(1)$$

Given, $f(x, y) = \sin xy$

and $(a, b) = \left(1, \frac{\pi}{2}\right)$, $f\left(1, \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

$f_x(x, y) = \cos xy \cdot y = y \cos xy$

$f_x\left(1, \frac{\pi}{2}\right) = \frac{\pi}{2}\cos\frac{\pi}{2} = 0$

$f_y(x, y) = \cos xy \cdot x = x \cos xy$

$f_y\left(1, \frac{\pi}{2}\right) = (1)\cos\frac{\pi}{2} = 0$

$f_{xx}(x, y) = y[-\sin xy](y) = -y^2 \sin xy$

$f_{xx}\left(1, \frac{\pi}{2}\right) = -\left(\frac{\pi}{2}\right)^2 \sin\frac{\pi}{2} = -\frac{\pi^2}{4}(1) = -\frac{\pi^2}{4}$

$= -\frac{\pi^2}{4}(1) = -\frac{\pi^2}{4}$

$f_{xy}(x, y) = y[-\sin xy \cdot x] + \cos xy (1)$

$= -xy \sin xy + \cos xy$

$f_{xy}\left(1, \frac{\pi}{2}\right) = -(1)\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2} + \cos\frac{\pi}{2}$

$= -\frac{\pi}{2}(1) + 0 = -\frac{\pi}{2}$

$f_{yy}(x, y) = x[-\sin xy]\, x$

$= -x^2 \sin xy$

$f_{yy}\left(1, \frac{\pi}{2}\right) = -(1)^2 \sin\frac{\pi}{2}$

$= -1$

Step II : Substitute these values in Equation (1),

$$f(x, y) = 1 + \left[(x-1)(0) + \left(y - \frac{\pi}{2}\right)(0) \right] + \frac{1}{2!}\left[(x-1)^2 \left(\frac{-\pi^2}{4}\right) + 2(x-1)\left(y-\frac{\pi}{2}\right) + \left(\frac{-\pi}{2}\right) + \left(y - \frac{\pi}{2}\right)^2 (-1) \right] + \ldots$$

$$\sin xy = 1 + 0 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2$$

$$\sin xy = 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 \checkmark \qquad \ldots\text{Ans.}$$

Example 5.8.5

Expand $e^x \cos y$ in powers of x, $\left(y - \frac{\pi}{2}\right)$ upto terms of degree three.

Solution :

Step I : We know, Taylor's expansion of f(x, y) about (a, b) is,

$$f(x, y) = f(a, b) + [(x-a) f_x(a, b) + (y-b) f_y(a, b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)] + \frac{1}{3!}[(x-a)^3 f_{xxx}(a, b) + 3(x-a)^2 (y-b) f_{xxy}(a, b) + 3(x-a)(y-b)^2 f_{xyy} + (y-b)^3 f_{yyy}(a, b)] + \ldots \quad \ldots(1)$$

Given : $f(x, y) = e^x \cos y$ and $(a, b) = \left(0, \frac{\pi}{2}\right)$,

$f\left(0, \frac{\pi}{2}\right) = e^0 \cos\left(\frac{\pi}{2}\right) = 0$

$f_x(x, y) = e^x \cos y$

$f_x\left(0, \frac{\pi}{2}\right) = e^0 \cos\left(\frac{\pi}{2}\right) = 0$

$= (1)(0) = 0$

$f_y(x, y) = -e^x \sin y$

$f_y\left(0, \frac{\pi}{2}\right) = -e^0 \sin\left(\frac{\pi}{2}\right)$

$= -(1)(1) = -1$

$f_{xx}(x, y) = e^x \cos y$

$f_{xx}\left(0, \frac{\pi}{2}\right) = e^0 \cos\left(\frac{\pi}{2}\right)$

$= (1)(0) = (0)$

$f_{xy}(x, y) = -e^x \sin y$

$f_{xy}\left(0, \frac{\pi}{2}\right) = -e^0 \sin\left(\frac{\pi}{2}\right) = -1$

$f_{yy}(x, y) = -e^x \cos y$

$f_{yy}\left(0, \frac{\pi}{2}\right) = -e^0 \cos\left(\frac{\pi}{2}\right) = -(1)(0) = 0$

$f_{xxx}(x, y) = e^x \cos y$

$f_{xxx}\left(0, \frac{\pi}{2}\right) = e^0 \cos\left(\frac{\pi}{2}\right)$

$= (1)(0) = 0$

$f_{xxy}(x, y) = -e^x \sin y$

$f_{xxy}\left(0, \frac{\pi}{2}\right) = -e^0 \sin\left(\frac{\pi}{2}\right)$

$= -(1)(1) = -1$

$f_{xyy}(x, y) = -e^x \cos y$

$f_{xyy}\left(0, \frac{\pi}{2}\right) = -e^0 \cos\left(\frac{\pi}{2}\right) = 0$

$f_{yyy}(x, y) = -e^x(-\sin y) = e^x \sin y$

$f_{yyy}\left(0, \frac{\pi}{2}\right) = e^0 \sin\left(\frac{\pi}{2}\right) = (1)(1) = i$

Step II : Substitute these values in Equation (1),

$$f(x, y) = 0 + \left[(x-0)(0) + \left(y - \frac{\pi}{2}\right)(-1) \right] + \frac{1}{2!}\left[(x-0)^2(0) + 2(x-0)\left(y - \frac{\pi}{2}\right)(-1) + \left(y - \frac{\pi}{2}\right)^2 (0) \right] + \frac{1}{3!}\left[(x-0)^3(0) + 3(x-0)^2\left(y - \frac{\pi}{2}\right)(-1) + 3(x-0)\left(y-\frac{\pi}{2}\right)^2(0) + \left(y - \frac{\pi}{2}\right)^3 (1) \right] + \ldots$$

$$f(x, y) = -\left(y - \frac{\pi}{2}\right) + \frac{1}{2}\left[0 - 2x\left(y - \frac{\pi}{2}\right) + 0\right] + \frac{1}{6}\left[0 - 3x^2\left(y - \frac{\pi}{2}\right) + 0 + \left(y - \frac{\pi}{2}\right)^3\right] + \ldots$$

$$e^x \cos y = -\left(y - \frac{\pi}{2}\right) - x\left(y - \frac{\pi}{2}\right) - \frac{1}{2}x^2\left(y - \frac{\pi}{2}\right) + \frac{1}{6}\left(y - \frac{\pi}{2}\right)^3 + \ldots \checkmark \qquad \ldots\text{Ans.}$$

Example 5.8.6

Expand e^x siny in powers of x and y as far as terms of third degree.

Solution :

Step I : We know, Taylor's expansion of f(x, y) about (a, b) is,

$$f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \frac{1}{3!}[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2 (y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy} + (y-b)^3 f_{yyy}(a,b)] + \ldots \quad \ldots(1)$$

Given, $f(x,y) = e^x \cos y$ and $(a, b) = \left(0, \frac{\pi}{2}\right)$,

$$f\left(0,\frac{\pi}{2}\right) = e^0 \cos\left(\frac{\pi}{2}\right) = 0$$

$$f_x(x,y) = e^x \cos y$$

$$f_x\left(0,\frac{\pi}{2}\right) = e^0 \cos\left(\frac{\pi}{2}\right) = 0 = (1)(0) = 0$$

$$f_y(x,y) = -e^x \sin y$$

$$f_y\left(0,\frac{\pi}{2}\right) = -e^0 \sin\left(\frac{\pi}{2}\right) = -(1)(1) = -1$$

$$f_{xx}(x,y) = e^x \cos y$$

$$f_{xx}\left(0,\frac{\pi}{2}\right) = e^0 \cos\left(\frac{\pi}{2}\right) = (1)(0) = (0)$$

$$f_{xy}(x,y) = -e^x \sin y$$

$$f_{xy}\left(0,\frac{\pi}{2}\right) = -e^0 \sin\left(\frac{\pi}{2}\right) = -1$$

$$f_{yy}(x,y) = -e^x \cos y$$

$$f_{yy}\left(0,\frac{\pi}{2}\right) = -e^0 \cos\left(\frac{\pi}{2}\right) = -(1)(0) = 0$$

$$f_{xxx}(x,y) = e^x \cos y$$

$$f_{xxx}\left(0,\frac{\pi}{2}\right) = e^0 \cos\left(\frac{\pi}{2}\right) = (1)(0) = 0$$

$$f_{xxy}(x,y) = -e^x \sin y$$

$$f_{xxy}\left(0,\frac{\pi}{2}\right) = -e^0 \sin\left(\frac{\pi}{2}\right) = -(1)(1) = -1$$

$$f_{xyy}(x,y) = -e^x \cos y$$

$$f_{xyy}\left(0,\frac{\pi}{2}\right) = -e^0 \cos\left(\frac{\pi}{2}\right) = 0$$

$$f_{yyy}(x,y) = -e^x(-\sin y)$$

$$f_{yyy}\left(0,\frac{\pi}{2}\right) = e^0 \sin\left(\frac{\pi}{2}\right) = e^x \sin y = (1)(1) = 1$$

Step II : Substitute these values in Equation (1),

$$f(x,y) = 0 + \left[(x-0)(0) + \left(y-\frac{\pi}{2}\right)(-1)\right] + \frac{1}{2!}\left[(x-0)^2(0) + 2(x-0)\left(y-\frac{\pi}{2}\right)(-1) + \left(y-\frac{\pi}{2}\right)^2(0)\right] + \frac{1}{3!}\left[(x-0)^3(0) + 3(x-0)^2\left(y-\frac{\pi}{2}\right)(-1) + 3(x-0)\left(y-\frac{\pi}{2}\right)^2(0) + \left(y-\frac{\pi}{2}\right)^3(1)\right] +$$

$$f(x,y) = -\left(y-\frac{\pi}{2}\right) + \frac{1}{2}\left[0 - 2x\left(y-\frac{\pi}{2}\right) + 0\right] + \frac{1}{6}\left[0 - 3x^2\left(y-\frac{\pi}{2}\right) + 0 + \left(y-\frac{\pi}{2}\right)^3\right] + \ldots$$

$$e^x \cos y = -\left(y-\frac{\pi}{2}\right) - x\left(y-\frac{\pi}{2}\right) - \frac{1}{2}x^2\left(y-\frac{\pi}{2}\right) + \frac{1}{6}\left(y-\frac{\pi}{2}\right)^3 + \ldots ✓$$

...Ans.

Example 5.8.7

Obtain Taylor's expansion of $\tan^{-1}\left(\frac{y}{x}\right)$ about (1, 1) upto the third degree terms.

Solution :

Step I : We know, Taylor's expansion of f(x, y) about (a, b) is,

$$f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \frac{1}{3!}[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2 (y-b) f_{xxy}(a,b) + 3(y-b)^2 (x-a) f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)] \ldots \quad \ldots(1)$$

Given,

$$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right) \text{ and } (a,b) = (1,1)$$

$$\therefore f(1,1) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$f_x(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right)$$

$$f_x(1,1) = \frac{-1}{1+1} = \frac{-1}{2}$$

$$= \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$f_y(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$f_y(1,1) = \frac{1}{1+1} = \frac{1}{2}$$

$$f_{xy} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$f_{xy}(1,1) = \frac{1-1}{(1+1)^2} = 0$$

$$f_{xx} = -y\left[\frac{-1}{(x^2+y^2)^2}2x\right]$$

$$f_{xx}(1,1) = \frac{2(1)(1)}{(1+1)^2}$$

$$= \frac{2xy}{(x^2+y^2)^2} = \frac{2}{4} = \frac{1}{2}$$

$$f_{yy} = x\left[\frac{-1}{(x^2+y^2)^2}2y\right] = \frac{-2xy}{(x^2+y^2)^2}$$

$$f_{yy}(1,1) = \frac{-2(1)(1)}{(1+1)^2} = \frac{-2}{4} = \frac{-1}{2}$$

Step II : Substitute these values in Equation (1)

$$\therefore f(x,y) = \frac{\pi}{4} + \left[(x-1)\left(\frac{-1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right] + \frac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + 2(x-1)(y-1)(0) + (y-1)^2\left(\frac{-1}{2}\right)\right] + \frac{1}{3!}\left[(x-1)^3\left(\frac{-1}{2}\right) + 3(x-1)^2(y-1)\left(\frac{-1}{2}\right) + 3(x-1)(y-1)^2\left(\frac{1}{2}\right) + (y-1)^3\left(\frac{1}{2}\right) + \ldots\right]$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2 + -\frac{1}{12}(x-1)^3 - \frac{1}{4}(x-1)^2(y-1) + \frac{1}{4}(x-1)(y-1)^2 + \frac{1}{12}(y-1)^3 + \ldots$$

$$f_{xxx} = \frac{(x^2+y^2)^2(2y) - 2xy(2)(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$f_{xxx} = \frac{2y(x^2+y^2) - 8x^2y}{(x^2+y^2)^3}$$

$$f_{xxy} = \frac{(x^2+y^2)^2(2x) - 2xy(2)(x^2+y^2)(2y)}{(x^2+y^2)^4}$$

$$= \frac{2x(x^2+y^2) - 8xy^2}{(x^2+y^2)^3}$$

$$f_{xxx}(1,1) = \frac{2(1)(1+1) - 8(1)(1)}{(1+1)^3} = \frac{(2)(2)-8}{(2)^3}$$

$$= \frac{4-8}{8} = \frac{-4}{8} = \frac{-1}{2}$$

$$f_{xxy}(1,1) = \frac{2(1)(1+1) - 8(1)(1)}{(1+1)^3} = \frac{(2)(2)-8}{(2)^3}$$

$$= \frac{4-8}{8} = \frac{-4}{8} = \frac{-1}{2}$$

$$f_{xyy} = \frac{(x^2+y^2)^2(2y) - (y^2-x^2)2(x^2+y^2)(2y)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2) - 4y(y^2-x^2)}{(x^2+y^2)^3}$$

$$f_{yyy} = \frac{(x^2+y^2)^2(-2x) - (-2xy)2(x^2+y^2)(2y)}{(x^2+y^2)^4}$$

$$= \frac{-2x(x^2+y^2) + 8xy^2}{(x^2+y^2)^3}$$

$$f_{xyy}(1,1) = \frac{2(1)(1+1) - 4(1)(1)}{(1+1)^3} = \frac{(2)(2)-4(0)}{(2)^3}$$

$$= \frac{4-0}{8} = \frac{4}{8} = \frac{1}{2}$$

$$f_{yyy}(1,1) = \frac{-2(1)(1+1) + 8(1)(1)}{(1+1)^3} = \frac{-2(2)+8}{(2)^3}$$

$$= \frac{-4+8}{8} = \frac{4}{8} = \frac{1}{2} ✓$$

...Ans.

5.9 Maxima and Minima of Functions of Two Variables u = f(x, y)

Introduction

We are very much familiar with maxima and minima of a function of single variable (y = f(x)). Now we extend this idea to the functions of two independent variables, u = f (x, y).

Let u = f (x, y) be a continuous and differentiable function of two independent variables x and y.

The function u = f (x, y) is said to have maximum value at x = a, y = b

if f (a, b) > f (a + h, b + k) and it have minimum value if f (a, b) < f (a + h, b + k), here, h and k are very small positive or negative values.

Geometrically, $u = f(x, y)$ represents a surface the point. The point of maximum may be compared with the highest point of a dome, and the minimum point may be compared with lowest point of a bowl. Sometimes at the point (a, b), the tangent plane is horizontal and the surface descends in certain directions and ascends in other directions, such a point is called a saddle point. The point at which function $u = f(x, y)$ is either maximum or minimum is known as stationary point. The maximum or minimum value of a function is called its extreme value.

5.9.1 Method of Finding Maxima and Minima

Let f is function of x and y of two variables which is continuous and differentiable in x and y. To find maximum or minimum value, use following procedure.

Step I : Given function, $f(x, y) = 0$

Step II : Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

Step III : Solve : $\frac{\partial f}{\partial x} = 0$; $\frac{\partial f}{\partial y} = 0$

Solve these equations simultaneously and find values of x and y.

$(x, y) \equiv (a_1, b_1), (a_2, b_2), (a_3, b_3)....$ are called stationary values.

Step IV : Find $r = \frac{\partial^2 f}{\partial x^2}$; $s = \frac{\partial^2 f}{\partial x\, \partial y}$; $t = \frac{\partial^2 f}{\partial y^2}$

Step V : Find r, s, t for each pair.

For (a_1, b_1)

Find $r_{(a_1, b_1)}, s_{(a_1, b_1)}, t_{(a_1, b_1)}$

Step VI : (i) If $(rt - s^2) > 0$ and $r > 0$, then function is minimum at (a_1, b_1).

$\therefore$ $f_{min} = f(a_1, b_1)$ is called extreme value.

(ii) If $(rt - s^2) > 0$ and $r < 0$ then function is maximum at (a_1, b_1).

$f_{max} = f(a_1, b_1)$ is called Extreme value

(iii) If $(rt - s^2) < 0$, then function is neither maximum nor minimum, then the point (a_1, b_1) is called **saddle** point.

Step VII : If $(rt - s^2) = 0$; we need more investigation / require more study.

Note : 1. If $(rt - s^2) > 0$ and $r < 0$ then the given function $f(x, y)$ is maximum at $(x_1, y_1) \equiv (a_1, b_1)$.

2. If $(rt - s^2) > 0$ and $r > 0$ then the given function $f(x, y)$ is minimum at $(x_1, y_1) \equiv (a_1, b_1)$

3. If $(rt - s^2) < 0$ then $f(x, y)$ is neither maximum nor minimum at $(x, y) \equiv (a, b)$

Type I : Example of Maxima, Minima of Two Independent Variables

Example 5.9.1

Discuss maxima and minima of the function $x^2 + y^2 + 6x + 12$.

Solution :

Step I : Given, $f(x, y) = x^2 + y^2 + 6x + 12$ ...(1)

Differentiate given function f - by using standard rules of derivatives, we get

Step II : $\frac{\partial f}{\partial x} = 2x + 6$

and $\frac{\partial f}{\partial y} = 2y$

For maxima and minima,

$$\Rightarrow \quad \frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y = 0$$

$$\therefore \quad 2x + 6 = 0 \quad | \quad 2y = 0$$

$$x = -3 \quad | \quad y = 0$$

$$(x, y) \equiv (-3, 0)$$

Step III : $r = \frac{\partial^2 f}{\partial x^2} = 2$;

$s = \frac{\partial^2 f}{\partial x\, \partial y} = 0$; $\quad t = \frac{\partial^2 f}{\partial y^2} = 2$

$$(rt - s^2)_{(-3, 0)} = (2 \times 2 - 0) = 4 > 0 \text{ and } r = 2 > 0$$

This shows that function is **minimum**.

Step IV : From Equation (1)

$f_{min} = (-3)^2 + 0 + 6(-3) + 12 = 9 - 18 + 12$

$f_{min} = 3$ be the extreme value ✓ ...**Ans.**

Type II : Examples of Maxima, Minima of Two Independent Variables

Example 5.9.2

Discuss the stationary points for maxima and minima of $x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$.

Solution :

Step I : Given, $f(x, y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$

Since, for maxima and minima,

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

Differentiate given function f - by using standard rules of derivatives, we get

$$\therefore \quad \frac{\partial f}{\partial x} = 3x^2 + y^2 - 24x + 21$$

$$\text{and} \quad \frac{\partial f}{\partial y} = 2xy - 4y$$

$\Rightarrow 3x^2 + y^2 - 24x + 21 = 0$ ...(1)

and $2xy - 4y = 0$ ...(2)

Step II : From Equation (2), $2xy - 4y = 0$

$\Rightarrow$ $2y(x - 2) = 0 \Rightarrow$ $y = 0$; $x = 2$

Put y = 0 in Equation (1), $\Rightarrow 3x^2 - 24x + 21 = 0$ gives,

$\Rightarrow$ $x^2 - 8x + 7 = 0$ $\quad \Rightarrow$ $(x - 7)(x - 1) = 0$

$\Rightarrow$ $x = 1, 7$

The stationary values are (1, 0), (7, 0)

Now, Put x = 2 in Equation (1) gives.

$12 + y^2 - 48 + 21 = 0 \Rightarrow y^2 - 15 = 0$

$\Rightarrow$ $y = \pm\sqrt{15}$

The stationary values are $(2, \sqrt{15}), (2, -\sqrt{15})$

Thus, **the stationary values are**

$(1, 0), (7, 0), (2, \sqrt{15}), (2, -\sqrt{15})$

Step III :

$r = \frac{\partial^2 f}{\partial x^2} = 6x - 24$;

$s = \frac{\partial^2 f}{\partial x\, \partial y} = 2y$;

$t = \frac{\partial^2 f}{\partial y^2} = 2x - 4$

Step IV :

(i) For (1, 0) $\equiv (x, y) \Rightarrow x = 1$; $y = 0$

$r = 6 - 24 = -18$;

$s = 0$; $\quad t = 2 - 4 = -2$

$\therefore$ $rt - s^2 = 36 - 0 = 36 > 0$ and $r = -18 < 0$

This show that, function is **maximum** at (1, 0)

$\therefore$ $f_{max} = [f(x, y)]_{(1, 0)}$

$= (x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10)_{(1, 0)}$

$f_{max} = 20$

Step V :

(ii) For (7, 0) $\equiv (x, y) \Rightarrow x = 7, y = 0$

$r = 42 - 24 = 18, s = 0$;

$t = 14 - 4 = 10$

$\therefore$ $rt - s^2 = 180 - 0 = 180 > 0$ and $r = 18 > 0$

This show that, function is **minimum** at (7, 0)

$f_{min} = (f(x, y))_{(7, 0)}$

$= (x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10)_{(7, 0)}$

$= 343 + 0 - 588 - 0 + 147 + 10 = -88$

$\therefore$ $f_{min} = -88$

Step VI :

(iii) For $(2, \sqrt{15}) \equiv (x, y) \Rightarrow x = 2, y = \sqrt{15}$

$r = \frac{\partial^2 f}{\partial x^2} = 12 - 24 = -12$,

$s = 2\sqrt{15}$;

and $t = \frac{\partial^2 f}{\partial y^2} = 4 - 4 = 0$

$\therefore$ $rt - s^2 = 0 - (2\sqrt{15})^2 = -4 \times 15 = -60 <$

This shows that, at $(2\sqrt{15})$, function is neith **maximum nor minimum.**

The stationary point $(2\sqrt{15})$ is a **saddle point.**

Step VII :

(iv) For $(2 - \sqrt{15}) \equiv (x, y) \Rightarrow x = 2, y = -\sqrt{15}$

$r = \frac{\partial^2 f}{\partial x^2} = 12 - 24 = -12$,

$s = \frac{\partial^2 f}{\partial x\, \partial y} = 2\sqrt{15}$;

and $t = \frac{\partial^2 f}{\partial y^2} = 4 - 4 = 0$

$\therefore$ $rt - s^2 = 0 - 60 = -60 < 0$

This shows that at $(2, -\sqrt{15})$, function is neithe **maximum nor minimum.**

The stationary point $(2, -\sqrt{15})$ is a **saddle point.** ✓ ...An

Example 5.9.3

Discuss maxima and minima of function $3x^2 - y^2 + x^3$.

Solution :

Step I : Given, $f(x, y) = 3x^2 - y^2 + x^3$ ...(

Differentiate given function f - by using standar rules of derivatives, we get

Step II : $\frac{\partial f}{\partial x} = 6x - 0 + 3x^2$

and $\frac{\partial f}{\partial y} = -2y$

$$\therefore \quad \frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

Step III : i.e. $6x + 3x^2 = 0$ and $-2y = 0$

$x(6+3x) = 0$ and $\mathbf{y = 0}$

$\mathbf{x = 0}$ and $6 + 3x = 0$

$3x = -6$ $\therefore \mathbf{x = -2}$

$\therefore x = 0;\ y = 0 \Rightarrow (0, 0)$

$x = -2;\ y = 0 \Rightarrow (-2, 0)$; are the stationary points.

Step IV : Now, $r = \dfrac{\partial^2 f}{\partial x^2} = 6 + 6x;$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 0; \quad t = \frac{\partial^2 f}{\partial y^2} = -2$$

and $(rt - s^2) = (6+6x)(-2) - 0 = (-12 - 12x)$

Step V :

For (0, 0) $(rt - s^2)_{(0,0)} = -12 - 12(0) = -12 < 0$

$\therefore$ Function is neither maximum nor minimum. Therefore (0, 0) is a saddle point.

For (– 2, 0) $(rt - s^2) = -12 - 12(-2)$

$= -12 + 24 = 12 > 0$

And $r = 6 + 6(-2) = 6 - 12 = -6 < 0$

$\therefore$ Function is maximum.

Step VI : From Equation (1)

$f(-2, 0) = f_{max}$

$= 3(-2)^2 - (-2)^3 = 12 - 8$

$=$ **4** be the extreme value. ✓ **...Ans.**

Example 5.9.4

Find extreme values of the function ;

$\mathbf{x^3 + 3xy^2 - 3x^2 - 3y^2 + 7}$

Solution :

Step I : Given $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$ ...(1)

Step II : For maximum and minimum function,

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

Differentiate given function f - by using standard rules of derivatives, we get

$\therefore \dfrac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x + 0$

and $\dfrac{\partial f}{\partial y} = 6xy - 6y$

Step III : $\therefore 3x^2 + 3y^2 - 6x = 0$

$\Rightarrow x^2 + y^2 - 2x = 0$ ...(2)

and $6xy - 6y = 0$ ...(3)

$6y(x-1) = 0$

$\Rightarrow y = 0, \quad x = 1$

At, $x = 1$, From Equation (2)

$1 + y^2 - 2 = 0$; $y^2 - 1 = 0$

$\therefore$ we get, $y = \pm 1 \Rightarrow (1, 1)$; $(1, -1)$

and at $y = 0$; From Equation (2)

$x^2 - 2x = 0$; $x(x-2) = 0$

$x = 0$; $x = 2 \Rightarrow (0, 0)\ (2, 0)$

$\therefore$ **stationary points are ,** (0, 0); (2, 0); (1, 1); (1, – 1)

Step IV : $r = \dfrac{\partial^2 f}{\partial x^2} = 6x - 6$;

$s = \dfrac{\partial^2 f}{\partial x\, \partial y} = 6y$;

$t = \dfrac{\partial^2 f}{\partial y^2} = 6x - 6$

For (x, y) ≡ (0, 0)

$r = -6$; $s = 0$;

$t = -6$

$\mathbf{(rt - s^2)} = (-6)(-6) - (0) = \mathbf{36 > 0}$

and $r = -6 < 0$

$\Rightarrow$ This show that function is maximum.

$\therefore$ From Equation (1), $f_{max} = f(0, 0) = 7$, be the **extreme value**

Step V : For (x, y) ≡ (2, 0)

$r = 12 - 6 = 6$; $s = 0$;

$t = 6$

$(rt - s^2) = 6 \times 6 - 0 = \mathbf{36 > 0}$

$r = 6 > 0$

This show that Function is **minimum** ; From Equation (1)

$f(2, 0) = f_{min} = 8 + 0 - 12 + 7 = 8 - 5$

$f_{min} = 3$ be the extreme value

Step VI : For (x, y) ≡ (1, 1)

$r = 6(1) - 6 = 0$; $s = 6$; $t = 0$

$(rt - s^2) = -36 < 0$

At (1, 1) the function is **neither maximum nor minimum.** The point (1. 1) is called saddle point.

Step VII : For (x, y) ≡ (1, – 1)

$r = 0$;

$s = -6$; $t = 0$

$(rt - s^2) = (-6)^2 = -36 < 0$

This shows that the function **neither maximum nor minimum.** The point (1, – 1) is a **saddle point.** ✓ **...Ans.**

Example 5.9.5

Discuss the maxima and minima for $\mathbf{x^3 + y^3 - 3axy}$. a > 0.

Solution :

Step I : Let, $f = x^3 + y^3 - 3axy$ ...(1)

For maxima, minima,

Step II : $\dfrac{\partial f}{\partial x} = 0 \qquad \dfrac{\partial f}{\partial y} = 0$

Differentiate given function f - by using standard rules of derivatives, we get

$\dfrac{\partial f}{\partial x} = 3x^2 - 3ay$ | And $\dfrac{\partial f}{\partial y} = 3y^2 - 3ax$

$3x^2 - 3ay = 0$ | $3y^2 - 3ax = 0$

$x^2 - ay = 0$ | $y^2 - ax = 0$

$x^2 = ay$ | $y^2 = ax$

$\Rightarrow y = \sqrt{ax}$

$x^2 = a\sqrt{ax}$

$x^4 = a^2(ax) \quad \Rightarrow \quad x^4 = a^3 x$

Hence, $x^4 - a^3 x = 0$; $\Rightarrow x(x^3 - a^3) = 0$

$\Rightarrow x = 0$; and $x = a$

$x = a \Rightarrow y = \sqrt{a \cdot a} \Rightarrow y = a$

$x = a$; $y = a$

$x = 0 \Rightarrow y = 0$

Hence, $(x, y) \equiv (0, 0), (a, a)$ are the stationary points.

Step III : Now, $r = \dfrac{\partial^2 f}{\partial x^2} = 6x$;

$s = \dfrac{\partial^2 f}{\partial x\, \partial y} = -3a$; $t = \dfrac{\partial^2 f}{\partial y^2} = 6y$

Step IV : For $(x, y) \equiv (0, 0)$

$r = 0$;

$s = -3a$;

$t = 0$

$\mathbf{(rt - s^2)} = 0 - (-3a)^2 = \mathbf{-9a^2 < 0.}$

The function is **neither maximum nor minimum.** The point (0, 0) is **saddle point.**

Step V : For (x, y) ≡ (a, a)

$r = 6a$; $s = -3a$;

$t = 6a$

$(rt - s^2) = 36a^2 - 9a^2 = 27a^2 > 0$;

And $r = 6a > 0$

This shows that function is **minimum.**

$f(a, a) = f_{min} = a^3 + a^3 - 3a^3 = 2a^3 - 3a^3$

$= -a^3$ be the **extreme value.** ✓ **...Ans.**

Example 5.9.6

Determine stationary points where the function $\mathbf{x^3 y^2 (1 - x - y)}$ has a maximum value.

Solution :

Step I : Let, $f(x, y) = x^3 y^2 (1 - x - y)$

$f(x, y) = x^3 y^2 - x^4 y^2 - x^3 y^3$ ...(1)

Differentiate given function f - by using standard rules of derivatives, we get

Step II : $\dfrac{\partial f}{\partial x} = 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3$

And $\dfrac{\partial f}{\partial y} = 2x^3 y - 2x^4 y - 3x^3 y^2$

Step III : For maxima, minima, $\dfrac{\partial f}{\partial x} = \mathbf{0}$ and $\dfrac{\partial f}{\partial y} = \mathbf{0}$.

$\therefore 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3 = 0$

$\Rightarrow x^2 y^2 (3 - 4x - 3y) = 0$

$\Rightarrow x = 0, y = 0$ and $3 - 4x - 3y = 0$ ...(2)

And

$2x^3 y - 2x^4 y - 3x^3 y^2 = 0$

$\Rightarrow x^3 y (2 - 2x - 3y) = 0$

$\Rightarrow x = 0, y = 0$ and $2 - 2x - 3y = 0$ ...(3)

From Equation (2) and from Equation (3), we get

$x = 0$; $y = 0$

$x = 0$; $2x + 3y = 2 \Rightarrow y = 2/3$

$y = 0$; $2x + 3y = 2 \Rightarrow x = 1$

From Equation (3) and from Equation (2)

$x = 0$, ; $4x + 3y = 3 \Rightarrow y = 1$

$y = 0$; $4x + 3y = 3 \Rightarrow x = 3/4$

And $2x + 3y = 2$; $4x + 3y = 3 \Rightarrow x = 1/2, y = 1/3$

Hence **stationary values are :**

(0, 0), (0, 2/3), (1, 0), (0, 1), (3/4, 0), (1/2, 1/3)

Step IV : Now, $r = \dfrac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2 y^2 - 6xy^3$;

$s = \dfrac{\partial^2 f}{\partial x\, \partial y} = 6x^2 y - 8x^3 y - 9x^2 y^2$

$t = \dfrac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3 y$

Step V : At $(x, y) = (0, 0), (0, 2/3), (0, 1)$

$r = 0, s = 0, t = 0 \Rightarrow rt - s^2 = 0$

Need further investigation.

Step VI : At $(x, y) \equiv (1, 0)$, and $(3/4, 0)$

$r = 0, s = 0, \ t = 0$, and $t = \frac{63}{128}$

But $rt - s^2 = 0$ So, at this point also **we need more information.**

Step VII : At $(x, y) \equiv (1/2, 1/3)$

$r = 1/9, \ s = -1/12, \ t = -1/8$

$$rt - s^2 = \left(\frac{1}{9}\right)\left(\frac{-1}{8}\right) - \left(\frac{-1}{12}\right)^2$$

$$= \frac{1}{144} > 0 \ ; \text{ and } \ r = \frac{1}{19} > 0.$$

This shows that, function is **minimum**.

$$\therefore F_{min} = f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{432} \checkmark \quad \text{...Ans.}$$

Example 5.9.7

Discuss the maxima and minima of

$$f(x, y) = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$$

Solution :

Step I : Let, $f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$ ...(1)

Step II : For maxima, minima, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

Differentiate given function f - by using standard rules of derivatives, we get

$$y - \frac{a^3}{x^2} = 0 \quad \text{...(2)}$$

$$x - \frac{a^3}{y^2} = 0 \quad \text{...(3)}$$

From Equations (2) and (3)

$$x^2y = a^3$$

$$xy^2 = a^3$$

$$\Rightarrow x = \frac{a^3}{y^2} \quad \text{Put in Equation (2)}$$

$$\left(\frac{a^3}{y^2}\right)^2 y = a^3 \Rightarrow a^3 = y^3 \Rightarrow y = a$$

(Remaining two values are imaginary)

From Equation (3) $x = a$, also

Therefore, **(a, a) be the stationary point.**

Step III : Now,

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{a^3}{x^4} \cdot 2x = \frac{2a^3}{x^3} ;$$

$$s = \frac{\partial^2 f}{\partial x \, \partial y} = 1 ;$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{a^3}{y^4}(2y) = \frac{2a^3}{y^3}$$

Step IV : At $(x, y) \equiv (a, a)$

$r = 2, \ s = 1, \ t = 2$

$\Rightarrow$ $\mathbf{rt - s^2 = 4 - 1 = 3}$ and $\mathbf{r = 2 > 0}$

This shows that, function is **minimum.**

$\therefore F_{min}$ From Equation (1)

$$F_{min} = f(a, a) = (a)(a) + a^3\left(\frac{1}{a} + \frac{1}{a}\right)$$

$$= a^2 + 2a^2 = 3a^2 \checkmark \quad \text{...Ans.}$$

Type III : Examples on Maxima and Minima of f(x. y) is to be formed from given condition

Example 5.9.8

Divide 120 into 3 parts so that the sum of their products taken two at a time shall be maximum.

Solution :

Step I : Let x, y, z be three parts of 120.

$\therefore x + y + z = 120$ ...(given) ...(1)

Also given, $xy + yz + zx$ is maximum ...(2)

From Equation (1),

$z = 120 - x - y$ and from Equation (2)

$$f(x, y) = xy + y(120 - x - y) + x(120 - x - y)$$

$$= xy + 120y - xy - y^2 + 120x - x^2 - xy$$

$$\therefore f(x, y) = -xy - x^2 - y^2 + 120y + 120x$$

Step II : For maxima and minima, $\frac{\partial f}{\partial x} = 0$; $\frac{\partial f}{\partial y} = 0$

Differentiate given function f - by using standard rules of derivatives, we get

$$\frac{\partial f}{\partial x} = -y - 2x + 120$$

and $$\frac{\partial f}{\partial y} = -x - 2y + 120$$

Step III : Now, $-y + 120 - 2x = 0$

and $-x + 120 - 2y = 0$

$$2x + y = 120 \quad \text{...(3)}$$

and $$x + 2y = 120 \quad \text{...(4)}$$

$\therefore 2x + y = x + 2y$

$\Rightarrow x = y$ Put in Equation (3)

$2y + y = 120 ; \quad 3y = 120$

$\Rightarrow y = 40 \Rightarrow x = 40$

We get $(x, y) \equiv (40, 40)$

Step IV : From Equation (1)

$\therefore z = 120 - x - y ;$

$\therefore z = 120 - 40 - 40$

$z = 40$

$\therefore (x, y, z) \equiv (40, 40, 40)$ be the **stationary point.**

Step V : $r = \frac{\partial^2 f}{\partial x^2} = -2$;

$s = \frac{\partial^2 f}{\partial x \, \partial y} = -1$; $t = \frac{\partial^2 f}{\partial y^2} = -2$

$\mathbf{(rt - s^2)} = (-2)(-2) - 1 = 4 - 1 = \mathbf{3 > 0}$

and $\mathbf{r = -2 < 0}$

$\therefore$ This shows that function is **maximum.** ✓ ...Ans.

Example 5.9.9

Find the stationary value of [sin x sin y sin (x + y)]

Solution :

Step I : Let $f(x, y) = \sin x \sin y \sin(x + y)$ ...(1)

Step II : For stationary values, $\frac{\partial f}{\partial x} = 0$

Differentiate given function f - by using standard rules of derivatives, we get

$\Rightarrow \sin y [\sin x \cdot \cos(x + y) + \cos x \sin(x + y)] = 0$

$\Rightarrow \sin y \sin(2x + y) = 0$

($\because$ use $\sin(A + B) = \sin A \cos B + \cos A \sin B$)

$\frac{1}{2}[\cos(-2x) - \cos(2x + 2y)] = 0$

($\because$ use $\sin A + \sin B = \cos(A - B) - \cos(A + B)$

$\Rightarrow \frac{1}{2}[\cos 2x - \cos(2x + 2y)] = 0$

$\Rightarrow \cos 2x - \cos(2x + 2y) = 0$ ...(2)

and $\frac{\partial f}{\partial y} = 0$

Differentiate given function f - by using standard rules of derivatives, we get

i.e. $\sin x [\cos y \sin(x + y) + \sin y \cos(x + y)] = 0$

$\Rightarrow \sin x \sin(x + 2y) = 0$

$\Rightarrow \frac{1}{2}[\cos 2y - \cos(2x + 2y)] = 0$

$\Rightarrow \cos 2y - \cos(2x + 2y) = 0$ ...(3

From Equations (2) and (3)

$\cos 2x - \cos(2x + 2y) = \cos 2y - \cos(2x + 2y)$

$\Rightarrow \cos 2x = \cos 2y$

$\Rightarrow \mathbf{x = y}$

Put $x = y$ in Equation (3)

$\cos 2y = \cos(4y) = 2\cos^2 2y - 1$

($\because \cos 2\theta = 2\cos^2\theta - 1$

$\Rightarrow 2\cos^2 2y - \cos 2y - 1 = 0$

It is like $ax^2 + bx + c = 0$

Therefore, $\cos 2y = \frac{1 \pm \sqrt{1 - 4 \times 2 \times (-1)}}{2 \times 2}$

$= \frac{1 \pm \sqrt{1 + 8}}{4}$

$\cos 2y = \frac{1 \pm 3}{4} = 1, -\frac{1}{2}$

$\Rightarrow \cos 2y = 1$ and $\cos 2y = -\frac{1}{2}$

$\Rightarrow 2y = 0$ and $2y = \frac{2\pi}{3}$

$\mathbf{y = 0}$ $\quad$ $\mathbf{y = \frac{\pi}{3}}$

Hence $(0, 0)$, $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ be the **stationary values.**

Step III : Now, $r = \frac{\partial^2 f}{\partial x^2}$;

$r = 2 \sin y \cos(2x + y)$

$s = \frac{\partial^2 f}{\partial x \, \partial y}$

$= [\cos x \sin(x + 2y) + \cos(x + 2y) \cdot \sin x$

$= \sin(2x + 2y)$

$t = \frac{\partial^2 f}{\partial y^2} = 2 \sin x \cos(x + 2y)$

Step IV : For $(x, y) \equiv (0, 0)$; $r = 0$, $s = 0, \ t = 0$

$rt - s^2 = 0$; Need more information

Step V : For $(x, y) \equiv \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$;

$$r = 2\sin\left(\frac{\pi}{3}\right)\cdot\cos(\pi)$$

$$= 2\left(\frac{\sqrt{3}}{2}\right)(-1) = -\sqrt{3}$$

$$s = \sin\left(\frac{4\pi}{3}\right) = \sin\left(\pi+\frac{\pi}{3}\right)$$

$$= -\frac{\sin\pi}{3} = \frac{-\sqrt{3}}{2}$$

$$t = 2\sin\left(\frac{\pi}{3}\right)\cos(\pi) = -\sqrt{3}$$

$$rt - s^2 = 3 - \left(\frac{3}{4}\right) = \frac{9}{4} > 0$$

and $\quad r = -\sqrt{3} < 0$

This shows that, the function is **maximum** at $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$

$$\therefore\ F_{max} = f\left(\frac{\pi}{3},\frac{\pi}{3}\right)$$

$$= \frac{\sin\pi}{3}\times -\frac{\sin\pi}{3}\times\sin\left(\frac{2\pi}{3}\right)$$

$$= \frac{3\sqrt{3}}{8}$$ ✓ **...Ans.**

Exercise 3

Ex. 1 : Discuss maxima and minima of the following functions

(i) $x^2+y^2+6x+12$ **Ans. :** $f_{min} = 3$

(ii) $xy(a-x-y)$ **Ans. :** $f_{min} = \frac{a^3}{27}$

(iii) $x^3y^2(12-3x-4y)$ **Ans. :** $f_{max} = 16$

Ex. 2 : Find the points on the surface $z^2 = xy+1$ nearest to the origin. Also find that distance. **Ans. :** $(0,0,\pm1)$

Ex. 3 : Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

Ex. 4 : Find a point in the plane, $x+2y+3z = 13$, nearest to the point (1,1,1) **Ans. :** $\left(\frac{3}{2},2,\frac{5}{2}\right)$

Ex. 5 : Find the largest product of the number x y and z when $x^2+y^2+z^2 = 9$.

5.10 Lagrange's Method of Undetermined Multipliers

Many times in practical and theoretical problems, we have to find stationary values of the functions of several variables, not all independent variables but, these variables are connected by some relations. Lagranges method of undetermined multipliers is useful for such functions.

Let f(x, y, z) be a function of three variables and these variables are also connected by relation $\phi(x,y,z) = 0$.

By Lagranges method, we get only stationary points, not the nature of the function.

There are two types :

(1) Functions having only one constraint.

(2) Functions having two or more constraints.

(3) This method can be used for two variables also.

Type I : Functions Having Only One Constraint

Working Rule

Step I : Let $u = f(x,y,z)$ ...(5.10.1)

be given function under the condition $\phi(x,y,z) = 0$ ...(5.10.2)

Step II : Construct a new function $F = u + \lambda\phi$; λ is undetermined multiplier.

For stationary values ; $\frac{\partial F}{\partial x} = 0$; $\frac{\partial F}{\partial y} = 0$; $\frac{\partial F}{\partial z} = 0$

Step III : Solve these equations simultaneously and find values of x, y, z interms of k.

$$\left.\begin{aligned} x &= f_1(k) \\ y &= f_1(k) \\ z &= f_1(k) \end{aligned}\right\} \quad ...(5.10.3)$$

Step IV : Substitute these values in Equation (5.10.2) in given condition and find value of k.

Step V : Again put value of k in Equation (5.10.3) we get, x, y, z.

Example 5.10.1

Find the minimum value of : x^2+y^2, subject to the condition : $ax+by = c$

Solution :

Step I : Let, $u = x^2+y^2$ ...(1)

and $\phi \equiv ax+by-c = 0$ ...(2)

Let, $F = u + \lambda\phi$

$$\Rightarrow F = (x^2+y^2)+\lambda(ax+by-c)$$

Find, $\frac{\partial f}{\partial x} = 0$; $\frac{\partial f}{\partial y} = 0$; $\frac{\partial f}{\partial z} = 0$

Differentiate F w.r.t. x, y, z by using standard rules of derivatives.

$$\therefore 2x+\lambda a = 0 \Rightarrow \frac{x}{a} = -\frac{\lambda}{2}$$

$$2y+\lambda b = 0 \Rightarrow \frac{y}{b} = -\frac{\lambda}{2}$$

Step II :

Let, $\frac{x}{a} = \frac{y}{b} = k$ (say)

$$\Rightarrow x = ak;\qquad y = bk$$

Put these values of x and y in Equation (2)

$$ax+by = c$$

$$a^2k+b^2k = c$$

$$\Rightarrow \mathbf{k} = \frac{\mathbf{c}}{\mathbf{a^2+b^2}}$$

Hence, $\quad x = a\cdot\frac{c}{a^2+b^2}$

and $\quad y = b\cdot\frac{c}{(a^2+b^2)}$

Therefore, $\quad f_{min} = (x^2+y^2) = \frac{c^2}{(a^2+b^2)^2}(a^2+b^2)$

$$= \frac{c^2}{(a^2+b^2)}$$ ✓ **...Ans.**

Example 5.10.2

Find the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid : $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

Solution :

Step I : Let, 2x, 2y, 2z are the sides of parallelepiped which are parallel to co-ordinate axes.

V = be the volume of parallelopiped.

$$\therefore \mathbf{V} = \mathbf{8xyz} \quad ...(1)$$

Given conditions : $\phi \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1 = 0$ ...(2)

Step II : Let, $F = v + \lambda\phi$

$$\therefore f = 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

$$\frac{\partial f}{\partial x} = 0; \qquad \frac{\partial f}{\partial y} = 0;$$

$$\frac{\partial f}{\partial z} = 0$$

Differentiate F w.r.t. x, y, z by using standard rules of derivatives.

$$\therefore 8yz+\lambda\left(\frac{2x}{a^2}\right) = 0 \Rightarrow -\lambda = \frac{4yza^2}{x}$$

$$8xz+\lambda\left(\frac{2y}{b^2}\right) = 0 \Rightarrow -\lambda = \frac{4xzb^2}{y}$$

and $\quad 8xy+\lambda\left(\frac{2z}{c^2}\right) = 0 \Rightarrow -\lambda = \frac{4xyc^2}{z}$

This implies, $\frac{4yza^2}{x} = \frac{4xzb^2}{y} = \frac{4xyc^2}{z}$

$$\Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Step III : Put these values in equation (2), we get

$$\frac{x^2}{a^2}+\frac{x^2}{a^2}+\frac{x^2}{a^2} = 1 \Rightarrow 3\frac{x^2}{a^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} = \frac{1}{3}$$

Therefore, $\quad \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}$

This gives, $\quad x = \frac{a}{\sqrt{3}} = y = z$

Step IV : Given parallelopiped is a rectangular, so, x = 0, means volume (v) = 0.

As x increases, volume also increasing continuously.

Hence, the volume must be largest at

$$x = \frac{a}{\sqrt{3}} = y = z;$$

$$\therefore V = 8xyz = \frac{\mathbf{8abc}}{\mathbf{3\sqrt{3}}}$$ ✓ **...Ans.**

Example 5.10.3

Show that rectangular solid of maximum value that can be inscribed in sphere is a cube.

Solution :

Step I : Let 2x, 2y, 2z are length, breadth and height of rectangular solid and r be radius of sphere.

$\therefore$ volume of solid $\quad V = 8xyz$

Equation of sphere is ; $x^2+y^2+z^2 = r^2$

$u = f(x,y,z) = V = 8xyz$ ...(1)

And $\quad \phi(x,y,z) = x^2+y^2+z^2-r^2 = 0$ ...(2)

Step II : Let, $F = u+\lambda\phi$

$$= (8xyz)+\lambda(x^2+y^2+z^2-r^2)$$

Differentiate F w.r.t. x, y, z by using standard rules of derivatives.

$$\frac{\partial F}{\partial x} = 8yz + 2\lambda x = 0$$

$$\frac{\partial F}{\partial y} = 8xz + 2\lambda y = 0$$

$$\frac{\partial F}{\partial z} = 8xy + 2\lambda z = 0$$

$8yz + 2\lambda x = 0$	$8xz + 2\lambda y = 0$
$8yz = -2\lambda x$	$8xz = -2\lambda y$
$\frac{8yz}{2x} = -\lambda$	$\frac{8xz}{2y} = -\lambda$
$\mathbf{\frac{4yz}{x} = -\lambda}$	$\mathbf{\frac{4xz}{y} = -\lambda}$

and $8xy + 2\lambda z = 0$

$$8xy = -2\lambda z$$

$$\frac{8xy}{2z} = -\lambda$$

$$\mathbf{\frac{4xy}{z}} = -\lambda$$

Step III : $\therefore \quad \frac{4yz}{x} = \frac{4xz}{y} = \frac{4xy}{z}$;

$$\frac{yz}{x} = \frac{xz}{y} = \frac{xy}{z}$$

Multiplying $\frac{1}{xyz}$, we get

$$\frac{1}{x^2} = \frac{1}{y^2} = \frac{1}{z^2} = k$$

$$x^2 = \frac{1}{k} ; \quad y^2 = \frac{1}{k} ; \quad z^2 = \frac{1}{k} ; \qquad ...(3)$$

Step IV : Equation (1) becomes,

$$\frac{1}{k} + \frac{1}{k} + \frac{1}{k} = r^2 ; \quad \frac{3}{k} = r^2 \Rightarrow \mathbf{k = \frac{3}{r^2}}$$

Step V : Put the value of k in Equation (3)

$$x^2 = \frac{r^2}{3} ; \quad y^2 = \frac{r^2}{3} ; \quad z^2 = \frac{r^2}{3}$$

$$x = \frac{r}{\sqrt{3}} ; y = \frac{r}{\sqrt{3}} ; z = \frac{r}{\sqrt{3}}$$

Since, $x = y = z$.

Hence the rectangular solid is a cube.✓ ...Hence proved.

Example 5.10.4

Find points on surface $z^2 = xy + 1$ nearest to origin, by using Langrange's method.

Solution :

Step I : Let, O = (0, 0, 0) be the origin

and P = (x, y, z) be any point on the surface

$$d(OP) = \sqrt{x^2 + y^2 + z^2}$$

$$d(OP)^2 = x^2 + y^2 + z^2$$

$$\therefore u = x^2 + y^2 + z^2 \qquad ...(1)$$

$$\phi = z^2 - xy - 1 = 0 \qquad ...(2)$$

Step II : Let, $F = u + \lambda\phi$

$$F = (x^2 + y^2 + z^2) + \lambda(z^2 - xy - 1) = 0$$

To find : $\frac{\partial F}{\partial x} = 0 ; \quad \frac{\partial F}{\partial y} = 0 ; \quad \frac{\partial F}{\partial z} = 0$

Differentiate F w.r.t. x, y, z by using standard rules of derivatives.

$$\therefore \frac{\partial F}{\partial x} = 2x - \lambda y = 0 ;$$

$$\frac{\partial F}{\partial y} = 2y - \lambda x = 0 ;$$

$$\frac{\partial F}{\partial z} = 2z + 2\lambda z = 0$$

Step III : $\Rightarrow 2x - \lambda y = 0$

$$\frac{2x}{y} = \lambda ; \quad \frac{2y}{x} = \lambda ; \quad \frac{2z}{2z} = -\lambda$$

$$\frac{2x}{y} = \lambda ; \quad \frac{2y}{x} = \lambda ; \quad \mathbf{-1 = \lambda}$$

Step IV : $\therefore \quad \frac{2x}{y} = -1$ and $\frac{2y}{x} = -1$

$$\Rightarrow y = -2x \text{ and } x = -2y$$

$$\therefore \Rightarrow y = -2(-2y) = 4y \Rightarrow 3y = 0$$

$$\Rightarrow y = 0 \text{ and we get } x = 0 \text{ also}$$

Equation (2) becomes

$$z^2 - 1 = 0 ; \quad z^2 = 1 ; \quad z = +1$$

Hence, (0, 0, ± 1) is the nearest point from origin. ✓ ...Ans.

Example 5.10.5

Find the maximum value of $x^m y^n z^p$ when $x + y + z = a$

Solution :

Step I : Let $u = x^m y^n z^p \qquad ...(1)$

and $\phi \equiv x + y + z - a = 0 \qquad ...(2)$

Step II : Let $F = u + \lambda\phi$

$$\Rightarrow \quad F = x^m y^n z^p + \lambda(x + y + z - a)$$

Find $\mathbf{\frac{\partial F}{\partial x} = 0 ; \frac{\partial F}{\partial y} = 0}$ and $\mathbf{\frac{\partial F}{\partial z} = 0}$

Differentiate F w.r.t. x, y, z by using standard rules of derivatives.

$$\Rightarrow mx^{m-1} y^n z^p + \lambda = 0$$

$$\Rightarrow mx^{m-1} y^n z^p = -\lambda \qquad ...(3)$$

$$x^m ny^{n-1} z^p + \lambda = 0$$

$$\Rightarrow nx^m y^{n-1} z^p = -\lambda \qquad ...(4)$$

and $px^m y^n z^{p-1} + \lambda = 0$

$$\Rightarrow px^m y^n z^{p-1} = -\lambda \qquad ...(5)$$

Step III : From Equations (3), (4) and (5), we get,

$$mx^{m-1} y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}$$

$$\Rightarrow \frac{mx^{m-1} y^n z^p}{x^m y^n z^p} = \frac{nx^m y^{n-1} z^p}{x^m y^n z^p} = \frac{px^m y^n z^{p-1}}{x^m y^n z^p}$$

$$\Rightarrow \frac{m}{x} = \frac{n}{y} = \frac{p}{z} = k \text{ (say)}$$

$$\Rightarrow x = \frac{m}{k} ; \quad y = \frac{n}{k} ; \quad z = \frac{p}{k} \qquad ...(6)$$

Step IV : Put these values of x, y, z in Equation (2),

$$\frac{m}{k} + \frac{n}{k} + \frac{p}{k} = a \quad \Rightarrow \quad \frac{m + n + p}{k} = a$$

$$\Rightarrow \mathbf{k = \frac{m + n + p}{a}}$$

Step V : $\therefore$ From Equation (6),

$$x = \frac{am}{m + n + p} ; \quad y = \frac{an}{m + n + p} ;$$

$$z = \frac{ap}{m + n + p}$$

Maximum value of $x^m y^n z^n$

$$= \left(\frac{am}{m + n + p}\right)^m \left(\frac{an}{m + n + p}\right)^n \left(\frac{ap}{m + n + p}\right)^p$$

$$= \mathbf{\frac{a^{m+n+p} \cdot m^m n^n p^p}{(m + n + p)^{m+n+p}}} ✓ \qquad \textbf{...Ans.}$$

Example 5.10.6

Using Lagrange's method of multipliers, show that the stationary value of $a^3x^2 + b^3y^2 + c^3z^2$ where, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ occur at $x = \frac{a + b + c}{a}$; $y = \frac{a + b + c}{b}$; $z = \frac{a + b + c}{c}$.

Solution :

Step I : Let, $u = a^3x^2 + b^3y^2 + c^3z^2 \qquad ...(1)$

and $\phi \equiv \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0 \qquad ...(2)$

Step II : Let $F = u + \lambda\phi$

$$F = (a^3x^2 + b^3y^2 + c^3z^2) + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right)$$

Find, $\mathbf{\frac{\partial F}{\partial x} = 0 ; \frac{\partial F}{\partial y} = 0}$ and $\mathbf{\frac{\partial F}{\partial z} = 0}$

Differentiate F w.r.t. x, y, z by using standard rules o[f] derivatives.

$$\therefore 2a^3x + \lambda\left(-\frac{1}{x^2}\right) = 0$$

$$\Rightarrow \quad -\lambda = 2a^3x^3 \qquad ...(3)$$

$$2b^3y + \lambda\left(-\frac{1}{y^2}\right) = 0 \quad \Rightarrow -\lambda = 2b^3y^3 \qquad ...(4)$$

and $2c^3z + \lambda\left(-\frac{1}{z^2}\right) = 0 \quad \Rightarrow -\lambda = 2c^3z^3 \qquad ...(5)$

Step III : From Equation (3), (4) and (5), we get,

$$2a^3x^3 = 2b^3y^3 = 2c^3z^3 = k^3 \text{ (say)}$$

$$\Rightarrow \quad x^3 = \frac{k^3}{a^3} ; \quad y^3 = \frac{k^3}{b^3} ;$$

and $z^3 = \frac{k^3}{c^3} \qquad ...($

$$x = \frac{k}{a} ; \quad y = \frac{k}{b} ; \quad z = \frac{k}{c}$$

Step IV : Put these values of x, y, z in Equation (2),

$$\frac{a}{k} + \frac{b}{k} + \frac{c}{k} = 1 \quad \Rightarrow \quad \mathbf{a + b + c = k}$$

Therefore, from Equation (6),

$$\mathbf{x = \frac{a + b + c}{a} ; \quad y = \frac{a + b + c}{b} ; \quad z = \frac{a + b + c}{c}} ✓$$

...Hence prov[ed]

Example 5.10.7

Use Lagrange's method to find the maximum a[nd] minimum distance of the point (3, 4, 12) from the sph[ere] $x^2 + y^2 + z^2 = 1$

Solution :

Step I : Let P (x, y, z) be any point on the surface of given curve, $x^2 + y^2 + z^2 = 1$

The distance from the point (3, 4, 12) is,

$$d = \sqrt{(x - 3)^2 + (y - 4)^2 + (z - 12)^2}$$

$$\therefore u = d^2 = (x - 3)^2 + (y - 4)^2 + (z - 12)^2 \; ..$$

And $\phi \equiv x^2 + y^2 + z^2 - 1 = 0$

Step II : Let $F = u + \lambda\phi$

$$= [(x - 3)^2 + (y - 4)^2 + (z - 12)^2] + \lambda(x^2 + y^2 + z^2 -$$

Find, $\frac{\partial F}{\partial x} = 0$; $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$

Differentiate F w.r.t. x, y, z by using standard rules of derivatives.

$$\Rightarrow \quad 2(x-3) + 2\lambda x = 0$$

$$\Rightarrow \quad \left(\frac{x-3}{x}\right) = -\lambda \quad \text{...(3)}$$

$$2(y-4) + 2\lambda y = 0$$

$$\Rightarrow \quad \left(\frac{y-4}{y}\right) = -\lambda \quad \text{...(4)}$$

$$\text{And} \quad 2(z-12) + 2\lambda z = 0$$

$$\Rightarrow \quad \left(\frac{z-12}{z}\right) = -\lambda \quad \text{...(5)}$$

Step III : From Equation (3), (4) and (5) we get,

$$\frac{x-3}{x} = \frac{y-4}{y} = \frac{z-12}{z}$$

$$\Rightarrow 1 - \frac{3}{x} = 1 - \frac{4}{y} = 1 - \frac{12}{z} = k \text{ (say)}$$

$$\Rightarrow 1-k = \frac{3}{x} \quad 1-k = \frac{4}{y} \text{ and } 1-k = \frac{12}{z}$$

$$\Rightarrow \quad x = \frac{3}{1-k}; \; y = \frac{4}{1-k}; \; z = \frac{12}{1-k} \quad \text{....(6)}$$

Step IV : Put these values of x, y, z in Equation (2),

$$\left(\frac{3}{1-k}\right)^2 + \left(\frac{4}{1-k}\right)^2 + \left(\frac{12}{1-k}\right)^2 = 1$$

$$\Rightarrow \quad 9 + 16 + 144 = (1-k)^2$$

$$\Rightarrow \quad 169 = (1-k)^2 \;\Rightarrow\; 1-k = \pm 13$$

$$\Rightarrow \quad k = 1-13 \text{ and } k = 1+13$$

$$\Rightarrow \quad k = -12 \text{ and } k = 14$$

From Equation (6) we get,

For $k = -12$	For $k = 14$
$x = \frac{3}{13}; y = \frac{4}{13}; z = \frac{12}{13}$	$x = \frac{3}{-13}; y = \frac{4}{-13}; z = \frac{12}{-13}$

Step V : Minimum distance

$$= \sqrt{\left(\frac{3}{13}-3\right)^2 + \left(\frac{4}{13}-4\right)^2 + \left(\frac{12}{13}-12\right)^2}$$

$$= \sqrt{\left(\frac{-36}{13}\right)^2 + \left(\frac{-48}{13}\right)^2 + \left(\frac{-144}{13}\right)^2}$$

$$= \frac{1}{13}\sqrt{1296 + 2304 + 20736}$$

$$= \frac{156}{13} = 12$$

Maximum distance

$$= \sqrt{\left(\frac{-3}{13}-3\right)^2 + \left(\frac{-4}{13}-4\right)^2 + \left(\frac{-12}{13}-12\right)^2}$$

$$= \sqrt{\left(\frac{42}{13}\right)^2 + \left(\frac{56}{13}\right)^2 + \left(\frac{168}{13}\right)^2}$$

$$= \frac{1}{13}\sqrt{1764 + 3136 + 28224}$$

$$= \frac{182}{13} = 14 \checkmark \quad \text{...Ans.}$$

Example 5.10.8

Find the minimum distance from origin to the plane 3x + 2y + z = 12.

Solution :

Step I : Let, P (x, y, z) - be any point on the surface $3x + 2y + z - 12 = 0$

$\therefore$ The distance from origin to that point

$$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$$= \sqrt{x^2 + y^2 + z^2}$$

$$u = d^2 = x^2 + y^2 + z^2 \quad \text{...(1)}$$

$$\text{and} \quad \phi \equiv 3x + 2y + z - 12 = 0 \quad \text{...(2)}$$

Step II : Let $F = u + \lambda\phi$

$$F = (x^2 + y^2 + z^2) + \lambda(3x + 2y + z - 12) = 0$$

Find, $\frac{\partial F}{\partial x} = 0$; $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$

Differentiate F w.r.t. x, y, z by using standard rules of derivatives.

$$\therefore 2x + 3\lambda = 0 \Rightarrow \quad \frac{2x}{3} = -\lambda \quad \text{...(3)}$$

$$2y + 2\lambda = 0 \Rightarrow \quad y = -\lambda \quad \text{...(4)}$$

$$\text{and} \quad 2z + \lambda = 0 \Rightarrow \quad 2z = -\lambda \quad \text{...(5)}$$

Step III : From Equations (3), (4), (5) we get,

$$\frac{2x}{3} = y = 2z = -\lambda = k$$

$$\Rightarrow x = \frac{3k}{2}, \; y = k; \; z = \frac{k}{2} \quad \text{...(6)}$$

Put these values of x in Equation (2),

$$3\left(\frac{3k}{2}\right) + 2(k) + \left(\frac{k}{2}\right) = 12$$

$$\Rightarrow 7k = 12 \Rightarrow \quad k = \frac{12}{7}$$

Step IV : Put this value of k in Equation (6),

$$x = \frac{3k}{2} = \frac{3}{2}\cdot\frac{12}{7} = \frac{18}{7}$$

$$y = \frac{12}{7}; \quad z = \frac{12}{7 \times 2} = \frac{6}{7}$$

$$\therefore \text{ The minimum distance (d)} = \sqrt{\left(\frac{18}{7}\right)^2 + \left(\frac{12}{7}\right)^2 + \left(\frac{6}{7}\right)^2}$$

$$= \sqrt{\frac{504}{7}} \checkmark \quad \text{...Ans.}$$

Example 5.10.9

If $u = \frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2}$, where x + y + z = 1, then find the stationary values.

Solution :

$$\textbf{Step I :} \text{ Let} \quad u = \frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2} \quad \text{...(1)}$$

$$\text{and} \quad \phi \equiv x + y + z - 1 = 0 \quad \text{...(2)}$$

Step II : Let $F = u + \lambda\phi$;

$$F = \left[\frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2}\right] + \lambda(x + y + z - 1)$$

Find, $\frac{\partial F}{\partial x} = 0$; $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$

Differentiate F w.r.t. x, y, z by using standard rules of derivatives.

$$\left.\begin{aligned} &\therefore \frac{-a^3}{x^4}\cdot 2x + \lambda = 0 \quad \Rightarrow \frac{2a^3}{x^3} = +\lambda \\ &\text{Similarly, } \frac{-b^3}{y^4}(2y) + \lambda = 0 \quad \Rightarrow \frac{2b^3}{y^3} = \lambda \\ &\frac{-c^3}{z^4}(2z) + \lambda = 0 \quad \Rightarrow \frac{2c^3}{z^3} = \lambda \end{aligned}\right\} \quad \text{...(3)}$$

Step III : From Equation (3) we get,

$$\frac{2a^3}{x^3} = \frac{2b^3}{y^3} = \frac{2c^3}{z^3} = \lambda = k \quad \text{(say)}$$

$$x^3 = \frac{2a^3}{k}; \; y^3 = \frac{2b^3}{k}; \; z^3 = \frac{2c^3}{k}$$

$$x = a\left(\frac{2}{k}\right)^{1/3}; \; y = b\left(\frac{2}{k}\right)^{1/3}; \; z = c\left(\frac{2}{k}\right)^{1/3} \quad \text{...(4)}$$

Step IV : Put these values of x, y, z in Equation (2),

$$a\left(\frac{2}{k}\right)^{1/3} + b\left(\frac{2}{k}\right)^{1/3} + c\left(\frac{2}{k}\right)^{1/3} = 1$$

$$(a+b+c)\left(\frac{2}{k}\right)^{\frac{1}{3}} = 1$$

$$\therefore \left(\sqrt[3]{2}\right)(a+b+c) = k^{\frac{1}{3}}$$

$$\mathbf{k = 2(a+b+c)^3}$$

Step V : $\therefore$ Put this value of k in Equation (4)

$$x = \frac{a\,2^{\frac{1}{3}}}{2^{1/3}(a+b+c)};$$

$$y = \frac{b\,2^{\frac{1}{3}}}{2^{1/3}(a+b+c)};$$

$$z = \frac{c\,2^{\frac{1}{3}}}{2^{1/3}(a+b+c)}$$

$$\Rightarrow x = \frac{\mathbf{a}}{\mathbf{(a+b+c)}}; \; y = \frac{\mathbf{b}}{\mathbf{(a+b+c)}}; \; z = \frac{\mathbf{c}}{\mathbf{(a+b+c)}} \checkmark$$

...Ans.

Example 5.10.10

As the dimension of a triangle ABC are varied, show that the maximum value of cos A · cos B · Cos C, is obtained when the triangle is equilateral.

Solution :

Step I : Let, $u = \cos A \cos B \cos C$ ...(1)

Since, Δ ABC

$$m\angle A + m\angle B + m\angle C = 180$$

$$\Rightarrow \phi \equiv A + B + C - 180 = 0 \quad \text{...(2)}$$

Step II : Let $\mathbf{F = u + \lambda\phi}$

$$F = \cos A \cdot \cos B \cdot \cos C + \lambda(A + B + C - 180)$$

Find, $\frac{\partial F}{\partial A} = 0$; $\frac{\partial F}{\partial B} = 0$ and $\frac{\partial F}{\partial C} = 0$

Differentiate F w.r.t. x, y, z by using standard rules of derivatives.

$$\Rightarrow -\sin A \cdot \cos B \cdot \cos C + \lambda = 0$$

$$\Rightarrow \lambda = \sin A \cos B \cos C \quad \text{...(3)}$$

$$-\cos A \sin B \cos C + \lambda = 0$$

$$\Rightarrow \lambda = \cos A \sin B \cos C \quad \text{...(4)}$$

$$\text{and, } -\cos A \cos B \sin C + \lambda = 0$$

$$\Rightarrow \lambda = \cos A \cos B \sin C \quad \text{...(5)}$$

Step III : From Equations (3), (4) and (5), we get,

$$\sin A \cos B \cos C = \cos A \sin B \cos C$$

$$= \cos A \cos B \sin C = \lambda = k \text{ (say)}$$

$$\Rightarrow \frac{\sin A \cos B \cos C}{\cos A \cos B \cos C} = \frac{\cos A \sin B \cos C}{\cos A \cos B \cos C}$$

$$= \frac{\cos A \cos B \sin C}{\cos A \cos B \cos C} = k$$

$$\Rightarrow \tan A = \tan B = \tan C = k$$

$$\Rightarrow A = \tan^{-1}(k)\ ;\ B = \tan^{-1}(k)$$

$$\text{and } C = \tan^{-1}(k) \quad \ldots(6)$$

Step IV : $\therefore$ Put in Equation (2), we get

$$3\tan^{-1}(k) = 180$$

$$\Rightarrow \tan^{-1} k = 60$$

Hence, $A = 60,\ B = 60,\ C = 60$

Thus, A = B = C, shows that the Δ ABC is a equilateral triangle. ✓ ...Hence proved.

Example 5.10.11

Using Lagrange's method divide 24 into three parts such that, the continued product of the first, square of the second and cube of the third may be maximum.

Solution :

Step I : Let x, y, z are the three parts of 24

$$\therefore u \equiv x \cdot y^2 z^3 \quad \ldots(1)$$

$$\text{and, } \phi \equiv x + y + z - 24 \quad \ldots(2)$$

Step II : Let, $F = u + \lambda\phi = (x \cdot y^2 z^3) + (x + y + z - 24)$

Find, $\frac{\partial F}{\partial x} = 0\ ;\ \frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$

Differentiate F w.r.t. x, y, z by using standard rules of derivatives.

$$\Rightarrow y^2 z^3 + \lambda = 0 \Rightarrow y^2 z^3 = -\lambda \quad \ldots(3)$$

$$\therefore 2xy z^3 + \lambda = 0 \Rightarrow 2xyz^3 = -\lambda \quad \ldots(4)$$

and $$3xy^2 z^2 + \lambda = 0 \Rightarrow 3x y^2 z^2 = -\lambda \quad \ldots(5)$$

Step III : From Equations (3), (4) and (5) we get

$$y^2 z^3 = 2xyz^3 = 3xy^2 z^2$$

$$\Rightarrow \frac{y^2 z^3}{xy^2 z^3} = \frac{2xyz^3}{xy^2 z^3} = \frac{3xy^2 z^2}{xy^2 z^3}$$

$$\Rightarrow \frac{1}{x} = \frac{2}{y} = \frac{3}{z} = k \text{ (say)}$$

$$x = \frac{1}{k}\ ;\ y = \frac{2}{k},\ z = \frac{3}{k} \quad \ldots(6)$$

Step IV : Put these values of x, y, z in Equation (2)

$$\frac{1}{k} + \frac{2}{k} + \frac{3}{k} = 24$$

$$\Rightarrow \frac{6}{k} = 24 \Rightarrow k = \frac{6}{24} = \frac{1}{4}$$

Step V : Therefore, from Equation (6)

x = 4 ; y = 8 ; z = 12 are the parts of 24 ✓ **...Ans.**

Exercise 4

Ex. 1 : Discuss maxima and minima of the following functions

(i) $x^2 + y^2 + 6x + 12$ **Ans. :** $f_{min} = 3$

(ii) $xy(a - x - y)$ **Ans. :** $f_{min} = \frac{a^3}{27}$

(iii) $x^3 y^2 (12 - 3x - 4y)$ **Ans. :** $f_{max} = 16$

Ex. 2 : Find the points on the surface $Z^2 = xy + 1$ nearest to the origin. Also find that distance. **Ans. :** $(0, 0, \pm 1)$

Ex. 3 : Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

Ex. 4 : Find the minimum value of $x^2 + y^2 + z^2$ with the constraint $xy + yz + zn = 3a^2$

Ex. 5 : Find the point on the plane $ax + by + ez = p$ at which the function $F = x^2 + y^2 + z^2$ has a minimum value and find this minimum value of f.

Ex. 6 : Use the method of the Lagrange's multipliers to find volume of larges and rectangular parallelepiped that can be incised in the euipsoid. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Ex. 7 : Find the length and breadth of a rectangle of maximum area that can be inscribed in the euipsoid $4x^2 + y^2 = 36$. **Ans. :** Area 12

Ex. 8 : Find the maximum value of $x^m y^h z^p$ when $x + y + z = a$ **Ans. :** $\frac{a^{m+n+p} m^m n^n p^p}{(m+n+n)^{m+n+p}}$

Ex. 9 : Find a point in the plane, $x + 2y + 3z = 13$, nearest to the point (1,1,1) **Ans. :** $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$

Ex. 10 : Find the largest product of the number x y and z when $x^2 + y^2 + z^2 = 9$.

5.11 University Questions and Answers

→ Oct 17

Q. 1 The Maclaurin's series of $\tan^{-1} x$ is———

(i) $x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$ (ii) $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

(iii) $1 - x + x^2 - \ldots$ (iv) $1 + x + x^2 + \ldots$ **(1 Mark)**

Ans. (ii) $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

Q. 2 Find the approximate value of $\tan^{-1}(1.003)$ correct upto four decimal places by using Taylor's theorem. **(4 Marks)**

Soln. :

Step I : By Taylor's series expansion of f(x + h);

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots(1)$$

Step II : Let, $f(x + h) = \tan^{-1}(x + h)$

$$\therefore f(x) = \tan^{-1} x$$

Step III : Differentiating f(x) successively with respect to x, and put x = 1

$$f'(x) = \frac{1}{1 + x^2}\ ;$$

$$f''(x) = \frac{-2x}{(1 + x^2)^2}\ ;$$

$$f'''(x) = -\left[\frac{(1 + x^2)^2 (2) - 2x \times 2(1 + x^2)(2x)}{(1 + x^2)^4}\right] = -\left[\frac{2(1 + x^2) - 8x^2}{(1 + x^2)^3}\right]$$

Step IV : Using the values in Equation (1)

$$\tan^{-1}(x + h) = \tan^{-1} x + h\left(\frac{1}{1 + x^2}\right) + \frac{h^2}{2!}\left(-\frac{2x}{(1 + x^2)^2}\right) + \frac{h^3}{3!}\left[-\frac{2(1 + x^2) - 8x^2}{(1 + x^2)^3}\right] \ldots \quad \ldots(2)$$

Step V : Let, x = 1, h = 0.003

$\therefore$ From Equation (2),

$$\tan^{-1}(1 + 0.003) = \tan^{-1}1 + (0.003)\left(\frac{1}{1 + (1)^2}\right) + \frac{(0.003)^2}{2!}\left(-\frac{2(1)}{(1 + (1)^2)^2}\right) + \frac{(0.003)^3}{3!}\left[-\frac{2(1 + (1)^2) - 8(1)^2}{(1 + (1)^2)^3}\right] \ldots$$

$$\tan^{-1}(1.003) = \tan^{-1}1 + (0.003)\left(\frac{1}{1 + 1}\right) - \frac{(0.000009)}{2!}\left(\frac{2}{(1 + 1)^2}\right) - \frac{(0.000000027)}{3!}\left[\frac{2(1 + 1) - 8(1)}{(1 + 1)^3}\right] \ldots \ldots(3)$$

Step VI : Using standard value in equation (3)

$$\tan^{-1} 1 = \frac{\pi}{4} = \frac{3.14}{4} = 0.785$$

$$\tan^{-1}(1.003) = 0.785 + (0.003)\left(\frac{1}{2}\right) - (0.0000045)\left(\frac{2}{(2)^2}\right) - (0.0000000045)\left[\frac{2(2) - 8(1)}{(2)^3}\right] \ldots$$

$$= 0.785 + (0.0015) - (0.0000045)\left(\frac{2}{4}\right) - (0.0000000045)\left(\frac{4 - 8}{8}\right) \ldots$$

$$= 0.785 + 0.0015 - (0.0000045)\left(\frac{1}{2}\right) - (0.0000000045)\left(\frac{-4}{8}\right) \ldots$$

$$= 0.785 + 0.0015 - (0.00000225) + (0.0000000045)\left(\frac{1}{2}\right) \ldots$$

$$= 0.785 + 0.0015 - (0.00000225) + (0.0000000023) \ldots$$

$$= 0.7865000023 - 0.00000225 = 0.7864977523 = 0.7865$$

Approximately

→ May 18

Q. 3 Using Taylor's theorem for two variables, expand the function $f(x,y) = e^x \cos y$ in the powers of $(x - 1)$ and $\left(y - \frac{\pi}{4}\right)$. **(4 Marks)**

Ans. : The function $f(x, y) = e^x \cos y$ is required to be expanded in the powers of $(x - 1)$ and $(y - \pi/4)$ by using Taylor's theorem.

$$f(x, y) = e^x \cos y \Rightarrow f(1, \pi/4) = e \cos\frac{\pi}{4} = \frac{e}{\sqrt{2}}$$

$$f_x(x, y) = e^x \cos y \Rightarrow f_x(1, \pi/4) = e \cos \pi/4 = \frac{e}{\sqrt{2}}$$

$$f_y(x, y) = -e^x \sin y \Rightarrow f_y(1, \pi/4) = -e \sin \pi/4 = -\frac{e}{\sqrt{2}}$$

$$f_{xx}(x, y) = e^x \cos y \Rightarrow f_{xx}(1, \pi/4) = e \cos \pi/4 = \frac{e}{\sqrt{2}}$$

$$f_{xy}(x, y) = -e^x \sin y \Rightarrow f_{xy}(1, \pi/4) = -e \sin \pi/4 = -\frac{e}{\sqrt{2}}$$

$$f_{yy}(x, y) = -e^x \cos y \Rightarrow f_{yy}(1, \pi/4) = -e \cos \pi/4 = -\frac{e}{\sqrt{2}}$$

$$f_{xxx}(x, y) = e^x \cos y \Rightarrow f_{xxx}(1, \pi/4) = e \cos \pi/4 = \frac{e}{\sqrt{2}}$$

Hence, by taylor's theorem we have

$$f(x, y) = e^x \cos y$$

$$= f(a, b) + [(x-a) f_x(a, b) + (y-b) f_y(a, b)]$$

$$+ \frac{1}{2!} [(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b)$$

$$+ (y-b)^2 f_{yy}(a, b)] + \frac{1}{3!} [(x-a)^3 f_{xxx}(a, b)$$

$$+ 3(x-a)^2 (y-b) f_{xxy}(a, b)$$

$$+ 3(x-a)(y-b)^2 f_{xyy}(a, b) + (y-b)^3 f_{yyy}(a, b)] + \ldots$$

$$= \frac{e}{\sqrt{2}} + \frac{e}{\sqrt{2}} \left[(x-1) - \left(y - \frac{\pi}{4} \right) \right]$$

$$+ \frac{e}{2!\sqrt{2}} \left[(x-1)^2 - 2(x-1)\left(y - \frac{\pi}{\sqrt{4}} \right) - \left(y - \frac{\pi}{4} \right)^2 \right]$$

$$+ \frac{e}{3!\sqrt{2}} \left[(x-1)^3 - 3(x-1)^2 \left(y - \frac{\pi}{\sqrt{4}} \right) - 3(x-1)\left(y - \frac{\pi}{4} \right)^2 + \left(y - \frac{\pi}{4} \right)^3 \right]$$

Q. 4 If $u = a^3 x^2 + b^3 y^2 + c^3 z^2$ where $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, show that the stationary value of u is given by $x = \frac{\Sigma a}{a}$, $y = \frac{\Sigma a}{b}$, $z = \frac{\Sigma a}{c}$ **(4 Marks)**

Ans. :

$$u = a^3 x^2 + b^3 y^2 + c^3 z^2$$

$$= f(x, y, z) \text{ and } g(x, y, z)$$

$$= \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

Let us define the function

$$f(x, y, z) = f(x, y, z) + \lambda\, g(x, y, z)$$

$$= a^3 x^2 + b^3 y^2 + c^3 z^2 + \lambda \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \quad \ldots(1)$$

Differentiating equation (1) w.r.to x, y, z partially and equating them to zero.

$$F_x = \frac{\partial F}{\partial x} = 2a^3 x - \frac{\lambda}{x^2} = 0 \quad \ldots(2)$$

$$F_y = \frac{\partial F}{\partial y} = 2b^3 y - \frac{\lambda}{y^2} = 0 \quad \ldots(3)$$

$$F_z = \frac{\partial F}{\partial z} = 2c^3 z - \frac{\lambda}{z^2} = 0 \quad \ldots(4)$$

From equation (2), (3) and (4) we have,

$$2a^3 x^3 = 2b^3 y^3 = 2c^3 z^3$$

$$\Rightarrow ax = by = cz = k$$

$$\Rightarrow x = \frac{k}{a}, y = \frac{k}{b}, z = \frac{k}{c}$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \Rightarrow \frac{a}{k} + \frac{b}{k} + \frac{c}{k} = 1$$

$$\Rightarrow k = a + b + c$$

$$\therefore x = \frac{a+b+c}{a},$$

$$y = \frac{a+b+c}{b},$$

$$z = \frac{a+b+c}{c}$$

$$\therefore x = \frac{\sum a}{a}, y = \frac{\sum a}{b}, z = \frac{\sum a}{c}$$

Q. 5 If the sides and angles of a plane triangle vary in such a way that its circum-radius remains constant, prove that **(4 Marks)**

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0,$$

Where da, db and dc are smaller increments in the sides a, b and c, respectively.

Ans. :

From the sine rule we have,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The Circum radius is given by,

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{C}{2 \sin C}$$

$$\Rightarrow a = 2R \sin A,\ b = 2R \sin B,\ C = 2R \sin C$$

$$\Rightarrow da = 2R \cos A\, dA,\ db = 2R \cos B\, dB,\ dc = 2R \cos C\, dC$$

$$\Rightarrow \frac{da}{\cos A} = 2R\, dA,\ \frac{db}{\cos B} = 2R\, dB,\ \frac{dc}{\cos C} = 2R\, dC$$

$$\therefore \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R\,(dA + dB + dC)$$

$$= 2R d(A + B + C)$$

$$= 2R\,(0) = 0$$

Chapter Ends

□□□

Multiple Choice Questions (MCQ)

UNIT 3

Applications of Partial Differentiations

- Short Questions and Answers
- Fill in the Blanks
- Multiple Choice Questions

UNIT III

Applications of Partia Differentiations

Short Questions and Answers

Ex. 1 : If $u = x(1-y)$ and $v = xy$ find $\frac{\partial(x,y)}{\partial(u,v)}$

Soln. : Given $u = x - xy$; $v = xy$

By definition $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix}$

Solving the determinant

$$\frac{\partial(u,v)}{\partial(x,y)} = x(1-y) + xy = x - xy + xy = x$$

Since $\frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(u,v)} = 1$

$$\therefore \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{x} \checkmark$$

Ex. 2 : Find $\frac{\partial(x,y)}{\partial(u,v)}$ for $x = e^u \cos v$; $y = e^u \sin v$.

Soln. : Step 1 : By definition

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix}$$

Step 2 : Solving the determinant

$J = e^{2u} \cos^2 v + e^{2u} \sin^2 v = e^{2u}(\cos^2 v + \sin^2 v)$

$= e^{2u}$ **Since, $\cos^2 v + \sin^2 v = 1$**

$$\frac{\partial(x,y)}{\partial(u,v)} = e^{2u} \checkmark$$

Ex. 3 : Find $\frac{\partial(x,y)}{\partial(u,v)}$ for $x = u(1-v)$ and $y = uv$

Soln. : Step 1 : By definition :

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

Step 2 : Solving the determinant.

$$J = u(1-v) + uv = u - uv + uv = u$$

$$\therefore \quad \frac{\partial(x,y)}{\partial(u,v)} = u \checkmark$$

Ex. 4 : If $u = \frac{x-y}{x+y}$, $v = \frac{x+y}{x}$ are functionally dependen then find a relation between them

Soln. : Given, u and v are functionally dependent

To find a relation between u and v by eliminatin x and y.

Given, $u = \frac{x-y}{x+y}$, $v = \frac{x+y}{x}$

$uv = \frac{x-y}{x+y} \times \frac{x+y}{x}$ (Multiply u and v)

$= \frac{x-y}{x} = 1 - \frac{y}{x} = 1 - (v-1)$

$\therefore v = 1 + \frac{y}{x}$,

$uv = 2 - v$ Relation between u and v. ✓

Ex. 5 : If $u = y + z$, $v = x + 2z^2$, $w = x - 4yz - 2y^2$ ar functionally dependent then find a relation betwee them

Soln. : To find a relation between u, v, w by eliminatin x, y, z.

Givem, $u = y + z$, $v = x + 2z^2$, $w = x - 4yz - 2y^2$

Consider $w = x - 4yz - 2y^2$

$= x - 4yz - 2y^2 - 2z^2 + 2z^2$

(Adding and subtracting $2z^2$)

$= \underline{x + 2z^2} - 4yz - 2y^2 - 2z^2$

$= v - 2(y^2 + 2yz + z^2)$

$(\because \; x + 2z^2 = v)$

$= v - 2(y+z)^2$

$(\because \; u = y + z)$

$w = v - 2u^2$ Relation between u, v, w. ✓

Ex. 6 : If $x = u^2 - v^2$; $y = uv$, find $\frac{\partial u}{\partial x}$

Soln. :

Let implicit relations between u,v,x and ybe,

$$f_1 \equiv u^2 - v^2 - x = 0 \; ; \quad f_2 \equiv uv - y = 0 \qquad \ldots(1)$$

$$\therefore \quad \frac{\partial u}{\partial x} = -\left[\frac{\dfrac{\partial (f_1, f_2)}{\partial (x, v)}}{\dfrac{\partial (f_1, f_2)}{\partial (u, v)}}\right] = -\frac{N_r}{D_r}$$

$$\text{Here, } N_r = \frac{\partial (f_1, f_2)}{\partial (x, v)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -2v \\ 0 & u \end{vmatrix} \text{(Differentiate Partially Eqn(1))}$$

$$\therefore N_r = -u$$

$$D_r = \frac{\partial (f_1, f_2)}{\partial (u, v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} \text{(Differentiate Partially Eqn(1))}$$

By solving determinant, we get,

$$\therefore \quad D_r = 2u^2 + 2v^2 = 2(u^2 + v^2)$$

$$\text{Hence, } \frac{\partial u}{\partial x} = -\left[\frac{-u}{2(u^2 + v^2)}\right] = \frac{u}{2(u^2 + v^2)} \checkmark$$

Ex. 7 : If $x = \rho \cos \phi$; $y = \rho \sin \phi$, $z = z$

find $\frac{\partial (x, y, z)}{\partial (\rho, \phi, z)}$

Soln. : Given , $x = \rho \cos \phi$; $y = \rho \sin \phi$; $z = z$

$$\frac{\partial (x, y, z)}{\partial (\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos \phi & -\rho \sin \phi & 0 \\ \sin \phi & \rho \cos \phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1 [\rho \cos^2 \phi + \rho \sin^2 \phi] = \rho$$

$$\therefore \quad \frac{\partial (x, y, z)}{\partial (\rho, \phi, z)} = \rho \quad (\because \sin^2 \theta + \cos^2 \theta = 1) \checkmark$$

Ex. 8 : Find the stationary point of the function $x^2 + y^2 + 6x + 12$.

Soln. :

Given, $f(x, y) = x^2 + y^2 + 6x + 12 \qquad \ldots(1)$

$$\frac{\partial f}{\partial x} = 2x + 6 \quad \text{and} \quad \frac{\partial f}{\partial x} = 2y$$

For maxima and minima (i.e. to find stationary points),

$$\Rightarrow \quad \frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y = 0$$

$$\therefore \; 2x + 6 = 0 \;\; \Big| \;\; 2y = 0$$

$$x = -3 \qquad y = 0$$

$\therefore$ **Stationary point is, $(x, y) \equiv (-3, 0)$ ✓**

Ex. 9 : Find the stationary points of function $3x^2 - y^2 + x^3$.

Soln. :

Given , $f(x, y) = 3x^2 - y^2 + x^3 \qquad \ldots(1)$

$$\frac{\partial f}{\partial x} = 6x - 0 + 3x^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = -2y$$

To find stationary points (i.e. for maxima and minima)

$$\therefore \; \frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

i.e. $6x + 3x^2 = 0$ and $-2y = 0$

$x(6 + 3x) = 0$ and $y = 0$

$x = 0$ and $6 + 3x = 0$

$3x = -6 \quad \therefore x = -2$

$\therefore \; x = 0 ; y = 0 \Rightarrow (0, 0)$

$x = -2$; $y = 0 \Rightarrow$ $(-2, 0)$ are the stationary points. ✓

Ex. 10 : Find the maximum and minimum values of $f(x, y) = xy(a - x - y)$; $a > 0$ at stationary point $(a/3, a/3)$

Soln. :

Given, $f(x, y) = xy(a - x - y) \qquad \ldots(1)$

Now,

$$r = \frac{\partial^2 f}{\partial x^2} = -2y$$

$$s = \frac{\partial^2 f}{\partial x \, \partial y} = a - 2x - 2y$$

$$t = \frac{\partial^2 f}{\partial y^2} = -2x$$

For $(x, y) \equiv (a/3, a/3)$

$$r = \frac{-2a}{3}, \qquad s = \frac{-a}{3}, \qquad t = \frac{-2a}{3}$$

$$\therefore \; rt - s^2 = \left(\frac{-2a}{3}\right)\left(\frac{-2a}{3}\right) - \left(\frac{-a}{3}\right)^2 = \frac{a^2}{3} > 0$$

and $r < 0$, if $a > 0$, function is maximum.

$$\text{For extreme value} = f\left(\frac{a}{3}, \frac{a}{3}\right) = \left(\frac{a}{3}\right)\left(\frac{a}{3}\right)\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^2}{27}$$

Fill in the Blanks / Multiple Choice Questions

Q. 1 The functions u and v are functionally dependent if ______.

(a) $\frac{\partial(u, v)}{\partial(x, y)} = 0$ (b) $\frac{\partial(u, v)}{\partial(x, y)} \neq 0$

(c) $\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(x, y)}{\partial(u, v)}$ (d) none of these **Ans. : (a)**

Q. 2 If $u = \frac{x + y}{1 - xy}$, $v = \tan^{-1} x + \tan^{-1} y$ are functionally dependent then relation between u and v is ______.

(a) $\tan u = v$ (b) $\tan x = y$

(c) $\tan v = u$ (d) $\tan y = x$ **Ans. : (c)**

Q. 3 If u_1 , u_2 be implicit functions of x_1, x_2 connected by f_1, f_2 such that

$f_1 (u_1, u_2, x_1, x_2) = 0$;

$f_2 (u_1, u_2, x_1, x_2) = 0$ then, $\frac{\partial (u_1, u_2)}{\partial (x_1, x_2)} =$ ______.

Soln. :

(a) $\dfrac{\frac{\partial (f_1, f_2)}{\partial (x, v)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}}$ (b) $-\dfrac{\frac{\partial (f_1, f_2)}{\partial (x_1, x_2)}}{\frac{\partial (f_1, f_2)}{\partial (u_1, u_2)}}$

(c) $(-1)^2 \dfrac{\frac{\partial (f_1, f_2)}{\partial (x, v)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}}$ (d) $(-1)^2 \dfrac{\frac{\partial (f_1, f_2)}{\partial (x_1, x_2)}}{\frac{\partial (f_1, f_2)}{\partial (u_1, u_2)}}$ **Ans. : (d)**

Q. 4 If u and v are functions of r, s and r, s are functions of x, y then, $\frac{\partial (u, v)}{\partial (x, y)} =$ ______.

(a) $\frac{\partial (u, v)}{\partial (r, s)} \times \frac{\partial (r, s)}{\partial (x, y)}$ (b) $\frac{\partial (u, v)}{\partial (x, y)} \times \frac{\partial (x, y)}{\partial (u, v)}$

(c) $\frac{\partial (u, v)}{\partial (x, y)} \times \frac{\partial (r, s)}{\partial (u, v)}$ (d) $\frac{\partial (u, v)}{\partial (x, y)} \times \frac{\partial (r, s)}{\partial (x, y)}$ **Ans. : (a)**

Q. 5 u and v are functions of x, y and x, y are functions of u, v then, $\frac{\partial (u, v)}{\partial (x, y)} \times \frac{\partial (x, y)}{\partial (u, v)} =$ ______.

(a) 0 (b) –1 (c) ∞ (d) 1 **Ans. : (d)**

Q. 6 If u and v are continuous and differentiable functions of two independent variables x and y then the Jacobian of u, v w.r.t. x and y is ______.

(a) $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} & \frac{\partial u}{\partial y} \end{vmatrix}$ (b) $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

(c) $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ (d) $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix}$ **Ans. : (b)**

Q. 7 If u, v are implicit functions of x, y such that $f_1 (u, v, x, y) = 0$; $f_2 (u, v, x, y) = 0$ then $\frac{\partial u}{\partial x} =$ ______.

(a) $\dfrac{\frac{\partial (f_1, f_2)}{\partial (x, v)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}}$ (b) $\dfrac{\frac{\partial (f_1, f_2)}{\partial (x, y)}}{\frac{\partial (f_1, f_2)}{\partial (y, v)}}$

(c) $\left(\dfrac{\frac{\partial (f_1, f_2)}{\partial (x, y)}}{\frac{\partial (f_1, f_2)}{\partial (y, v)}}\right)$ (d) $-\left(\dfrac{\frac{\partial (f_1, f_2)}{\partial (x, v)}}{\frac{\partial (f_1, f_2)}{\partial (u, v)}}\right)$ **Ans. : (d)**

Q. 8 If u , v, w are implicit functions of x, y, z such that $f_1 (u, v, w, x, y, z) = 0$; $f_2 (u, v, w, x, y, z) = 0$;

$f_3 (u, v, w, x, y, z) = 0$ then $\frac{\partial y}{\partial w}$

(a) $-\left[\dfrac{\frac{\partial (f_1, f_2, f_3)}{\partial (x, w, z)}}{\frac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)}}\right]$ (b) $\dfrac{\frac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)}}{\frac{\partial (f_1, f_2, f_3)}{\partial (u, v, w)}}$

(c) $\dfrac{\frac{\partial (f_1, f_2, f_3)}{\partial (u, v, w)}}{\frac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)}}$ (d) $-\dfrac{\frac{\partial (f_1, f_2, f_3)}{\partial (u, v, w)}}{\frac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)}}$

Ans. : (a)

Q. 9 The percentage error in x, y are ______.

(a) $\frac{dx}{x}, \frac{dy}{y}$ (b) 100 dx , 100 dy

(c) $\frac{dx}{x} \times 100$, $\frac{dy}{y} \times 100$ (d) 100 x, 100 y **Ans. : (c)**

Q. 10 The relative error in x, y are ______.

(a) dx, dy (b) x dx, y dy

(c) $\frac{dy}{dx}$, $\frac{dx}{dy}$ (d) $\frac{dx}{x}$, $\frac{dy}{y}$ **Ans. : (d)**

Q. 11 For the function f(x, y) = 0 if $r = \frac{\partial^2 f}{\partial x^2}$;

$s = \frac{\partial^2 f}{\partial x \, \partial y}$; $t = \frac{\partial^2 f}{\partial y^2}$ and $(rt - s^2) > 0$ then function is minimum at (a_1, b_1) if ______.

(a) $r > 0$ (b) $r = 0$ (c) $r < 0$ (d) $s < 0$ **Ans. : (a)**

Q. 12 For the function f(x, y) = 0 if $r = \frac{\partial^2 f}{\partial x^2}$; $s = \frac{\partial^2 f}{\partial x \, \partial y}$;

$t = \frac{\partial^2 f}{\partial y^2}$ and $(rt - s^2)\, s > 0$ then function is maximum at (a_1, b_1) if ______.

(a) $r > 0$ (b) $s > 0$ (c) $r < 0$ (d) $s < 0$ **Ans. : (c)**

Q. 13 For the function $x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$ if $r = \frac{\partial^2 f}{\partial x^2}$;

$s = \frac{\partial^2 f}{\partial x \, \partial y}$; $t = \frac{\partial^2 f}{\partial y^2}$; $(rt - s^2) > 0$ and $r = -6$ at (0, 0) then function at (0, 0) is ______.

(a) minimum (b) maximum

(c) linear (d) increasing **Ans. : (b)**

□□□

UNIT 4

Reduction Formulae and Curve Tarcing

➢➢ Syllabus :

Reduction formulae for $\int_0^{\pi/2} \sin^n x \, dx$, $\int_0^{\pi/2} \cos^n x \, dx$, $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

Curve Tracing : Tracing of the Curves given in Cartesian, Parametric & Polar forms

- **Chapter 6 : Reduction Formulae**
- **Chapter 7 : Curve Tracing and Rectification of Curves**

$$[1+n-1]\, I_n = -\sin^{n-1} x \cos x + (n-1)\, I_{n-2}$$

$$n\, I_n = -\sin^{n-1} x \cos x + (n-1)\, I_{n-2}$$

$$I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n} I_{n-2} \quad ...(5)$$

This is required reduction formula for $\int \sin^n x\, dx$

6.3 Reduction Formula for $\int_0^{\pi/2} \sin^n x\, dx$ where n is Any Positive Integer

From equation (5),

$$I_n = \int \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n} I_{n-2}$$

Taking limits of integration as $x : 0 \to \frac{\pi}{2}$

$$I_n = \int_0^{\pi/2} \sin^n x\, dx = -\left[\frac{\sin^{n-1} x \cos x}{n}\right]_0^{\pi/2} + \frac{(n-1)}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$= -\left[\frac{\sin^{n-1}\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)}{n} - \frac{\sin^{n-1} 0 \cos 0}{n}\right]$$

$$+\frac{(n-1)}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx \quad ...(6)$$

→ Using standard values

$$...\left[\sin\frac{\pi}{2} = 1, \cos\frac{\pi}{2} = 0, \sin 0 = 0, \cos 0 = 1 \text{ in equation (6)}\right]$$

$$I_n = -\left[\frac{(1)(0)}{n} - \frac{(0)(1)}{n}\right] + \frac{(n-1)}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$= -(0-0) + \frac{n-1}{n} I_{n-2}$$

$$\therefore \quad I_n = \frac{n-1}{n} I_{n-2} \quad ...(7)$$

We have,

$$I_n = \frac{\text{The value at suffix of I} - 1}{\text{The value at suffix of I}}\; I_{(\text{The value at suffix of I} - 2)}$$

As $I_n = \frac{n-1}{n} I_{n-2}$, is applicable to all positive integers n.

If n is positive integer greater than 2 then $(n-2)$ is also positive integer, so the formula can be applied to I_{n-2} also.

$$\therefore I_{n-2} = \frac{\text{The value at suffix of I} - 1}{\text{The value at suffix of I}}\; I_{(\text{The value at suffix of I} - 2)}$$

$$= \frac{(n-2)-1}{(n-2)} I_{((n-2)-2)} = \frac{n-3}{n-2} I_{n-4}$$

From equation (7),

$$I_n = \frac{n-1}{n}\left[\frac{n-3}{n-2} I_{n-4}\right]$$

Repeated application of formula (1)

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot I_{n-6}$$

$$I_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6} I_{n-8} \quad ...(8)$$

and so on.

Case I : When n is even positive integer

In this case, $I_2 = \frac{2-1}{2} I_{2-2} = \frac{1}{2} I_0$ ...(9)

Now, $I_4 = \frac{4-1}{4} I_{4-2} = \frac{3}{4} I_2$

But from equation (9), $I_2 = \frac{1}{2} I_0$

$$\therefore I_4 = \frac{3}{4}\cdot\frac{1}{2} I_0 \quad ...(10)$$

Now, $I_6 = \frac{6-1}{6} I_{6-2} = \frac{5}{6} I_4$

But from equation (10),

$$I_4 = \frac{3}{4}\cdot\frac{1}{2} I_0$$

$$\therefore I_6 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2} I_0 \quad ...(11)$$

And so on.

From above formula we observe that,

$$I_n = \left[\left(\frac{\text{The value at suffix of I} - 1}{\text{The value at suffix of I}}\right)\left(\frac{\text{Previous term in numerator} - 2}{\text{Previous term in denominator} - 2}\right)\left(\frac{\text{Previous term in numerator} - 2}{\text{Previous term in denominator} - 2}\right)\right.$$

......continue till we get positive nonzero term in both numerator and denominator $\Big] \times I_0$

$$\therefore \text{In general, } I_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{1}{2} I_0 \quad ...(12)$$

$$\text{Now } I_0 = \int_0^{\pi/2} \sin^0 x\, dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$\therefore$ From equation (12),

$$\boxed{\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \quad \text{if n is an even positive integer}}$$

For ex. $\int_0^{\pi/2} \sin^2 x\, dx = \frac{1}{2}\cdot\frac{\pi}{2}$;

$$\int_0^{\pi/2} \sin^4 x\, dx = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$\int_0^{\pi/2} \sin^6 x\, dx = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \text{ and so on.}$$

Case II : If n is odd positive integer

In this case, $I_3 = \frac{3-1}{3} I_{3-2} = \frac{2}{3} I_1$ ...(13)

Now, $I_5 = \frac{5-1}{5} I_{5-2} = \frac{4}{5} I_3$

But from Equation (13),

$$I_3 = \frac{2}{3} I_1$$

$$\therefore I_5 = \frac{4}{5}\cdot\frac{2}{3} I_1 \quad ...(14)$$

Now, $I_7 = \frac{7-1}{7} I_{7-2} = \frac{6}{7} I_5$

But from Equation (14),

$$I_5 = \frac{4}{5}\cdot\frac{2}{3} I_1$$

$$\therefore I_7 = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} I_1 \quad ...(15)$$

And so on.

From above formula we observe that,

$$I_n = \left[\left(\frac{\text{The value at suffix of I} - 1}{\text{The value at suffix of I}}\right)\left(\frac{\text{Previous term in numerator} - 2}{\text{Previous term in denominator} - 2}\right)\left(\frac{\text{Previous term in numerator} - 2}{\text{Previous term in denominator} - 2}\right)\right.$$

......continue till we get positive nonzero term in both numerator and denominator $\Big] \times I_1$

$$\therefore \text{In general, } I_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3} I_1 \quad ...(16)$$

$$\text{Now} \quad I_1 = \int_0^{\pi/2} \sin x\, dx = [-\cos x]_0^{\pi/2}$$

$$= -\left(\cos\frac{\pi}{2} - \cos 0\right) = -(0-1) = 1$$

From equation (16),

$$\boxed{I_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{4}{5}\,\frac{2}{3}\; 1, \text{ if n is an odd positive integer}}$$

$$\text{Thus, } \int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},$$

if n is even positive integer

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1, \text{ if n is odd}$$

positive integer

6.4 Reduction Formula for $\int \cos^n x\, dx$ where n is any Positive Integer

Let, $\quad I_n = \int \cos^n x\, dx,$

where n is any positive integer ...(1)

As $\int \cos^n x\, dx$ cannot be evaluated directly, we write $\cos^n x$ as $\cos x \cos^{n-1} x$

$$\therefore I_n = \int \cos x \cos^{n-1} x\, dx \quad ...(2)$$

→ Using standard integrating by parts rule in equation(2)

$$...\left[\int u\, v\, dx = u\int v\, dx - \int\left(\frac{d}{dx}u\right)\left(\int v\, dx\right) dx\right]$$

Here taking $u = \cos^{n-1} x$, $v = \cos x$

$$\therefore I_n = \cos^{n-1} x \int \cos x\, dx$$

$$-\int\left(\frac{d}{dx}\cos n - 1\, x\right)\left(\int \cos x\, dx\right) dx \quad ...(3)$$

→ Using standard formula in equation (3)

$$...\left[\frac{d}{dx}(\cos^{n-1} x) = (n-1)\cos^{n-2} x\,\frac{d}{dx}(\cos x) = (n-1)\cos^{n-2} x(-\sin x) \qquad \int \cos x\, dx = \sin x\right]$$

$$\therefore I_n = \cos^{n-1} x\, (\sin x) - \int [(n-1)\cos^{n-2}(x)(-\sin x)](\sin x)\, dx$$

$$= \cos^{n-1} x\, \sin x - (-)(n-1)\int \cos^{n-2} x\, (\sin x)(\sin x)\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\, dx \qquad ...(4)$$

→ **Using standard trigonometric identity in equation (4)**

$$...[\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x]$$

$$\therefore I_n = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,(1-\cos^2 x)\, dx)$$

$$= \cos^{n-1} x \sin x + (n-1)\int (\cos^{(n-2)} x - \cos^{(n-2)} x \cos^2 x)\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int (\cos^{(n-2)} x - \cos^2 x)\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\left[\int \cos^{(n-2)} x\, dx - \int \cos^n dx\right]$$

From equation(1),

$$\int \cos^n x\, dx = I_n, \Rightarrow \int \cos^{n-2} x\, dx = I_{n-2}$$

$$\therefore I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$[1 + (n-1)] I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$[1 + n - 1] I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$n I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n} I_{n-2} \qquad ...(5)$$

This is required reduction formula for $\int \cos^n x\, dx$

Similarly we can prove that

$$\int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}.........\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},$$

if n is even positive integer

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}......\frac{4}{5}\cdot\frac{2}{3}\cdot 1,$$

if n is odd positive integer

Example 6.4.1

Find the value of $\int_0^{\pi/2} \cos^3 x\, dx$

Solution : Step I : Let $I = \int_0^{\pi/2} \cos^3 x\, dx \qquad ...(1)$

→ **Using standard formula in equation (1),**

$$...\left[\int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}....\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},\right.$$

if n is even positive integer

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}......\frac{2}{3}\cdot 1,$$

if n is odd positive integer]

Here n = 3, an odd integer

Step II : $\therefore \int_0^{\pi/2} \cos^3 x\, dx = \frac{3-1}{3}\cdot 1 = \frac{2}{3}\cdot 1 = \frac{2}{3}$

Example 6.4.2

Find the value of $\int_0^{\pi/2} \cos^8 x\, dx$

Solution :

Step I : Let $I = \int_0^{\pi/2} \cos^8 x\, dx \qquad ...(1)$

→ **Using standard formula in equation (1),**

$$...\left[\int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}.....\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},\right.$$

if n is even positive integer

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}......\frac{2}{3}\cdot 1,$$

if n is odd positive integer]

Here n = 8, an even integer

$I_n = \left[\left(\frac{\text{The value at suffix of } I - 1}{\text{The value at suffix of } I}\right)\right.$ $\left(\frac{\text{Previous term in numerator} - 2}{\text{Previous term in denominator} - 2}\right)$ $\left(\frac{\text{Previous term in numerator} - 2}{\text{Previous term in denominator} - 2}\right)$ ......continue till we get positive nonzero term in both numerator and denominator $\left.\right] \times I_0$

Step II :

$$\therefore \int_0^{\pi/2} \cos^8 x\, dx = \left(\frac{8-1}{8}\right)\cdot\left(\frac{8-3}{8-2}\right)\cdot\left(\frac{8-5}{8-4}\right)\cdot\left(\frac{8-7}{8-4}\right)\cdot\frac{\pi}{2}$$

Here n = 8, an even integer

$$\therefore \int_0^{\pi/2} \cos^8 x\, dx = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35\pi}{256}$$

Example 6.4.3

Find the value of $\int_0^{\pi/2} \sin^5 x\, dx$

Solution : Step I : Let $I = \int_0^{\pi/2} \sin^5 x\, dx \qquad ...(1)$

→ **Using standard formula in equation (1),**

$$...\left[\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}.......\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},\right.$$

if n is even positive integer

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}......\frac{2}{3}\cdot 1,$$

if n is odd positive integer]

Here n = 5, an odd integer

$I_n = \left[\left(\frac{\text{The value at suffix of } I - 1}{\text{The value at suffix of } I}\right)\right.$ $\left(\frac{\text{Previous term in numerator} - 2}{\text{Previous term in denominator} - 2}\right)$ $\left(\frac{\text{Previous term in numerator} - 2}{\text{Previous term in denominator} - 2}\right)$ ......continue till we get positive nonzero term in both numerator and denominator $\left.\right] \times I_1$

Step II :

$$\therefore \int_0^{\pi/2} \sin^5 x\, dx = \left(\frac{5-1}{5}\right)\cdot\left(\frac{5-3}{5-2}\right)\cdot 1 = \frac{4}{5}\cdot\frac{2}{3}\cdot 1 = \frac{8}{15}$$

Example 6.4.4

Find the value of $\int_0^{\pi/2} \sin^4 x\, dx$

Solution :

Step I : Let $I = \int_0^{\pi/2} \sin^4 x\, dx \qquad ...(1)$

→ **Using standard formula in equation (1),**

$$...\left[\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}..\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},\right.$$

if n is even positive integer

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}......\frac{2}{3}\cdot 1,$$

if n is odd positive integer]

Here n = 4, an even integer

$I_n = \left[\left(\frac{\text{The value at suffix of } I - 1}{\text{The value at suffix of } I}\right)\right.$ $\left(\frac{\text{Previous term in numerator} - 2}{\text{Previous term in denominator} - 2}\right)$ $\left(\frac{\text{Previous term in numerator} - 2}{\text{Previous term in denominator} - 2}\right)$ ......continue till we get positive nonzero term in both numerator and denominator $\left.\right] \times I_0$

Step II: $\therefore \int_0^{\pi/2} \sin^4 x\, dx = \left(\frac{4-1}{4}\right)\cdot\left(\frac{4-3}{4-2}\right)\cdot\frac{\pi}{2}$

$$\therefore \int_0^{\pi/2} \sin^4 x\, dx = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{16}$$

6.5 Reduction Formula for $\int_0^{\pi/2} \sin^m x \cos^n x\, dx$, m and n are Positive Integer

It can be prove that

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{[(m-1)(m-3)(m-5)...\times 2 \text{ or } 1]\,[(n-1)(n-3)(n-5)....\times 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)....2 \text{ or } 1} \times k$$

Where $k = \pi/2$ if m and n both are even integers

$= 1$ for all other values of m and n

6.5.1 Illustrative Examples

Example 6.5.1

Evaluate $\int_0^{\pi/2} \sin^3 \theta \cos^4 \theta\, d\theta$

Solution :

Step I : Let $I = \int_0^{\pi/2} \sin^3 \theta \cos^4 \theta\, d\theta \qquad ...(1)$

→ **Using standard reduction formula in equation (1)**

$$\because \left[\int_0^{\pi/2} \sin^m x \cos^n x\,dx\right] = \frac{[(m-1)(m-3)(m-5)\ldots\times 2 \text{ or } 1]\;[(n-1)(n-3)(n-5)\ldots\times 2 \text{ or } 1]}{(m+r)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1}\times k$$

Where $k = \pi/2$ if m and n both are even integers

= 1 for all other values of m and n

Here m = 3 , n = 4

Step II :

$$\therefore I = \int_0^{\pi/2} \sin^3\theta \cos^4\theta \, d\theta$$

$$= \frac{[(3-1)]\;[(4-1)(4-3)]}{(3+4)(3+4-2)(3+4-4)(3+4-6)} \times k$$

Since m and n both are not even, k = 1

Step III $\therefore I = \dfrac{[(2)]\;[(3)(1)]}{(7)(7-2)(7-4)(7-6)} \times 1$

$$= \frac{(2)\cdot(3)\cdot(1)}{(7)\cdot(5)\cdot(3)\cdot(1)} \times 1 = \mathbf{\frac{2}{35}}$$

Example 6.5.2

Evaluate $\displaystyle\int_0^{\pi/2} \sin^5 x \cdot \cos^3 \theta \, d\theta$

Solution :

Step I : Let $I = \displaystyle\int_0^{\pi/2} \sin^5\theta \cos^3\theta \, d\theta$ ...(1)

→ **Using standard reduction formula in equation (1)**

$$\because \left[\int_0^{\pi/2} \sin^m x \cos^n x\,dx\right] = \frac{[(m-1)(m-3)(m-5)\ldots\times 2 \text{ or } 1]\;[(n-1)(n-3)(n-5)\ldots\times 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1}\times k$$

Where $k = \pi/2$ if m and n both are even integers

= 1 for all other values of m and n

Here m = 5 , n = 3

Step II :

$$\therefore I = \int_0^{\pi/2} \sin^5\theta \cos^3\theta \, d\theta$$

$$= \frac{[(5-1)(5-3)]\;[(3-1)]}{(5+3)(5+3-2)(5+3-4)(5+3-6)} \times k$$

Since m and n both are not even, k = 1

Step III : $\therefore I = \dfrac{[(4)(2)]\;[(2)]}{(8)(8-2)(8-4)(8-6)} \times 1$

$$= \frac{(4\cdot 2)(2)}{8\cdot 6\cdot 4\cdot 2} \times 1 = \mathbf{\frac{1}{24}}$$

Example 6.5.3

Evaluate $\displaystyle\int_0^{\pi/2} \sin^6\theta \cdot \cos^8 \theta \, d\theta$

Solution :

Step I : Let $I = \displaystyle\int_0^{\pi/2} \sin^6\theta \cdot \cos^8\theta \, d\theta$ ...(1)

→ **Using standard reduction formula in equation (1)**

$$\because \left[\int_0^{\pi/2} \sin^m x \cos^n x\,dx\right] = \frac{[(m-1)(m-3)(m-5)\ldots\times 2 \text{ or } 1]\;[(n-1)(n-3)(n-5)\ldots\times 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1}\times k$$

Where $k = \pi/2$ if m and n both are even integers

= 1 for all other values of m and n

Here m = 6 , n = 8

Step II : $\therefore I = \displaystyle\int_0^{\pi/2} \sin^6\theta \cdot \cos^8\theta \, d\theta$

$$= \frac{[(6-1)(6-3)(6-5)]\;[(8-1)(8-3)(8-5)(8-7)]}{(6+8)(6+8-2)(6+8-4)(6+8-6)(6+8-8)(6+8-10)(6+8-12)} \times k$$

Since m and n both are even, $k = \dfrac{\pi}{2}$

Step III

$$\therefore I = \frac{[(5)(3)(1)]\;[(7)(5)(3)(1)]}{(14)(14-2)(14-4)(14-6)(14-8)(14-10)(14-12)} \times \frac{\pi}{2}$$

$$I = \frac{(5\cdot 3\cdot 1)\;(7\cdot 5\cdot 3\cdot 1)}{14\cdot 12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2} \times \frac{\pi}{2} = \frac{5\pi}{4096}$$

Example 6.5.4

Evaluate $\displaystyle\int_0^{\pi/2} \sin^6\theta \cdot \cos^5 \theta \, d\theta$

Solution :

Step I : Let $I = \displaystyle\int_0^{\pi/2} \sin^6\theta \cdot \cos^5\theta \, d\theta$ ...(1)

→ **Using standard reduction formula in equation (1)**

$$\because \left[\int_0^{\pi/2} \sin^m x \cos^n x\,dx\right] = \frac{[(m-1)(m-3)(m-5)\ldots\times 2 \text{ or } 1]\;[(n-1)(n-3)(n-5)\ldots\times 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1}\times k$$

Where $k = \pi/2$ if m and n both are even integers

= 1 for all other values of m and n

Here m = 6 , n = 5

Step II : $\therefore I = \displaystyle\int_0^{\pi/2} \sin^6\theta \cdot \cos^5\theta \, d\theta$

$$= \frac{[(6-1)(6-3)(6-5)]\;[(5-1)(5-3)]}{(6+5)(6+5-2)(6+5-4)(6+5-6)(6+5-8)(6+5-10)} \times k$$

Since m and n both are not even, k = 1

Step III

$$\therefore I = \frac{[(5)(3)(1)]\;[(4)(2)]}{(11)(11-2)(11-4)(11-6)(11-8)(11-10)} \times 1$$

$$I = \frac{(5\cdot 3\cdot 1)\;(4\cdot 2)}{11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1} \times 1 = \mathbf{\frac{8}{693}}$$

Example 6.5.5

Evaluate $\displaystyle\int_0^{\pi/2} \sin^4\theta \cdot \cos^8 \theta \, d\theta$

Solution :

Step I

Let $I = \displaystyle\int_0^{\pi/2} \sin^4\theta \cdot \cos^8\theta \, d\theta$ ...(1)

→ **Using standard reduction formula in equation (1)**

$$\because \left[\int_0^{\pi/2} \sin^m x \cos^n x\,dx\right] = \frac{[(m-1)(m-3)(m-5)\ldots\times 2 \text{ or } 1]\;[(n-1)(n-3)(n-5)\ldots\times 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1}\times k$$

Where $k = \pi/2$ if m and n both are even integers

= 1 for all other values of m and n

Here m = 4 , n = 8

Step II : $\therefore I = \displaystyle\int_0^{\pi/2} \sin^4\theta \cdot \cos^8 \, d\theta$

$$= \frac{[(4-1)(4-3)]\;[(8-1)(8-3)(8-5)(8-7)]}{(4+8)(4+8-2)(4+8-4)(4+8-6)(4+8-8)(4+8-10)} \times k$$

Since m and n both are even, $k = \dfrac{\pi}{2}$

Step III

$$\therefore I = \frac{[(3)(1)]\;[(7)(5)(3)(1)]}{(12)(12-2)(12-4)(12-6)(12-8)(12-10)} \times \frac{\pi}{2}$$

$$I = \frac{(3\cdot 1)\;(7\cdot 5\cdot 3\cdot 1)}{12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2\cdot 1} \times \frac{\pi}{2} = \mathbf{\frac{7\pi}{2048}}$$

Substitutions

Sr. No.	Algebraic function	Substitution
1.	$\sqrt{a^2-x^2}$ / $(a^2-x^2)^n$	$x = a\sin\theta$ or $x = a\cos\theta$
2.	$\sqrt{a-x}$ / $(a-x)^n$	$x = a\sin^2\theta$ or $x = a\cos^2\theta$
3.	$\sqrt{a^2+x^2}$ / $(a^2+x^2)^n$	$x = a\tan\theta$
4.	$\sqrt{a+x}$ / $(a+x)^n$	$x = a\tan^2\theta$
5.	$\sqrt{(x^2-a^2)}$ / $(x^2-a^2)^n$	$x = a\sec\theta$

6.6 Examples Based on Above Substitutions

Example 6.6.1

Evaluate $\displaystyle\int_0^1 \frac{x^7}{\sqrt{1-x^2}}\,dx$

Solution :

Step I : Let $I = \int_0^1 \frac{x^7}{\sqrt{1-x^2}}\, dx$

Since there is a term $\sqrt{1-x^2}$,

which is of the form $\sqrt{a^2-x^2}$

Comparing $\sqrt{1-x^2}$ and $\sqrt{a^2-x^2}$ we get, $a = 1$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$dx = \cos\theta\, d\theta$

When $x = 0, \quad \theta = \sin^{-1}0 = 0$

$x = 1, \quad \theta = \sin^{-1}1 = \frac{\pi}{2}$

$\therefore \theta : 0 \to \frac{\pi}{2}$

Step II : $\therefore \; I = \int_0^{\pi/2} \frac{\sin^7\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta\, d\theta \quad \ldots(1)$

→ **Using standard formula in equation (1)**

$\ldots[\sin^2\theta + \cos^2\theta = 1 \Rightarrow 1-\sin^2\theta = \cos^2\theta]$

Step III : $\therefore \; I = \int_0^{\pi/2} \frac{\sin^7\theta}{\sqrt{\cos^2\theta}} \cdot \cos\theta\, d\theta$

$$= \int_0^{\pi/2} \frac{\sin^7\theta}{\cos\theta} \cdot \cos\theta\, d\theta = \int_0^{\pi/2} \sin^7\theta\, d\theta \quad \ldots(2)$$

→ **Using standard formula in equation (2),**

$$\ldots\left[\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \text{ if n is even positive integer}\right.$$

$$\left. = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{2}{3} \cdot 1, \text{ if n is odd positive integer}\right]$$

Here $n = 7$, an odd positive integer

Step IV

$$I = \left(\frac{7-1}{7}\right) \cdot \left(\frac{7-3}{7-2}\right) \cdot \left(\frac{7-5}{7-4}\right) \cdot 1 = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = \mathbf{\frac{16}{35}}$$

Example 6.6.2

Evaluate $\int_0^1 \frac{x^2(4-x^2)}{\sqrt{1-x^2}}\, dx$

Solution :

Step I : Let $I = \int_0^1 \frac{x^2(4-x^2)}{\sqrt{1-x^2}}\, dx$

Since there is a term $\sqrt{1-x^2}$,

which is of the form $\sqrt{a^2-x^2}$

Comparing $\sqrt{1-x^2}$ and $\sqrt{a^2-x^2}$ we get, $a = 1$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$dx = \cos\theta\, d\theta$

When $x = 0, \quad \theta = \sin^{-1}0 = 0$

$x = 1, \quad \theta = \sin^{-1}1 = \frac{\pi}{2}$

$\therefore \theta : 0 \to \frac{\pi}{2}$

Step II : $\therefore \; I = \int_0^{\pi/2} \frac{(\sin^2\theta)(4-\sin^2\theta)}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta\, d\theta \quad \ldots(1)$

→ **Using standard formula in equation (1)**

$\ldots[\sin^2\theta + \cos^2\theta = 1 \Rightarrow 1-\sin^2\theta = \cos^2\theta]$

Step III : $\therefore \; I = \int_0^{\pi/2} \frac{(\sin^2\theta)(4-\sin^2\theta)}{\sqrt{\cos^2\theta}} \cdot \cos\theta\, d\theta$

$$= \int_0^{\pi/2} \frac{(\sin^2\theta)(4-\sin^2\theta)}{\cos\theta} \cdot \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} (\sin^2\theta)(4-\sin^2\theta)\, d\theta$$

$$= \int_0^{\pi/2} 4\sin^2\theta\, d\theta - \int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$= 4\int_0^{\pi/2} \sin^2\theta\, d\theta - \int_0^{\pi/2} \sin^4\theta\, d\theta \quad \ldots(2)$$

→ **Using standard formula in equation (2),**

$$\ldots\left[\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \text{ if n is even positive integer}\right.$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{2}{3} \cdot 1, \text{ if n is odd positive integer}$$

Here In first integral n = 2 In first integral n = 4, an even positive integer]

Step IV : $I = 4\left(\frac{2-1}{2}\right)\frac{\pi}{2} - \left(\frac{4-1}{4}\right) \cdot \left(\frac{4-3}{4-2}\right) \cdot \frac{\pi}{2}$

$$= 4\left(\frac{1}{2} \cdot \frac{\pi}{2}\right) - \left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right) = \pi - \frac{3\pi}{16}$$

$$= \frac{16\pi}{16} - \frac{3\pi}{16} = \frac{16\pi - 3\pi}{16} = \mathbf{\frac{13\pi}{16}}$$

Example 6.6.3

Evaluate $\int_0^a x^2(a^2-x^2)^{5/2}\, dx$

Solution :

Step I : Let $I = \int_0^a x^2(a^2-x^2)^{5/2}\, dx$

Since there is a term $\sqrt{a^2-x^2}$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$dx = \cos\theta\, d\theta$

When $x = 0, \quad \theta = \sin^{-1}0 = 0$

$x = 1, \quad \theta = \sin^{-1}1 = \frac{\pi}{2}$

$\therefore \theta : 0 \to \frac{\pi}{2}$

Step II: $\therefore \; I = \int_0^{\pi/2} a^2\sin^2\theta\,(a^2 - a^2\sin^2\theta)^{5/2}\, a\cos\theta\, d\theta$

$$= \int_0^{\pi/2} (a^2\sin^2\theta)(a^2)^{5/2}(1-\sin^2\theta)^{5/2}\, a\cos\theta\, d\theta \quad \ldots(1)$$

→ **Using standard formula in equation (1)**

$\ldots[\sin^2\theta + \cos^2\theta = 1 \Rightarrow 1-\sin^2\theta = \cos^2\theta]$

Step III

$$\therefore \; I = \int_0^{\pi/2} (a^2\sin^2\theta)(a^2)^{5/2}(\cos^2\theta)^{5/2}\, a\cos\theta\, d\theta \quad \ldots(2)$$

→ **Using standard formula of index rule (2)**

$\ldots[(a^m)^n = a^{mn}]$

$\therefore$ Here for the term $(a^2)^{5/2}$, $a = a^2$, $m = 2$, $n = \frac{5}{2}$

$\therefore (a^2)^{5/2} = a^{2 \times 5/2} = a^5$

Also for the term $(\cos^2\theta)^{5/2}$, $a = \cos^2\theta$, $m = 2$, $n = \frac{5}{2}$

$\therefore (\cos^2\theta)^{5/2} = \cos\theta^{2 \times 5/2} = \cos^5\theta$

Step IV: $\therefore \; I = \int_0^{\pi/2} (a^2\sin^2\theta)(a^5)(\cos^5\theta)\, a\cos\theta\, d\theta$

$$= \int_0^{\pi/2} (a^2 a^5 a)(\sin^2\theta)(\cos^5\theta\cos\theta)\, d\theta$$

$$= \int_0^{\pi/2} a^8(\sin^2\theta)(\cos^6\theta)\, d\theta = a^8\int_0^{\pi/2} (\sin^2\theta)(\cos^6\theta)\, d\theta \quad \ldots(3)$$

→ **Using standard reduction formula in equation (1)**

$$\ldots\left[\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{[(m-1)(m-3)(m-5)\ldots\times 2 \text{ or } 1]\,[(n-1)(n-3)(n-5)\ldots\times 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1} \times k\right.$$

Where $k = \pi/2$ if m and n both are even integers

$\left. = 1 \text{ for all other values of m and n}\right]$

Here $m = 2$, $n = 6$

Step V

$$\therefore I = \frac{[(2-1)]\,[(6-1)(6-3)(6-5)]}{(2+6)(2+6-2)(2+6-4)(2+6-6)} \times k$$

Here both m and n are even integers, $k = \frac{\pi}{2}$

Step VI : $\therefore \; I = a^8 \frac{[(1)]\,[(5)(3)(1)]}{(8)(8-2)(8-4)(8-6)} \times k$

$$= a^8 \frac{(1) \cdot (5 \cdot 3)}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \mathbf{\frac{5\pi a^8}{256}}$$

Example 6.6.4

Evaluate $\int_0^1 x^5 \sin^{-1} x \, dx$

Solution :

Step I $\quad I = \int_0^1 x^5 \sin^{-1} x \, dx \quad \ldots(1)$

→ **Using standard integrating by parts rule in equation(1)**

$$\ldots\left[\int u\, v\, dx = u \int v\, dx - \int\left(\frac{d}{dx}u\right)\left(\int v\, dx\right) dx\right]$$

Here taking $u = \sin^{-1} x$, $v = x^5$

Step II

$$I = \sin^{-1} x \int_0^1 x^5 dx - \int_0^1 \left(\left(\frac{d}{dx}\sin^{-1} x\right) \int x^5 dx\right) dx \quad \ldots(2)$$

→ **Using standard formula of derivative and integration in equation (2)**

$$\ldots\left[\int x^m dx = \frac{x^{m+1}}{m+1} \text{ .Here } m = 5 \Rightarrow \int x^5 dx = \frac{x^{5+1}}{5+1} = \frac{x^6}{6} \quad \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}\right]$$

Step III

$$\therefore I = \left[(\sin^{-1} x)\left(\frac{x^6}{6}\right)\right]_0^1 - \int_0^1 \left(\frac{1}{\sqrt{1-x^2}}\right)\left(\frac{x^6}{6}\right) dx$$

$$= \left[(\sin^{-1} 1)\left(\frac{1^6}{6}\right) - (\sin^{-1} 0)\left(\frac{0^6}{6}\right)\right] - \int_0^1 \left(\frac{1}{\sqrt{1-x^2}}\right)\left(\frac{x^6}{6}\right) dx \quad \ldots(3)$$

→ **Using standard values in equation (3)**

$$\ldots\left[\sin^{-1}(1) = \frac{\pi}{2}, \sin^{-1}(0) = 0\right]$$

Step IV

$$\therefore I = \left[\left(\frac{\pi}{2}\right)\left(\frac{1}{6}\right) - (0)\left(\frac{0}{6}\right)\right] - \int_0^1 \left(\frac{1}{\sqrt{1-x^2}}\right)\left(\frac{1}{6}\right) x^6 dx$$

$$= \left[\left(\frac{\pi}{12}\right) - 0\right] - \frac{1}{6}\int_0^1 \left(\frac{1}{\sqrt{1-x^2}}\right) x^6 dx$$

$$= \frac{\pi}{12} - \frac{1}{6}\int_0^1 \frac{x^6}{\sqrt{1-x^2}} dx \quad \ldots(4)$$

Step V : Since there is a term $\sqrt{1-x^2}$ in equation (4) , which is of the form $\sqrt{a^2 - x^2}$

Comparing $\sqrt{1-x^2}$ and $\sqrt{a^2-x^2}$ we get , a = 1

Put $\quad x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$dx = \cos\theta \, d\theta$

When $\quad x = 0, \quad \theta = \sin^{-1}0 = 0$

$x = 1, \quad \theta = \sin^{-1}1 = \frac{\pi}{2}$

$\therefore \theta : 0 \rightarrow \frac{\pi}{2}$

Step VI : ∴ Equation (4) ⇒

$$\therefore I = \int_0^{\pi/2} \frac{(\sin^6\theta)}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta \, d\theta \quad \ldots(5)$$

→ **Using standard formula in equation (5)**

$\ldots[\sin^2\theta + \cos^2\theta = 1 \Rightarrow 1 - \sin^2\theta = \cos^2\theta]$

Step VII

$$\therefore I = \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2} \frac{\sin^6\theta}{\sqrt{\cos^2\theta}} \cdot \cos\theta \, d\theta$$

$$= \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2} \frac{\sin^6\theta}{\cos\theta} \cdot \cos\theta \, d\theta$$

$$= \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2} \sin^6\theta \, d\theta \quad \ldots(6)$$

→ **Using standard formula in equation (1),**

$$\ldots\left[\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4} \cdots\cdots \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \text{ if n is even positive integer}\right.$$

$$\left. = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4} \cdots\cdots \frac{2}{3}\cdot 1, \text{ if n is odd positive integer}\right]$$

Here n = 6 , an even positive integer

Step VIII : $\therefore \quad I = \frac{\pi}{12} - \frac{1}{6}\left(\frac{6-1}{6}\right)\left(\frac{6-3}{6-2}\right)\cdot\left(\frac{6-5}{6-4}\right)\cdot\frac{\pi}{2}$

$$= \frac{\pi}{12} - \frac{1}{6}\left(\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) = \frac{\pi}{12} - \frac{1}{6}\left(\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= \frac{\pi}{12} - \frac{1}{6}\left(\frac{5\pi}{32}\right) = \mathbf{\frac{\pi}{12} - \frac{5\pi}{192}}$$

Example 6.6.5

Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$

Solution : Step I : Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$

Since there is a term $1 + x^2$, which is of the form $a^2 + x^2$

Comparing $1 + x^2$ and $a^2 + x^2$, we get , a = 1

Put $\quad x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$dx = \sec^2\theta \, d\theta$

When $\quad x = 0, \quad \theta = \tan^{-1}0 = 0$

$x = 1, \quad \theta = \tan^{-1}1 = \frac{\pi}{4}$

$\therefore \theta : 0 \rightarrow \frac{\pi}{4}$

Step II: $\quad \therefore I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{(1+\tan^2\theta)} \sec^2\theta \, d\theta \quad \ldots(1)$

→ **Using standard trigonometric identity in equation (1),**

$\ldots[1 + \tan^2\theta = \sec^2\theta]$

$$\therefore I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta} \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta \quad \ldots(2)$$

→ **Using property of definite integral in equation (2),**

$$\ldots\left[\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

Here $x = \theta$, $a = \frac{\pi}{4}$

Step III

$$\therefore I = \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - \theta\right)\right] d\theta \quad \ldots(3)$$

→ **Using standard trigonometric identity in equation (3),**

$$\ldots\left[\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}\right]$$

Here $A = \frac{\pi}{4}$, $B = \theta$

Step IV : $\therefore I = \int_0^{\pi/4} \log\left[1 + \frac{\tan\pi/4 - \tan\theta}{1 + \tan\pi/4 \cdot \tan\theta}\right] d\theta \ldots(4)$

→ **Using standard value in equation (4),**

$$\ldots\left[\tan\frac{\pi}{4} = 1\right]$$

Step V : $\quad \therefore I = \int_0^{\pi/4} \log\left[1 + \frac{1-\tan\theta}{1+\tan\theta}\right] d\theta$

$$= \int_0^{\pi/4} \log\left[\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right] d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta \quad \ldots(5)$$

→ **Using standard Logarithm formula in equation (4),**

$$\ldots\left[\log\frac{A}{B} = \log A - \log B\right]$$

Here A = 2 , B = 1 + tanθ

Step VI :∴ $\quad I = \int_0^{\pi/4} (\log 2 - \log(1+\tan\theta))\, d\theta$

$$= \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta \quad \ldots(6)$$

∵ From equation (1) $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = I$

Equation (6) ⇒

$$I = \int_0^{\pi/4} \log 2\, d\theta - I$$

$$I + I = \int_0^{\pi/4} \log 2\, d\theta$$

$$2I = \log 2 \int_0^{\pi/4} d\theta = \log 2\, [\theta]_0^{\pi/4}$$

$$= (\log 2)\left(\frac{\pi}{4} - 0\right) = (\log 2)\left(\frac{\pi}{4}\right)$$

$$\therefore I = \frac{\log 2}{2}\left(\frac{\pi}{4}\right) = \frac{\pi \log 2}{8}$$

6.7 Properties of Definite Integrals

I. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Proof : L.H.S. $= \int_0^a f(x)\,dx$

Put $x = a - t \Rightarrow dx = -dt$

When $x = 0,\ t = a$,

When $x = a,\ t = 0$

$\therefore$ L.H.S. $= \int_a^0 f(a-t)(-dt) = \int_0^a f(a-t)\,dt$

$\therefore \int_0^a f(x)\,dx = \int_0^a \mathbf{f(a-x)\,dx}$

II. $\int_{-a}^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(-x)\,dx$

Proof We know that,

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx \quad \ldots(1)$$

Consider, $I = \int_{-a}^0 f(x)\,dx$

Put $x = -t \Rightarrow dx = -dt$

When $x = -a,\ t = a$,

when $x = 0,\ t = 0$

$I = \int_a^0 f(-t)\,(-dt) = \int_0^a f(-t)\,dt$

$I = \int_0^a f(-x)\,dx$

(As value of definite integral is independent of variable changing t to x)

$\therefore$ Equation (1) becomes,

$$\int_{-a}^a f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx$$

Hence proof

Note : We can easily prove that

(i) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ **if f(x) is even function**

(ii) $\int_{-a}^a f(x)\,dx = 0$ **if f(x) is odd function**

III. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^a f(2a-x)\,dx$

Proof

We have, $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \quad \ldots(1)$

Consider, $I = \int_a^{2a} f(x)\,dx$

Put $x = 2a - t \quad dx = -dt$

When $x = a,\ t = a$

When, $x = 2a,\ t = 0$

$\therefore I = \int_0^a f(2a-t)\,(-dt) = \int_0^a f(2a-t)\,dt$

$= \int_0^a f(2a-x)\,dx$

Equation (1) becomes,

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

Hence proof.

IV. If $f(a-x) = -f(a+x)$ prove that $\int_0^{2a} f(x)\,dx = 0$

Proof : We have,

$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Also, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$\therefore \int_0^{2a} f(x)\,dx = \int_0^a f(a-x)\,dx + \int_0^a f[a-(2a-x)]\,dx$

Note :

If, any integral is not of the form,

$\int_0^{\pi/2} \sin^m x\,dx$ or $\int_0^{\pi/2} \cos^m x\,dx$

or $\int_0^{\pi/2} \sin^m x \cos^n x\,dx$

Use above properties and solve the examples by simplifying it.

i.e. if limits of integration are 0 to $\pi/2$ but integrand is not of the form $\sin^m x$, $\cos^m x$, $\sin^m x \cos^n x$ or if integrand are of the above form but limits are not 0 to $\pi/2$, then above formulae of definite integral should be used.

Formulae : From above properties of definite integral we obtain following results.Students are advised to remember these results as standard formulae.

1. $\int_0^{\pi} \sin^n x\,dx = 2\int_0^{\pi/2} \sin^n x\,dx \quad \forall$ all positive integer n.

2. $\int_0^{\pi} \cos^n x\,dx = 2\int_0^{\pi/2} \cos^n x\,dx$, if n is an even integer

   $= 0$ if n is an odd integer

3. $\int_0^{2\pi} \sin^n x\,dx = 4\int_0^{\pi/2} \sin^n x\,dx$, if n is an even integer

   $= 0$ if n is an odd integer

4. $\int_0^{2\pi} \cos^n x\,dx = 4\int_0^{\pi/2} \cos^n x\,dx$, if n is an even integer

   $= 0$ if n is an odd integer

5. $\int_0^{\pi} \sin^m x \cos^n x\,dx$

   $= 2\int_0^{\pi/2} \sin^m x \cos^n x\,dx$ if n = even integer

   $= 0$ if n = odd integer

   m, may be even or may be odd integer

6. $\int_0^{2\pi} \sin^m x \cos^n x\,dx = 4\int_0^{\pi/2} \sin^m x \cos^n x\,dx$ if both m, n are even integers

   $= 0$ otherwise

7. $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if f(x) is an even function

   $= 0$, if f(x) is an odd function

Type I : Examples of the form $\int_0^{\pi} f(x)\,dx$

Example 6.7.1

Evaluate $\int_0^{\pi} \sin^5 x\,dx$

Solution :

Step I : Let $I = \int_0^{\pi} \sin^5 x\,dx \quad \ldots(1)$

→ **Using standard formula in equation (1)**

$$\ldots\left[\int_0^{\pi} \sin^n x\,dx = 2\int_0^{\pi/2} \sin^n x\,dx, \quad \forall \text{ all positive integer n}\right]$$

Step II : $\therefore I = 2\int_0^{\pi/2} \sin^5 x\,dx \quad \ldots(2)$

→ **Using standard formula in equation (2),**

$$\ldots\left[\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \text{ if n is even positive integer}\right.$$

$$\left.= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\cdot 1, \text{ if n is odd positive integer}\right]$$

$$\therefore I = 2\left[\left(\frac{5-1}{5}\right)\left(\frac{5-3}{5-2}\right)\times k\right]$$

Here n = 5 , an odd positive integer $\Rightarrow k = 1$

Step III : $\therefore I = 2\left[\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)\times 1\right] = \frac{16}{15}$

Example 6.7.2

Evaluate $\int_0^{\pi} \sin^6 x \, dx$

Solution :

Step I : Let $I = \int_0^{\pi} \sin^6 x \, dx$ ...(1)

→ Using standard formula in equation (1)

$$\ldots\left[\int_0^{\pi} \sin^n x \, dx = 2\int_0^{\pi/2} \sin^n x \, dx, \quad \forall \text{ all positive integer } n\right]$$

$$\therefore \quad I = 2\int_0^{\pi/2} \sin^6 x \, dx \qquad \ldots(2)$$

→ Using standard formula in equation (2),

$$\ldots\left[\begin{aligned}\int_0^{\pi/2} \sin^n x \, dx &= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \quad \text{if n is even positive integer}\\ &= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\cdot 1, \quad \text{if n is odd positive integer}\end{aligned}\right]$$

Step II : $\therefore I = 2\left[\left(\frac{6-1}{6}\right)\left(\frac{6-3}{6-2}\right)\left(\frac{6-5}{6-4}\right)\times k\right]$

Here n = 6 , an even positive integer $\Rightarrow k = \frac{\pi}{2}$

$\therefore I = 2\left[\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\times\frac{\pi}{2}\right] = \frac{5\pi}{16}$

Example 6.7.3

Evaluate $\int_0^{\pi} \cos^4 x \, dx$

Solution :

Step I : Let $I = \int_0^{\pi} \cos^4 x \, dx$ ...(1)

→ Using standard formula in equation (1)

$$\ldots\left[\begin{aligned}\int_0^{\pi} \cos^n x \, dx &= 2\int_0^{\pi/2} \cos^n x \, dx, \quad \text{if n is an even integer}\\ &= 0 \quad \text{if n is an odd integer}\end{aligned}\right]$$

Here n = 4 , an even integer

$$\therefore I = 2\int_0^{\pi/2} \cos^4 x \, dx \qquad \ldots(2)$$

→ Using standard formula in equation (2),

$$\ldots\left[\begin{aligned}\int_0^{\pi/2} \cos^n x \, dx &= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \quad \text{if n is even positive integer}\\ &= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\cdot 1, \quad \text{if n is odd positive integer}\end{aligned}\right]$$

Step II : $\therefore I = 2\left[\left(\frac{4-1}{4}\right)\left(\frac{4-3}{4-2}\right)\times k\right]$

Here n = 4 , an even positive integer $\Rightarrow k = \frac{\pi}{2}$

Step III : $\therefore I = 2\left[\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\times\frac{\pi}{2}\right] = \frac{3\pi}{8}$

Example 6.7.4 Evaluate $\int_0^{\pi} \cos^7 x \, dx$

Solution :

Step I : Let $I = \int_0^{\pi} \cos^7 x \, dx$ ...(1)

→ Using standard formula in equation (1)

$$\ldots\left[\begin{aligned}\int_0^{\pi} \cos^n x \, dx &= 2\int_0^{\pi/2} \cos^n x \, dx, \quad \text{f n is an even integer}\\ &= 0 \quad \text{if n is an odd integer}\end{aligned}\right]$$

Here n = 7 , an odd integer

$\therefore I = 0$

Example 6.7.5

Evaluate $\int_0^{\pi} \sin^3 x \cos^5 x \, dx$

Solution :

Step I : Let $I = \int_0^{\pi} \sin^3 x \cos^5 x \, dx$ ...(1)

→ Using standard formula in equation (1)

$$\ldots\left[\left.\begin{aligned}\int_0^{\pi} \sin^m x \cos^n x \, dx &= 2\int_0^{\pi/2} \sin^m x \cos^n x \, dx \quad \text{if n = even integer}\\ &= 0 \quad \text{if n = odd integer}\end{aligned}\right\} \text{m, may be even or may be odd integer}\right]$$

Here m = 3 , n = 5

Since n is an odd i.e. power of cosine term is odd

$\therefore I = 0$

Example 6.7.6

Evaluate $\int_0^{\pi} \sin^4 x \cos^3 x \, dx$

Solution :

Step I : Let $I = \int_0^{\pi} \sin^4 x \cos^3 x \, dx$ ...(1)

→ Using standard formula in equation (1)

$$\ldots\left[\left.\begin{aligned}\int_0^{\pi} \sin^m x \cos^n x \, dx &= 2\int_0^{\pi/2} \sin^m x \cos^n x \, dx \quad \text{if n = even integer}\\ &= 0 \quad \text{if n = odd integer}\end{aligned}\right\} \text{m, may be even or may be odd integer}\right]$$

Here m = 4 , n = 3

Since n is an odd i.e. power of cosine term is odd

$\therefore I = 0$

Example 6.7.7

Evaluate $\int_0^{\pi} \sin^4 x \cos^6 x \, dx$

Solution :

Step I : Let $I = \int_0^{\pi} \sin^4 x \cos^6 x \, dx$ ...(1)

→ **Using standard formula in equation (1)**

$$\ldots\left[\begin{array}{ll}\int_0^{\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi/2} \sin^m x \cos^n x\,dx & \text{if } n = \text{even integer} \\ = 0 & \text{if } n = \text{odd integer}\end{array}\right\} \begin{array}{l}m\text{, may be even or} \\ \text{may be odd integer}\end{array}\right]$$

Here $m = 4$, $n = 6$

Since n is an even i.e. power of cosine term is even

Step II : $\therefore I = 2\int_0^{\pi/2} \sin^4 x \cos^6 x\ dx$ ...(2)

→ **Using standard reduction formula in equation (2)**

$$\ldots\left[\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{[(m-1)(m-3)(m-5)\ldots\times 2 \text{ or } 1]\,[(n-1)(n-3)(n-5)\ldots\times 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1}\times k\right.$$

Where $k = \pi/2$ if m and n both are even integer

s = 1 for all other values of m and n

Here $m = 4$, $n = 6$

Step III :

$$\therefore I = 2\times\frac{[(4-1)(4-3)]\,[(6-1)(6-3)(6-5)]}{(4+6)(4+6-2)(4+6-4)(4+6-6)(4+6-8)}\times k$$

Here both m and n are even integers, $k = \frac{\pi}{2}$

$$\therefore I = 2\times\frac{[(3)(1)]\,[(5)(3)(1)]}{(10)(10-2)(10-4)(10-6)(10-8)}\times\frac{\pi}{2} = 2\times\frac{(3)(1)\cdot(5\cdot 3)}{10\cdot 8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \mathbf{\frac{3\pi}{256}}$$

Example 6.7.8

Evaluate $\int_0^{\pi} \sin^3 x \cos^4 x\ dx$

Solution :

Step I : Let $I = \int_0^{\pi} \sin^3 x \cos^4 x\ dx$ ...(1)

→ **Using standard formula in equation (1)**

$$\ldots\left[\begin{array}{ll}\int_0^{\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi/2} \sin^m x \cos^n x\,dx & \text{if } n = \text{even integer} \\ = 0 & \text{if } n = \text{odd integer}\end{array}\right\} \begin{array}{l}m\text{, may be even or may be} \\ \text{odd integer}\end{array}\right]$$

Here $m = 3$, $n = 4$

Since n is an even i.e. power of cosine term is even

Step II :

$$\therefore I = 2\int_0^{\pi/2} \sin^4 x \cos^6 x\ dx \quad \ldots(2)$$

→ **Using standard reduction formula in equation (2)**

$$\ldots\left[\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{[(m-1)(m-3)(m-5)\ldots\times 2 \text{ or } 1]\,[(n-1)(n-3)(n-5)\ldots\times 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1}\times k\right.$$

Where $k = \pi/2$ if m and n both are even integers = 1 for all other values of m and n

Here $m = 4$, $n = 6$

Step III : $\therefore$

$$I = 2\times\frac{[(4-1)(4-3)]\,[(6-1)(6-3)(6-5)]}{(4+6)(4+6-2)(4+6-4)(4+6-6)(4+6-8)}\times k$$

Here both m and n are even integers, $k = \frac{\pi}{2}$

$$\therefore I = 2\times\frac{[(3)(1)]\,[(5)(3)(1)]}{(10)(10-2)(10-4)(10-6)(10-8)}\times\frac{\pi}{2} = 2\times\frac{(3)(1)\cdot(5\cdot 3)}{10\cdot 8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \mathbf{\frac{3\pi}{256}}$$

Type II : Examples of the form $\int_0^{2\pi} x\, f(x)\,dx$

Example 6.7.9

Evaluate $\int_0^{2\pi} \sin^8 x\ dx$

Solution :

Step I :

$$\text{Let } I = \int_0^{2\pi} \sin^8 x\ dx \quad \ldots(1)$$

→ **Using standard formula in equation (1)**

$$\ldots\left[\begin{array}{ll}\int_0^{2\pi} \sin^n x\,dx = 4\int_0^{\pi/2} \sin^n x\,dx, & \text{if } n \text{ is an even integer} \\ = 0, & \text{if } n \text{ is an odd integer}\end{array}\right]$$

Step II : $\therefore I = 4\int_0^{\pi/2} \sin^8 x\ dx$ ...(2)

→ **Using standard formula in equation (2),**

$$\ldots\left[\begin{array}{l}\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \\ \qquad\qquad \text{if n is even positive integer} \\ = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\cdot 1, \\ \qquad\qquad \text{if n is odd positive integer}\end{array}\right]$$

Step III :

$$\therefore I = 4\left[\left(\frac{8-1}{8}\right)\left(\frac{8-3}{8-2}\right)\left(\frac{8-5}{8-4}\right)\left(\frac{8-7}{8-6}\right)\times k\right]$$

Here $n = 8$, an even positive integer $\Rightarrow k = \frac{\pi}{2}$

$$\therefore I = 4\left[\left(\frac{7}{8}\right)\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\times\frac{\pi}{2}\right] = \mathbf{\frac{35\pi}{64}}$$

Example 6.7.10

Evaluate $\int_0^{2\pi} \sin^7 x\ dx$

Solution :

Step I : Let $I = \int_0^{2\pi} \sin^7 x\ dx$ ...(1)

→ **Using standard formula in equation (1)**

$$\therefore \int_0^{2\pi} \sin^n x\,dx = 4\int_0^{\pi/2} \sin^n x\,dx, \text{ if n is an even integer}$$
$$= 0, \quad \text{if n is an odd integer}$$

Here n = 7, an odd integer

$\therefore I = 0$

Example 6.7.11

Evaluate $\int_0^{2\pi} \cos^4 x\ dx$

Solution :

Step I : Let $I = \int_0^{2\pi} \cos^4 x\ dx$ ...(1)

→ **Using standard formula in equation (1)**

$$\therefore \int_0^{2\pi} \cos^n x\,dx = 4\int_0^{\pi/2} \cos^n x\,dx, \text{ if n is an even integer}$$
$$= 0, \quad \text{if n is an odd integer}$$

Step II : $\therefore I = 4\int_0^{\pi/2} \cos^4 x\ dx$ ...(2)

→ **Using standard formula in equation (2),**

$$\therefore \int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \quad \text{if n is even positive integer}$$
$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\cdot 1, \quad \text{if n is odd positive integer}$$

Step III : $\therefore I = 4\left[\left(\frac{4-1}{4}\right)\left(\frac{4-3}{4-2}\right)\times k\right]$

Here n = 4, an even positive integer $\Rightarrow k = \frac{\pi}{2}$

$$\therefore I = 4\left[\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\times\frac{\pi}{2}\right] = \frac{3\pi}{4}$$

Example 6.7.12

Evaluate $\int_0^{2\pi} \cos^4 x\ dx$

Solution :

Step I : Let $I = \int_0^{2\pi} \cos^4 x\ dx$ ...(1)

→ **Using standard formula in equation (1)**

$$\therefore \int_0^{2\pi} \cos^n x\,dx = 4\int_0^{\pi/2} \cos^n x\,dx, \text{ if n is an even integer}$$
$$= 0, \quad \text{if n is an odd integer}$$

Step II : $\therefore I = 4\int_0^{\pi/2} \cos^4 x\ dx$ ...(2)

→ **Using standard formula in equation (2),**

$$\therefore \int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \quad \text{if n is even positive integer}$$
$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\cdot 1, \quad \text{if n is odd positive integer}$$

Step III : $\therefore I = 4\left[\left(\frac{4-1}{4}\right)\left(\frac{4-3}{4-2}\right)\times k\right]$

Here n = 4, an even positive integer $\Rightarrow k = \frac{\pi}{2}$

$$\therefore I = 4\left[\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\times\frac{\pi}{2}\right] = \frac{3\pi}{4}$$

Example 6.7.13

Evaluate $\int_0^{2\pi} \cos^5 x\ dx$

Solution :

Step I : Let $I = \int_0^{2\pi} \cos^5 x\ dx$ ...(1)

→ **Using standard formula in equation (1)**

$$\therefore \int_0^{2\pi} \cos^n x\,dx = 4\int_0^{\pi/2} \cos^n x\,dx, \text{ if n is an even integer}$$
$$= 0, \quad \text{if n is an odd integer}$$

Here n = 5, an odd positive integer $\therefore I = 0$

Example 6.7.14

Evaluate $\int_0^{2\pi} \sin^4 x\cos^2 x\ dx$

Solution :

Step I : Let $I = \int_0^{2\pi} \sin^4 x\cos^2 x\ dx$ ...(1)

→ **Using standard formula in equation (1)**

$$\therefore \int_0^{2\pi} \sin^m x\cos^n x\,dx = 4\int_0^{\pi/2} \sin^m x\cos^n x\,dx \quad \text{if both m, n are even integers}$$
$$= 0$$

Here m = 4, n = 2

Since m and n both are an even positive integer

Step II : $\therefore I = 4\int_0^{2\pi} \sin^4 x\cos^2 x\ dx$ ...(2)

→ **Using standard reduction formula in equation (2)**

$$\therefore \int_0^{\pi/2} \sin^m x\cos^n x\,dx = \frac{[(m-1)(m-3)(m-5)\ldots 2 \text{ or } 1]\,[(n-1)(n-3)(n-5)\ldots 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1}\times k$$

Where $k = \pi/2$ if m and n both are even integers

$= 1$ for all other values of m and n

Here m = 4, n = 2

Step III : $\therefore I = 4 \times \frac{[(4-1)(4-3)]\ [(2-1)]}{(4+2)(4+2-2)(4+2-4)} \times k$

Here both m and n are even integers, $k = \frac{\pi}{2}$

Step IV : $\therefore I = 4 \times \frac{[(3)(1)]\ [(1)]}{(6)(6-2)(6-4)} \times \frac{\pi}{2} = 4 \times \frac{(3)(1)\cdot(1)}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{\pi}{8}$

Example 6.7.15

Evaluate $\int_0^{2\pi} \sin^6 x \cos^4 x \; dx$

Solution :

Step I : Let $I = \int_0^{2\pi} \sin^6 x \cos^4 x \; dx$ ...(1)

→ **Using standard formula in equation (1)**

$$\ldots\left[\begin{aligned}\int_0^{2\pi} \sin^m x \cos^n x\, dx &= 4 \int_0^{\pi/2} \sin^m x \cos^n x\, dx \quad \text{if both m, n are even integers} \\ &= 0\end{aligned}\right]$$

Here m = 6, n = 4

Since m and n both are an even positive integer

Step II : $\therefore I = 4 \int_0^{2\pi} \sin^6 x \cos^4 x \; dx$ ...(2)

→ **Using standard reduction formula in equation (2)**

$$\ldots\left[\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{[(m-1)(m-3)(m-5)\ldots 2 \text{ or } 1]\ [(n-1)(n-3)(n-5)\ldots 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1} \times k\right]$$

Where $k = \pi/2$ if m and n both are even integers

$= 1$ for all other values of m and n

Here m = 6, n = 4

Step III : $\therefore I = 4 \times \frac{[(6-1)(6-3)(6-5)]\ [(4-1)(4-3)]}{(6+4)(6+4-2)(6+4-4)(6+4-6)(6+4-8)} \times k$

Here both m and n are even integers, $k = \frac{\pi}{2}$

Step IV : $\therefore I = 4 \times \frac{[(5)(3)(1)]\ [(3)(1)]}{(10)(10-2)(10-4)(10-6)(10-8)} \times \frac{\pi}{2} = 4 \times \frac{(5)(3)(1)\cdot(3)}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{3\pi}{128}$

Example 6.7.16

Evaluate $\int_0^{2\pi} \sin^3 x \cos^6 x \; dx$

Solution :

Step I : Let $I = \int_0^{2\pi} \sin^3 x \cos^6 x \; dx$ ...(1)

→ **Using standard formula in equation (1)**

$$\ldots\left[\begin{aligned}\int_0^{2\pi} \sin^m x \cos^n x\, dx &= 4 \int_0^{\pi/2} \sin^m x \cos^n x\, dx \quad \text{if both m, n are even integers} \\ &= 0\end{aligned}\right]$$

Here m = 3, n = 6

Since m and n both are an even positive integer

$\therefore I = 0$

Type III : Examples of the form $\int_0^{\pi} x f(x)\, dx$

Example 6.7.17

Evaluate $\int_0^{\pi} x \cos^6 x\, dx$

Solution :

Step I : Let, $I = \int_0^{\pi} x \cos^6 x\, dx$ ...(1)

→ **Using standard formula of definite integral in equation (1)**

$$\ldots\left[\int_0^{a} f(x)\, dx = \int_0^{a} f(a-x) dx \quad \text{Here } a = \pi,\ f(x) = x\cos^6 x\right]$$

Step II : $\therefore I = \int_0^{\pi} (\pi - x) \cos^6 (\pi - x)\, dx$ ...(2)

→ **Using standard trigonometric formula in equation (2)**

...[$\cos(\pi - x) = \cos\pi \cos x + \sin\pi \sin x = -\cos x$

$\therefore \cos^n(\pi - x) = \cos^n x$, if n is even integer

$= -\cos^n x$, if n is odd integer]

Here n = 6, an even integer

$\therefore \cos^6(\pi - x) = \cos^6 x$

Step III : $\therefore I = \int_0^{\pi} (\pi - x) \cos^6 x\, dx$

$= \int_0^{\pi} (\pi \cos^6 x - x \cos^6 x)\, dx$

$I = \int_0^{\pi} \pi \cos^6 x\, dx - \int_0^{\pi} x \cos^6 x\, dx$ ...(3)

Step IV : Adding Equations (1) and (3) we get,

$I + I = \int_0^{\pi} x \cos^6 x\, dx + \int_0^{\pi} \pi \cos^6 x\, dx - \int_0^{\pi} x \cos^6 x\, dx$

$2I = \pi \int_0^{\pi} \cos^6 x\, dx$ ...(4)

→ **Using standard formula in equation (4)**

$$\ldots\left[\begin{aligned}\int_0^{\pi} \cos^n x\, dx &= 2 \int_0^{\pi/2} \cos^n x\, dx, \text{ if n is an even integer} \\ &= 0 \qquad \text{if n is an odd integer}\end{aligned} \quad \text{Here } n = 6, \text{ an even integer}\right]$$

Step V : $\therefore 2I = \pi \times 2 \int_0^{\pi/2} \cos^6 x \; dx$ ...(5)

→ **Using standard formula in equation (5),**

$$\ldots\left[\int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \quad \text{if n is even positive integer}\right.$$

$$\left.= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\cdot 1, \quad \text{if n is odd positive integer}\right]$$

Step VI : $\therefore 2I = 2\pi\left[\left(\frac{6-1}{6}\right)\left(\frac{6-3}{6-2}\right)\left(\frac{6-5}{6-4}\right)\times k\right]$

Here $n = 6$, an even positive integer $\Rightarrow k = \frac{\pi}{2}$

$$I = \pi\left[\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\times\frac{\pi}{2}\right]$$

$$I = \pi\left[\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$I = \frac{5\pi^2}{32}$$

Example 6.7.18

Evaluate $\int_0^{\pi} x\cos^8 x\, dx$

Solution :

Step I : Let, $\quad I = \int_0^{\pi} x\cos^8 x\, dx \quad \ldots(1)$

→ **Using standard formula of definite integral in equation (1)**

$$\ldots\left[\int_0^a f(x)\, dx = \int_0^a f(a-x)dx \quad \text{Here } a = \pi,\ f(x) = x\cos^8 x\right]$$

Step II : $\quad \therefore I = \int_0^{\pi} (\pi - x)\cos^8(\pi - x)\, dx \quad \ldots(2)$

→ **Using standard trigonometric formula in equation (2)**

$\ldots[\cos(\pi-x) = \cos\pi\cos x + \sin\pi\sin x = -\cos x$

$\therefore \cos^n(\pi-x) = \cos^n x$, if n is even integer

$= -\cos^n x$, if n is odd integer]

Here $n = 8$, an even integer

$\therefore \cos^8(\pi - x) = \cos^8 x$

Step III

$$\therefore I = \int_0^{\pi}(\pi - x)\cos^8 x\, dx = \int_0^{\pi}(\pi\cos^8 x - x\cos^8 x)\, dx$$

$$I = \int_0^{\pi}\pi\cos^8 x\, dx - \int_0^{\pi} x\cos^8 x\, dx \quad \ldots(3)$$

Step IV : Adding Equations (1) and (3) we get,

$$I + I = \int_0^{\pi} x\cos^8 x\, dx + \int_0^{\pi}\pi\cos^8 x\, dx - \int_0^{\pi} x\cos^8 x\, dx$$

$$2I = \pi\int_0^{\pi}\cos^8 x\, dx \quad \ldots(4)$$

→ **Using standard formula in equation (4)**

$$\ldots\left[\int_0^{\pi}\cos^n x\, dx = 2\int_0^{\pi/2}\cos^n x\, dx, \text{ if n is an even integer}\right.$$

$$\left.= 0 \quad \text{if n is an odd integer}\right]$$

Here $n = 8$, an even integer

Step V : $\therefore 2I = \pi\times 2\int_0^{\pi/2}\cos^8 x\, dx \quad \ldots(5)$

→ **Using standard formula in equation (5),**

$$\ldots\left[\int_0^{\pi/2}\cos^n x\, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \quad \text{if n is even positive integer}\right.$$

$$\left.= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\cdot 1, \quad \text{if n is odd positive integer}\right]$$

Step VI :

$$\therefore 2I = 2\pi\left[\left(\frac{8-1}{8}\right)\left(\frac{8-3}{8-2}\right)\left(\frac{8-5}{8-4}\right)\left(\frac{8-7}{8-6}\right)\times k\right]$$

Here n = 6, an even positive integer $\Rightarrow k = \frac{\pi}{2}$

$$I = \pi\left[\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\times\frac{\pi}{2}\right]$$

$$I = \pi\left[\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$I = \frac{35}{256}\pi^2$$

Example 6.7.19

Evaluate $\int_0^{\pi} x\cos^4 x\, dx$

Solution :

Step I : Let, $\quad I = \int_0^{\pi} x\cos^4 x\, dx \quad \ldots(1)$

→ **Using standard formula of definite integral in equation (1)**

$$\ldots\left[\int_0^a f(x)\, dx = \int_0^a f(a-x)dx \quad \text{Here } a = \pi,\ f(x) = x\cos^4 x\right]$$

Step II : $\quad \therefore I = \int_0^{\pi}(\pi - x)\cos^4(\pi - x)\, dx \quad \ldots(2)$

→ Using standard trigonometric formula in equation (2)

$\ldots[\cos(\pi-x) = \cos\pi\cos x + \sin\pi\sin x = -\cos x$

$\therefore \cos^n(\pi-x) = \cos^n x$, if n is even integer

$= -\cos^n x$, if n is odd integer]

Here $n = 4$, an even integer

$\therefore \cos^4(\pi - x) = \cos^4 x$

Step III :

$$\therefore I = \int_0^{\pi}(\pi - x)\cos^4 x\, dx = \int_0^{\pi}(\pi\cos^4 x - x\cos^4 x)\, dx$$

$$I = \int_0^{\pi}\pi\cos^4 x\, dx - \int_0^{\pi} x\cos^4 x\, dx \quad \ldots(3)$$

Step IV : Adding Equations (1) and (3) we get,

$$I + I = \int_0^{\pi} x\cos^4 x\, dx + \int_0^{\pi}\pi\cos^4 x\, dx - \int_0^{\pi} x\cos^4 x\, dx$$

$$2I = \pi\int_0^{\pi}\cos^4 x\, dx \quad \ldots(4)$$

→ **Using standard formula in equation (4)**

$$\ldots\left[\int_0^{\pi}\cos^n x\, dx = 2\int_0^{\pi/2}\cos^n x\, dx, \text{ if n is an even integer}\right.$$

$$\left.= 0 \quad \text{f n is an odd integer}\right]$$

Here $n = 4$, an even integer

Step V : $\quad \therefore 2I = \pi\times 2\int_0^{\pi/2}\cos^4 x\, dx \quad \ldots(5)$

→ **Using standard formula in equation (5),**

$$\ldots\left[\int_0^{\pi/2}\cos^n x\, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \quad \text{if n is even positive integer}\right.$$

$$\left.= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\cdot 1, \quad \text{if n is odd positive integer}\right]$$

Step VI : $\quad \therefore 2I = 2\pi\left[\left(\frac{4-1}{4}\right)\left(\frac{4-3}{4-2}\right)\times k\right]$

Here n = 4, an even positive integer $\Rightarrow k = \frac{\pi}{2}$

$$I = \pi\left[\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\times\frac{\pi}{2}\right]$$

$$I = \pi\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{3\pi^2}{16}$$

Example 6.7.20

Evaluate $\int_0^{\pi} x\sin^4 x\, dx$

Solution :

Step I : Let, $\quad I = \int_0^{\pi} x\sin^4 x\, dx \quad \ldots(1)$

→ **Using standard formula of definite integral in equation (1)**

$$\ldots\left[\int_0^a f(x)\, dx = \int_0^a f(a-x)dx\right.$$

$$\left[\text{Here } a = \pi,\ f(x) = x\sin^4 x\right]$$

Step II : $$\therefore I = \int_0^{\pi} (\pi - x)\sin^4(\pi - x)\,dx \qquad ...(2)$$

→ **Using standard trigonometric formula in equation (2)**

$$...\left[\sin(\pi-x) = \sin\pi\cos x - \cos\pi\sin x = \sin x \quad \therefore \sin^n(\pi-x) = \sin^n x,\ \text{for any integer } n\right]$$

Step III

$$\therefore I = \int_0^{\pi}(\pi - x)\sin^4 x\,dx = \int_0^{\pi}(\pi\sin^4 x - x\sin^4 x)\,dx$$

$$I = \int_0^{\pi}\pi\sin^4 x\,dx - \int_0^{\pi} x\sin^4 x\,dx \qquad ...(3)$$

Step IV : Adding Equations (1) and (3) we get,

$$I + I = \int_0^{\pi} x\sin^4 x\,dx + \int_0^{\pi}\pi\sin^4 x\,dx - \int_0^{\pi} x\sin^4 x\,dx$$

$$2I = \pi\int_0^{\pi}\sin^4 x\,dx \qquad ...(4)$$

→ **Using standard formula in equation (4)**

$$...\left[\int_0^{\pi}\sin^n x\,dx = 2\int_0^{\pi/2}\sin^n x\,dx,\ \text{for any integer } n\right]$$

Here n = 4 , an even integer

Step V : $$\therefore 2I = \pi\times 2\int_0^{\pi/2}\sin^4 x\,dx \qquad ...(5)$$

→ **Using standard formula in equation (5),**

$$...\left[\int_0^{\pi/2}\sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},\ \text{if n is even positive integer} \\ = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\cdot 1,\ \text{if n is odd positive integer}\right]$$

Step VI : $$\therefore 2I = 2\pi\left[\left(\frac{4-1}{4}\right)\left(\frac{4-3}{4-2}\right)\times k\right]$$

Here n = 4 , an even positive integer ⇒ $k = \frac{\pi}{2}$

$$I = \pi\left[\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\times\frac{\pi}{2}\right]$$

$$I = \pi\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{3\pi^2}{16}$$

Example 6.7.21

Evaluate $\int_0^{\pi} x\sin^5 x\,dx$

Solution :

Step I : Let, $$I = \int_0^{\pi} x\sin^5 x\,dx \qquad ...(1)$$

→ **Using standard formula of definite integral in equation (1)**

$$...\left[\int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx\right]$$

Here $a = \pi$, $f(x) = x\sin^5 x$

Step II : $$\therefore I = \int_0^{\pi}(\pi - x)\sin^5(\pi - x)\,dx \qquad ...(2)$$

→ **Using standard trigonometric formula in equation (2)**

$$...\left[\sin(\pi-x) = \sin\pi\cos x - \cos\pi\sin x = \sin x \quad \therefore \sin^n(\pi-x) = \sin^n x,\ \text{for any integer } n\right]$$

Step III

$$\therefore I = \int_0^{\pi}(\pi - x)\sin^5 x\,dx = \int_0^{\pi}(\pi\sin^5 x - x\sin^5 x)\,dx$$

$$I = \int_0^{\pi}\pi\sin^5 x\,dx - \int_0^{\pi} x\sin^5 x\,dx \qquad ...(3)$$

Step IV : Adding Equations (1) and (3) we get,

$$I + I = \int_0^{\pi} x\sin^5 x\,dx + \int_0^{\pi}\pi\sin^5 x\,dx - \int_0^{\pi} x\sin^5 x\,dx$$

$$2I = \pi\int_0^{\pi}\sin^5 x\,dx \qquad ...(4)$$

→ **Using standard formula in equation (4)**

$$...\left[\int_0^{\pi}\sin^n x\,dx = 2\int_0^{\pi/2}\sin^n x\,dx,\ \text{for any integer } n\right]$$

Step V : $$\therefore 2I = \pi\times 2\int_0^{\pi/2}\sin^5 x\,dx \qquad ...(5)$$

→ **Using standard formula in equation (5),**

$$...\left[\int_0^{\pi/2}\sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},\ \text{if n is even positive integer} \\ = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\cdot 1,\ \text{if n is odd positive integer}\right]$$

Step VI : $$\therefore 2I = 2\pi\left[\left(\frac{5-1}{5}\right)\left(\frac{5-3}{5-2}\right)\times k\right]$$

Here n = 5 , an odd positive integer ⇒ k = 1

$$I = \pi\left[\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)\times 1\right]$$

$$I = \frac{8\pi}{15}$$

Example 6.7.22

Evaluate $\int_0^{\pi} x\sin^7 x\cos^4 x\,dx$

Solution :

Step I : Let, $$I = \int_0^{\pi} x\sin^7 x\cos^4 x\,dx \qquad ...(1)$$

→ **Using standard formula of definite integral in equation (1)**

$$...\left[\int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx \quad \text{Here } a = \pi,\ f(x) = x\sin^7 x\cos^4 x\right]$$

Step II : $$\therefore I = \int_0^{\pi}(\pi - x)\sin^7(\pi - x)\cos^4(\pi - x)\,dx \qquad ...(2)$$

→ **Using standard trigonometric formula in equation (2)**

$$...\left[\begin{array}{r|r} \sin(\pi-x) = \sin\pi\cos x - \cos\pi\sin x = \sin x & \cos(\pi-x) = \cos\pi\cos x + \sin\pi\sin x = -\cos x \\ \therefore \sin^n(\pi-x) = \sin^n x,\ \text{for any integer } n & \therefore \cos^n(\pi-x) = \cos^n x,\ \text{if n is even integer} \\ & = -\cos^n x,\ \text{if n is odd integer}\end{array}\right]$$

Here n = 4 , an even integer. $\therefore \cos^4(\pi - x) = \cos^4 x$

Step III : $$\therefore I = \int_0^{\pi}(\pi - x)\sin^7 x\cos^4 x\,dx = \int_0^{\pi}(\pi\sin^7 x\cos^4 x - x\sin^7 x\cos^4 x)\,dx$$

$$I = \int_0^{\pi}\pi\sin^7 x\cos^4 x\,dx - \int_0^{\pi} x\sin^7 x\cos^4 x\,dx \qquad ...(3)$$

Step IV : Adding Equations (1) and (3) we get,

$$I + I = \int_0^{\pi} x\sin^7 x\cos^4 x\,dx + \int_0^{\pi}\pi\sin^7 x\cos^4 x\,dx - \int_0^{\pi} x\sin^7 x\cos^4 x\,dx$$

$$2I = \pi \int_0^{\pi} \sin^7 x \cos^4 x \qquad \ldots(4)$$

→ **Using standard formula in equation (4)**

$$\because \left[\int_0^{\pi} \sin^m x \cos^n x \, dx = 2 \int_0^{\pi/2} \sin^m x \cos^n x \, dx \quad \text{if n = even integer}; \quad = 0 \quad \text{if n = odd integer} \right\} \text{m, may be even or may be odd integer} \Bigg]$$

Here m = 7 , n = 4

Since n is an even i.e. power of cosine term is even

Step V : $\therefore 2I = \pi \times 2 \int_0^{\pi/2} \sin^7 x \cos^4 x \, dx \qquad \ldots(5)$

→ **Using standard reduction formula in equation (5)**

$$\because \left[\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{[(m-1)(m-3)(m-5)\ldots \times 2 \text{ or } 1]\,[(n-1)(n-3)(n-5)\ldots \times 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1} \times k \right]$$

Where $k = \pi/2$ if m and n both are even integers

$= 1$ for all other values of m and n

Here m = 7 , n = 4

Step VI :

$$\therefore 2I = \pi \times 2 \times \frac{[(7-1)(7-3)(7-5)]\,[(4-1)(4-3)]}{(7+4)(7+4-2)(7+4-4)(7+4-6)(7+4-8)(7+4-10)} \times k$$

Here both m and n are not even integers , k = 1

$$\therefore I = \pi \times \frac{[(6)(4)(2)]\,[(3)(1)]}{(11)(11-2)(11-4)(11-6)(11-8)(11-10)} \times 1 = \pi \left[\frac{(6 \cdot 4 \cdot 2)(3 \cdot 1)}{11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1} \cdot 1 \right] = \pi \left[\frac{16}{1155} \right] = \mathbf{\frac{16\pi}{1155}}$$

Example 6.7.23

Evaluate $\int_0^{\pi} x \sin^5 x \cos^4 x \, dx$

Solution :

Step I : Let, $I = \int_0^{\pi} x \sin^5 x \cos^4 x \, dx \qquad \ldots(1)$

→ **Using standard formula of definite integral in equation (1)**

$$\because \left[\int_0^{a} f(x)\, dx = \int_0^{a} f(a-x) dx \right]$$

Here $a = \pi$, $f(x) = x \sin^5 x \cos^4 x$

Step II : $\therefore I = \int_0^{\pi} (\pi - x) \sin^5 (\pi - x) \cos^4 (\pi - x) \, dx \qquad \ldots(2)$

→ **Using standard trigonometric formula in equation (2)**

$$\because \left[\begin{array}{l|l} \sin(\pi-x) = \sin\pi \cos x - \cos\pi \sin x = \sin x & \cos(\pi-x) = \cos\pi \cos x + \sin\pi \sin x = -\cos x \\ \therefore \sin^n (\pi-x) = \sin^n x \text{, for any integer n} & \therefore \cos^n (\pi-x) = \cos^n x \text{, if n is even integer} \\ & \quad = -\cos^n x \text{, if n is odd integer} \end{array} \right]$$

Here n = 4 , an even integer

$\therefore \cos^4 (\pi - x) = \cos^4 x$

Step III : $\therefore I = \int_0^{\pi} (\pi - x) \sin^5 x \cos^4 x \, dx = \int_0^{\pi} (\pi \sin^5 x \cos^4 x - x \sin^5 x \cos^4 x) \, dx$

$$I = \int_0^{\pi} \pi \sin^5 x \cos^4 x \, dx - \int_0^{\pi} x \sin^5 x \cos^4 x \, dx \qquad \ldots(3)$$

Step IV : Adding Equations (1) and (3) we get,

$$I + I = \int_0^{\pi} x \sin^5 x \cos^4 x \, dx + \int_0^{\pi} \pi \sin^5 x \cos^4 x \, dx - \int_0^{\pi} x \sin^5 x \cos^4 x \, dx$$

$$2I = \pi \int_0^{\pi} \sin^5 x \cos^4 x \qquad \ldots(4)$$

→ **Using standard formula in equation (4)**

$$\because \left[\int_0^{\pi} \sin^m x \cos^n x \, dx = 2 \int_0^{\pi/2} \sin^m x \cos^n x \, dx \quad \text{if n = even integer}; \quad = 0 \quad \text{if n = odd integer} \right\} \text{m, may be even or may be odd integer}$$

Here m = 5 , n = 4

Since n is an even i.e. power of cosine term is even

Step V : $\therefore 2I = \pi \times 2 \int_0^{\pi/2} \sin^5 x \cos^4 x \, dx \qquad \ldots(5)$

→ **Using standard reduction formula in equation (5)**

$$\because \left[\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{[(m-1)(m-3)(m-5)\ldots 2 \text{ or } 1]\,[(n-1)(n-3)(n-5)\ldots 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1} \times k \right.$$

Where $k = \pi/2$ if m and n both are even integers

$= 1$ for all other values of m and n

Here m = 5 , n = 4

Step VI : $\therefore\ 2I = \pi \times 2 \times \frac{[(5-1)(5-3)]\ [(4-1)(4-3)]}{(5+4)(5+4-2)(5+4-4)(5+4-6)(5+4-8)} \times k$

Here both m and n are not even integers , k = 1

$$\therefore\ I = \pi \times \frac{[(4)(2)]\ [(3)(1)]}{(9)(9-2)(9-4)(9-6)(9-8)} \times 1 = \pi \left[\frac{(4\cdot 2)(3\cdot 1)}{9\cdot 7\cdot 5\cdot 3\cdot 1}\cdot 1\right] = \pi\left[\frac{8}{315}\right] = \frac{8\pi}{315}$$

Example 6.7.24

Evaluate $\int_0^{\pi} x\ \sin^4 x \cos^4 x\ dx$

Solution :

Step I : Let, $I = \int_0^{\pi} x \sin^4 x \cos^4 x\ dx$ ...(1)

→ Using standard formula of definite integral in equation (1)

$$\ldots\left[\int_0^a f(x)\,dx = \int_0^a f(a-x)dx\ ,\ \text{Here } a = \pi\ ,\ f(x) = x\sin^4 x\cos^4 x\right]$$

Step II : $\therefore I = \int_0^{\pi} (\pi - x)\sin^4(\pi - x)\cos^4(\pi - x)\,dx$ ...(2)

→ Using standard trigonometric formula in equation (2)

$$\ldots\left[\begin{array}{l|l} \sin(\pi-x) = \sin\pi\cos x - \cos\pi\sin x = \sin x & \cos(\pi-x) = \cos\pi\cos x + \sin\pi\sin x = -\cos x \\ \therefore \sin^n(\pi-x) = \sin^n x\ \text{, for any integer n} & \therefore \cos^n(\pi-x) = \cos^n x\ \text{, if n is even integer} \\ & \qquad = -\cos^n x\ \text{, if n is odd integer} \end{array}\right]$$

Here n = 4 , an even integer

$\therefore \cos^4(\pi - x) = \cos^4 x$

Step III : $\therefore I = \int_0^{\pi} (\pi - x)\sin^4 x\cos^4 x\,dx = \int_0^{\pi} (\pi\sin^4 x\cos^4 x - x\sin^4 x\cos^4 x)\,dx$

$$I = \int_0^{\pi} \pi\sin^4 x\cos^4 x\,dx - \int_0^{\pi} x\sin^4 x\cos^4 x\,dx \qquad \ldots(3)$$

Step IV : Adding Equations (1) and (3) we get,

$$I + I = \int_0^{\pi} x\sin^4 x\cos^4 x\,dx + \int_0^{\pi}\pi\sin^4 x\cos^4 x\,dx - \int_0^{\pi} x\sin^4 x\cos^4 x\,dx$$

$$2I = \pi\int_0^{\pi}\sin^4 x\cos^4 x \qquad \ldots(4)$$

→ Using standard formula in equation (4)

$$\ldots\left[\begin{array}{ll} \int_0^{\pi}\sin^m x\cos^n x\,dx = 2\int_0^{\pi/2}\sin^m x\cos^n x\,dx & \text{if n = even integer} \\ = 0 & \text{if n = odd integer} \end{array}\right\}\ \text{m , may be even or may be odd integer}\Bigg]$$

Here m = 4 , n = 4

Since n is an even i.e. power of cosine term is even

Step V : $\therefore\ 2I = \pi \times 2\int_0^{\pi/2}\sin^4 x\cos^4 x\ dx$ ...(5)

→ Using standard reduction formula in equation (5)

$$\ldots\left[\int_0^{\pi/2}\sin^m x\cos^n x\,dx = \frac{[(m-1)(m-3)(m-5)\ldots 2 \text{ or } 1]\ [(n-1)(n-3)(n-5)\ldots 2\text{ or }1]}{(m+n)(m+n-2)(m+n-4)\ldots 2\text{ or }1}\times k\right.$$

Where $k = \pi/2$ if m and n both are even integers

= 1 for all other values of m and n]

Here m = 4 , n = 4

Step VI : $\therefore\ 2I = \pi \times 2\times \frac{[(4-1)(4-3)]\ [(4-1)(4-3)]}{(4+4)(4+4-2)(4+4-4)(4+4-6)}\times k$

Here both m and n are even integers , $k = \frac{\pi}{2}$

$$\therefore I = \pi\times\frac{[(3)(1)]\ [(3)(1)]}{(8)(8-2)(8-4)(8-6)}\times\frac{\pi}{2} = \pi\left[\frac{(3\cdot 1)(3\cdot 1)}{8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2}\right] = \pi\left[\frac{3\pi}{128}\right] = \frac{3}{128}\pi^2$$

Type I : Examples of the form $\int_0^{1/m\,(\pi/2)} f(mx)\,dx$

In these examples put mx = t and then simplify.

Example 6.7.25

Evaluate $\int_0^{\pi/6}\sin^5 3\theta\ d\theta$

Solution :

Step I : Let $I = \int_0^{\pi/6}\sin^5 3\theta\,d\theta$ ...(1)

Put $3\theta = t$ in equation (1)

$\therefore\ \theta = \frac{1}{3}t \quad \Rightarrow d\theta = \frac{1}{3}\,dt$

When $\theta = 0, t = (3)(0) = 0$

and when $\theta = \frac{\pi}{6}, t = (3)\left(\frac{\pi}{6}\right) = \frac{\pi}{2}$

$\therefore t : 0 \to \frac{\pi}{2}$

Step II : Equation (1) ⇒

$$\therefore\ I = \int_0^{\pi/2}\sin^5 t\left(\frac{1}{3}dt\right) = \frac{1}{3}\int_0^{\pi/2}\sin^5 t\,dt \qquad \ldots(2)$$

→ Using standard formula in equation (2),

$$\ldots\left[\int_0^{\pi/2}\sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\ ,\right.$$

if n is even positive integer

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\cdot 1\ ,$$

if n is odd positive integer]

Here n = 5 , an odd positive integer $\Rightarrow k = 1$

Step III :

$$I = \frac{1}{3}\left(\frac{5-1}{5}\right)\left(\frac{5-3}{5-2}\right)\cdot 1 = \frac{1}{3}\left(\frac{4}{5}\cdot\frac{2}{3}\right) = \frac{8}{45}$$

Example 6.7.26

Evaluate $\int_0^{\pi/6} \cos^6 3\theta \, d\theta$

Solution :

Step I : Let $I = \int_0^{\pi/6} \cos^6 3\theta \, d\theta$ ...(1)

Put $3\theta = t$ in equation (1)

$\therefore \theta = \frac{1}{3}t \Rightarrow d\theta = \frac{1}{3} dt$

When $\theta = 0, t = (3)(0) = 0$

and when $\theta = \frac{\pi}{6}, t = (3)\left(\frac{\pi}{6}\right) = \frac{\pi}{2}$

$\therefore t : 0 \to \frac{\pi}{2}$

Step II : Equation (1) $\Rightarrow$

$$\therefore I = \int_0^{\pi/2} \cos^6 t\left(\frac{1}{3}dt\right) = \frac{1}{3}\int_0^{\pi/2} \cos^6 t \, dt \quad ...(2)$$

→ **Using standard formula in equation (2),**

$$\because \int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \quad \text{if n is even positive integer}$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\cdot 1, \quad \text{if n is odd positive integer}$$

Here n = 6 , an even positive integer $\Rightarrow k = \frac{\pi}{2}$

Step III : $\therefore I = \frac{1}{3}\left(\frac{6-1}{6}\right)\left(\frac{6-3}{6-2}\right)\cdot\left(\frac{6-5}{6-4}\right)\cdot\frac{\pi}{2}$

$$= \frac{1}{3}\left(\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) = \frac{5\pi}{96}$$

Example 6.7.27

Evaluate $\int_0^{\pi/4} \sin^7 2\theta \, d\theta$

Solution :

Step I : Let $I = \int_0^{\pi/6} \sin^7 2\theta \, d\theta$ ...(1)

Put $2\theta = t$ in equation (1)

$\therefore \theta = \frac{1}{2}t \Rightarrow d\theta = \frac{1}{2} dt$

When $\theta = 0, t = (2)(0) = 0$

and when $\theta = \frac{\pi}{4}, t = (2)\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

$\therefore t : 0 \to \frac{\pi}{2}$

Step II : Equation (1) $\Rightarrow$

$$\therefore I = \int_0^{\pi/2} \sin^7 t\left(\frac{1}{2}dt\right) = \frac{1}{2}\int_0^{\pi/2} \sin^7 t \, dt \quad ...(2)$$

→ **Using standard formula in equation (2),**

$$\because \int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \quad \text{if n is even positive integer}$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\cdot 1, \quad \text{if n is odd positive integer}$$

Here n = 7 , an odd positive integer $\Rightarrow k = 1$

Step III : $\therefore I = \frac{1}{2}\left(\frac{7-1}{7}\right)\left(\frac{7-3}{7-2}\right)\left(\frac{7-5}{7-4}\right)\cdot 1$

$$= \frac{1}{2}\left[\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1\right] = \frac{8}{35}$$

Type II : Example of the form $\int_{-a}^{a} f(x)\,dx$

Here simplify the integral by checking whether the function f(x) is even or odd then use appropriate reduction formula.

Example 6.7.28

Evaluate $\int_{-\pi/2}^{\pi/2} \sin^7 x \, dx$

Solution :

Step I : Let $I = \int_{-\pi/2}^{\pi/2} \sin 7 x \, dx$ ...(1)

→ **Using standard result in equation (1)**

$$\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \quad \text{if f(x) is an even function}$$

$$= 0, \quad \text{if f(x) is an odd function}$$

$\because \sin^n(-x) = \sin^n(x)$ if n is even

$= -\sin^n(x)$ if n is odd

$\therefore$ since 7 is an odd integer , $\sin^7 x$ is odd function.

Step II : $\therefore \int_{-\pi/2}^{\pi/2} \sin^7 x \, dx = 0$

Example 6.7.29

Evaluate $\int_{-\pi/2}^{\pi/2} \sin^4 x \cos^2 x \, dx$

Solution :

Step I : Let $I = \int_{-\pi/2}^{\pi/2} \sin^4 x \cos^2 x \, dx$ ...(1)

→ **Using standard result in equation (1)**

$$\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \quad \text{if f(x) is an even function}$$

$$= 0, \quad \text{if f(x) is an odd function}$$

$\because \sin^n(-x) = \sin^n(x)$ if n is even

$= -\sin^n(x)$ if n is odd

$\cos^n x$ is even function for any integer

$\Rightarrow \cos^2 x$ is an even function

Since 4 is an even integer $\sin^4 x$ is even function

$\therefore \sin^4 x$ and $\cos^2 x$ both are even functions

$\Rightarrow \sin^4 x \cos^2 x$ is an even function

Step II : $\therefore I = 2\int_0^{\pi/2} \sin^4 x \cos^2 x \, dx$ ...(2)

→ **Using standard reduction formula in equation (2)**

$$\because \int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{[(m-1)(m-3)(m-5)\ldots\times 2 \text{ or } 1]\,[(n-1)(n-3)(n-5)\ldots\times 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1}\times k$$

Where $k = \pi/2$ if m and n both are even integers

$= 1$ for all other values of m and n

Here m = 4 , n = 2

Step III : $\therefore I = 2\times\frac{[(4-1)(4-3)]\,[(2-1)]}{(4+2)(4+2-2)(4+2-4)}\times k$

Here both m and n are even integers , $k = \frac{\pi}{2}$

$$\therefore I = 2\times\frac{[(3)(1)]\,[(1)]}{(6)(6-2)(6-4)}\times\frac{\pi}{2} = 2\times\frac{(3)(1)\cdot(1)}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{\pi}{16}$$

Example 6.7.30

Evaluate $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^4 x \, dx$

Solution :

Step I : Let $I = \int_{-\pi/2}^{\pi/2} \sin^2 x \cos^4 x \, dx$ ...(1)

→ **Using standard result in equation (1)**

$$\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \quad \text{f f(x) is an even function}$$

$$= 0, \quad \text{if f(x) is an odd function}$$

$\because \sin^n(-x) = \sin^n(x)$ if n is even

$= -\sin^n(x)$ if n is odd

$\cos^n x$ is even function for any integer

$\Rightarrow \cos^4 x$ is an even function

Since 2 is an even integer $\sin^2 x$ is even function

$\therefore \sin^2 x$ and $\cos^4 x$ both are even functions

$\Rightarrow \sin^2 x \cos^4 x$ is an even function

Step II : $\therefore I = 2\int_0^{\pi/2} \sin^2 x \cos^4 x \, dx$ ...(2)

➔ Using standard reduction formula in equation (2)

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \begin{cases} \dfrac{[(m-1)(m-3)(m-5)\ldots\times 2 \text{ or } 1][(n-1)(n-3)(n-5)\ldots\times 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1} \times k & \\ \quad \text{Where } k = \pi/2 \text{ if } m \text{ and } n \text{ both are even integers} & \\ \quad = 1 \text{ for all other values of } m \text{ and } n & \end{cases}$$

Here $m = 2$, $n = 4$

Step III : $\therefore\ I = 2 \times \dfrac{[(2-1)][(4-1)(4-3)]}{(2+4)(2+4-2)(2+4-4)} \times k$

Here both m and n are even integers, $k = \dfrac{\pi}{2}$

$$\therefore I = 2 \times \frac{[(1)][(3)(1)]}{(6)(6-2)(6-4)} \times \frac{\pi}{2}$$

$$= 2 \times \frac{(1)(3)(1)}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \mathbf{\frac{\pi}{16}}$$

Example 6.7.31

Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^4 x \cos^3 x\, dx$

Solution :

Step I : Let $I = \displaystyle\int_{-\pi/2}^{\pi/2} \sin^4 x \cos^3 x\, dx \quad \ldots(1)$

➔ Using standard result in equation (1)

$$\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\displaystyle\int_0^a f(x)\, dx, & \text{if } f(x) \text{ is an even function} \\ = 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$$

$\because\ \sin^n(-x) = \sin^n(x)$ if n is even

$= -\sin^n(x)$ if n is odd

$\cos^n x$ is even function for any integer

$\Rightarrow \cos^3 x$ is an even function

Since 4 is an even integer $\sin^4 x$ is even function

$\therefore\ \sin^4 x$ and $\cos^3 x$ both are even functions

$\Rightarrow \sin^4 x \cos^3 x$ is an even function

Step II : $\therefore\ I = 2\displaystyle\int_0^{\pi/2} \sin^4 x \cos^3 x\, dx$

➔ Using standard reduction formula in equation (2)

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \begin{cases} \dfrac{[(m-1)(m-3)(m-5)\ldots\times 2 \text{ or } 1][(n-1)(n-3)(n-5)\ldots\times 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1} \times k & \\ \quad \text{Where } k = \pi/2 \text{ if } m \text{ and } n \text{ both are even integers} & \\ \quad = 1 \text{ for all other values of } m \text{ and } n & \end{cases}$$

Here $m = 4$, $n = 3$

Step III

$$\therefore I = 2 \times \frac{[(4-1)(4-3)][(3-1)]}{(4+3)(4+3-2)(4+3-4)(4+3-6)} \times k$$

Here both m and n are not even integers, $k = 1$

Step IV : $\therefore I = 2 \times \dfrac{[(3)(1)][(2)]}{(7)(7-2)(7-4)(7-6)} \times 1$

$$= 2 \times \frac{(3)(1)(2)}{7 \cdot 5 \cdot 3 \cdot 1} \cdot 1 = \frac{4}{35}$$

Example 6.7.32

Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^3 x \cos^4 x\, dx$

Solution :

Step I : Let $I = \displaystyle\int_{-\pi/2}^{\pi/2} \sin^3 x \cos^4 x\, dx \quad \ldots(1)$

➔ Using standard result in equation (1)

$$\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\displaystyle\int_0^a f(x)\, dx, & \text{if } f(x) \text{ is an even function} \\ = 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$$

$\because\ \sin^n(-x) = \sin^n(x)$ if n is even

$= -\sin^n(x)$ if n is odd

$\cos^n x$ is even function for any integer

$\Rightarrow \cos^4 x$ is an even function

Since 3 is an odd integer $\sin^3 x$ is odd function

$\therefore\ \sin^3 x$ is an odd function and $\cos^4 x$ is an even functions

$\Rightarrow \sin^3 x \cos^4 x$ is an odd function

Step II : $\therefore\ I = 0$

Example 6.7.33

Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} \cos^5 x\, dx$

Solution :

Step I : Let $I = \displaystyle\int_{-\pi/2}^{\pi/2} \cos^5 x\, dx \quad \ldots(1)$

➔ Using standard result in equation (1)

$$\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\displaystyle\int_0^a f(x)\, dx, & \text{if } f(x) \text{ is an even function} \\ = 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$$

$\cos^n x$ is even function for any integer n

Step II : $\therefore\ I = 2\displaystyle\int_0^{\pi/2} \cos^5 x\, dx \quad \ldots(2)$

➔ Using standard formula in equation (2),

$$\int_0^{\pi/2} \cos^n x\, dx = \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots\cdots \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2}, & \text{if n is even positive integer} \\ = \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots\cdots \dfrac{2}{3} \cdot 1, & \text{if n is odd positive integer} \end{cases}$$

Here $n = 5$, an odd positive integer $\Rightarrow k = 1$

Step III : $\therefore I = 2\left(\dfrac{5-1}{5}\right)\left(\dfrac{5-3}{5-2}\right) \cdot 1 = 2\left[\dfrac{4}{5} \cdot \dfrac{2}{3} \cdot 1\right] = \mathbf{\dfrac{16}{15}}$

Example 6.7.34

Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^6 x\, dx$

Solution :

Step I : Let $I = \displaystyle\int_{-\pi/2}^{\pi/2} \sin^6 x\, dx \quad \ldots(1)$

➔ Using standard result in equation (1)

$$\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\displaystyle\int_0^a f(x)\, dx, & \text{if } f(x) \text{ is an even function} \\ = 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$$

$\because\ \sin^n(-x) = \sin^n(x)$ if n is even

$= -\sin^n(x)$ if n is odd

Here $n = 6$ is an even integer therefore $\sin^6 x$ is an even function

Step II : $\therefore\ I = 2\displaystyle\int_0^{\pi/2} \sin^6 x\, dx \quad \ldots(2)$

➔ Using standard formula in equation (2),

$$\int_0^{\pi/2} \sin^n x\, dx = \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots\cdots \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2}, & \text{if n is even positive integer} \\ = \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots\cdots \dfrac{2}{3} \cdot 1, & \text{if n is odd positive integer} \end{cases}$$

Here n = 6 , an odd positive integer $\Rightarrow k = \dfrac{\pi}{2}$

Step III : $\therefore I = 2\left(\dfrac{6-1}{6}\right)\left(\dfrac{6-3}{6-2}\right)\left(\dfrac{6-5}{6-4}\right) \cdot \dfrac{\pi}{2}$

$$= 2\left[\frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right] = \frac{5\pi}{16}$$

Example 6.7.35

Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^3 x\, dx$

Solution :

Step I : Let $I = \displaystyle\int_{-\pi/2}^{\pi/2} \sin^3 x\, dx \quad \ldots(1)$

→ **Using standard result in equation (1)**

$$\ldots\left[\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx, \text{ if } f(x) \text{ is an even function} \quad = 0, \text{ if } f(x) \text{ is an odd function}\right]$$

$\because \sin^n(-x) = \sin^n(x)$ if n is even

$= -\sin^n(x)$ if n is odd

Here n = 3 is an odd integer therefore $\sin^3 x$ is an odd function

Step II : $\therefore$ I = 0

Example 6.7.36

Evaluate $\int_{-\pi}^{\pi} \sin^2 x \cos^3 x\, dx$

Solution :

Step I : Let $I = \int_{-\pi}^{\pi} \sin^2 x \cos^3 x\, dx$ ...(1)

→ **Using standard result in equation (1)**

$$\ldots\left[\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx, \text{ if } f(x) \text{ is an even function } = 0, \text{ if } f(x) \text{ is an odd function}\right]$$

$\because \sin^n(-x) = \sin^n(x)$ if n is even

$= -\sin^n(x)$ if n is odd

$\cos^n x$ is even function for any integer

$\Rightarrow \cos^3 x$ is an even function

Since 2 is an even integer $\sin^2 x$ is even function

$\therefore$ $\sin^2 x$ and $\cos^3 x$ both are even functions

$\Rightarrow \sin^2 x \cos^3 x$ is an even function

Step II : $\therefore I = \int_{0}^{\pi} \sin^2 x \cos^3 x\, dx$ ...(2)

→ **Using standard reduction formula in equation (2)**

$$\left[\int_{0}^{\pi} \sin^m x \cos^n x\, dx = 2\int_{0}^{\pi/2} \sin^m x \cos^n x\, dx \quad \text{if } n = \text{even integer}; \quad = 0 \quad \text{if } n = \text{odd integer} \quad \Big\}\ m \text{, may be even or may be odd integer}\right]$$

Here m = 2 , n = 3

Since n is an odd integer i.e. power of cosine term is odd

$\therefore$ 2I = 2 (0) = 0 $\quad\therefore$ **I = 0 = 0**

Example 6.7.37

Evaluate $\int_{-\pi}^{\pi} \sin^6 x \cos^4 x\, dx$

Solution :

Step I : Let $I = \int_{-\pi}^{\pi} \sin^6 x \cos^4 x\, dx$ ...(1)

→ **Using standard result in equation (1)**

$$\ldots\left[\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx, \quad \text{if } f(x) \text{ is an even function}; \quad = 0, \quad \text{if } f(x) \text{ is an odd function}\right]$$

$\because \sin^n(-x) = \sin^n(x)$ if n is even

$= -\sin^n(x)$ if n is odd

$\cos^n x$ is even function for any integer

$\Rightarrow \cos^4 x$ is an even function

Since 6 is an even integer $\sin^6 x$ is even function

$\therefore$ $\sin^6 x$ and $\cos^4 x$ both are even functions

$\Rightarrow \sin^6 x \cos^4 x$ is an even function

Step II : $\therefore$ $I = 2\int_{0}^{\pi} \sin^6 x \cos^4 x\, dx$ ...(2)

→ **Using standard reduction formula in equation (2)**

$$\ldots\left[\int_{0}^{\pi} \sin^m x \cos^n x\, dx = 2\int_{0}^{\pi/2} \sin^m x \cos^n x\, dx \quad \text{if } n = \text{even integer}; \quad = 0 \quad \text{if } n = \text{odd integer} \quad \Big\}\ m \text{, may be even or may be odd integer}\right]$$

Here m = 6 , n = 4

Since n is an even integer i.e. power of cosine term is even

Step III : $\therefore I = 2\left[2\int_{0}^{\pi/2} \sin^6 x \cos^4 x\, dx\right] = 4\int_{0}^{\pi/2} \sin^6 x \cos^4 x\, dx$ ...(3)

→ **Using standard reduction formula in equation (3)**

$$\ldots\left[\int_{0}^{\pi/2} \sin^m x \cos^n x\, dx = \frac{[(m-1)(m-3)(m-5)\ldots 2 \text{ or } 1]\,[(n-1)(n-3)(n-5)\ldots 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1} \times k\right.$$

$$\left.\text{Where} \quad k = \pi/2 \text{ if m and n both are even integers } = 1 \text{ for all other values of m and n}\right]$$

Here m = 6 , n = 4

Step IV : $\therefore I = 4 \times \dfrac{[(6-1)(6-3)(6-5)]\,[(4-1)(4-3)]}{(6+4)(6+4-2)(6+4-4)(6+4-6)(6+4-8)} \times k$

Here both m and n are even integers $\Rightarrow k = \dfrac{\pi}{2}$

$$\therefore I = 4 \times \frac{[(5)(3)(1)]\,[(3)(1)]}{(10)(10-2)(10-4)(10-6)(10-8)} \times \frac{\pi}{2}$$

$$= 4 \times \frac{(5)(3)(1)(3)(1)}{10\cdot 8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = 4\left[\frac{(5.3.1)\ (3.1)}{10\cdot 8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2}\right] = \frac{3\pi}{128}$$

Example 6.7.38

Evaluate $\displaystyle\int_{-\pi}^{\pi} \sin^4 x \cos^2 x\, dx$

Solution :

Step I : Let $I = \displaystyle\int_{-\pi}^{\pi} \sin^4 x \cos^2 x\, dx$ ...(1)

→ **Using standard result in equation (1)**

$$\ldots\left[\begin{array}{lll}\displaystyle\int_{-a}^{a} f(x)\,dx & = 2\displaystyle\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function} \\ & = 0, & \text{if } f(x) \text{ is an odd function}\end{array}\right]$$

$\because \sin^n(-x) = \sin^n(x)$ if n is even

$= -\sin^n(x)$ if n is odd

$\cos^n x$ is even function for any integer $\Rightarrow \cos^2 x$ is an even function

Since 4 is an even integer $\sin^4 x$ is even function

$\therefore \sin^4 x$ and $\cos^2 x$ both are even functions $\Rightarrow \sin^4 x \cos^2 x$ is an even function

Step II : $\therefore I = 2\displaystyle\int_0^{\pi} \sin^4 x \cos^2 x\, dx$ ...(2)

→ **Using standard reduction formula in equation (2)**

$$\ldots\left[\begin{array}{ll}\displaystyle\int_0^{\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi/2} \sin^m x \cos^n x\,dx & \text{if } n = \text{even integer} \\ \qquad = 0 & \text{if } n = \text{odd integer}\end{array}\right\} \text{m, may be even or may be odd integer}\Bigg]$$

Here m = 4 , n = 2

Since n is an even integer i.e. power of cosine term is even

Step III : $\therefore I = 2\left[2\displaystyle\int_0^{\pi/2} \sin^4 x \cos^2 x\, dx\right] = 4\displaystyle\int_0^{\pi/2} \sin^4 x \cos^2 x\, dx$ ...(3)

→ **Using standard reduction formula in equation (3)**

$$\ldots\left[\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{[(m-1)(m-3)(m-5)\ldots 2 \text{ or } 1]\,[(n-1)(n-3)(n-5)\ldots 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1} \times k\right.$$

$$\left.\text{Where } k = \pi/2 \text{ if m and n both are even integers} = 1 \text{ for all other values of m and n}\right]$$

Here m = 4 , n = 2

Step IV : $\therefore I = 4 \times \dfrac{[(4-1)(4-3)]\,[(2-1)]}{(4+2)(4+2-2)(4+2-4)} \times k$

Here both m and n are even integers $\Rightarrow k = \dfrac{\pi}{2}$

$$\therefore I = 4 \times \frac{[(3)(1)]\,[(1)]}{(6)(6-2)(6-4)} \times \frac{\pi}{2} = 4\left[\frac{(3.1)\ (1)}{6\cdot 4\cdot 2}\ \frac{\pi}{2}\right] = \frac{\pi}{8}$$

Example 6.7.39

Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} \cos^3 x\,(1+\sin x)^2\, dx$

Solution :

Step I : Let $I = \displaystyle\int_{-\pi/2}^{\pi/2} \cos^3 x\,(1+\sin x)^2\, dx$ ...(1)

→ **Using standard formula in equation (1)**

...$[(a+b)^2 = a^2 + 2ab + b^2]$

Here a = 1 , b = sinx

Step II : $I = \displaystyle\int_{-\pi/2}^{\pi/2} \cos^3 x\,(1 + 2\sin x + \sin^2 x)\,dx = \int_{-\pi/2}^{\pi/2} \cos^3 x + 2\cos^3 x \sin x + \cos^3 x \sin^2 x\, dx$

$= \displaystyle\int_{-\pi/2}^{\pi/2} \cos^3 x\,dx + 2\int_{-\pi/2}^{\pi/2} \cos^3 x \sin x\,dx + \int_{-\pi/2}^{\pi/2} \cos^3 x \sin^2 x\,dx$ ...(2)

→ **Using standard formula in equation (2)**

$$\begin{array}{lll}\displaystyle\int_{-a}^{a} f(x)\,dx & = 2\displaystyle\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function} \\ & = 0, & \text{if } f(x) \text{ is an odd function}\end{array}$$

$\because \sin^n(-x) = \sin^n(x)$ if n is even

$= -\sin^n(x)$ if n is odd

$\cos^n x$ is even function for any integer

Step III : $\therefore \cos^3 x$ is an even function $\Rightarrow \displaystyle\int_{-\pi/2}^{\pi/2} \cos^3 x\,dx = 2\int_0^{\pi/2} \cos^3 x\,dx$ ...(3)

Step IV : $\cos^3 x$ is an even function , sinx is an odd function $\Rightarrow \cos^3 x \sin x$ is an odd function

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \cos^3 x \sin x \, dx = 0 \quad ...(4)$$

Step V : $\cos^3 x$ and $\sin^2 x$ both are even functions $\Rightarrow \int_{-\pi/2}^{\pi/2} \cos^3 x \sin^2 x \, dx = 2 \int_0^{\pi/2} \cos^3 x \sin^2 x \, dx$ ...(5)

Step VI : Using values from equation (3) , (4),(5) in equation (2) we get,

$$\therefore I = 2 \int_0^{\pi/2} \cos^3 x \, dx + 0 + 2 \int_0^{\pi/2} \cos^3 x \sin^2 x \, dx \quad ...(6)$$

$$\therefore I = 2\left[\frac{2}{3} \cdot 1\right] + 2\left[\frac{(2)\ (1)}{5 \cdot 3 \cdot 1} \cdot 1\right] = \frac{4}{3} + \frac{4}{15} = \frac{4 \times 5}{3 \times 5} + \frac{4}{15} = \frac{20}{15} + \frac{4}{15} = \frac{20+4}{15} = \frac{24}{15} = \frac{8}{5}$$

Example 6.7.40

Evaluate $\int_0^{2\pi} \sin^2 \theta \, (1 + \cos \theta)^4 \, d\theta$

Solution :

Step I : Let $I = \int_0^{2\pi} \sin^2 \theta \, (1 + \cos \theta)^4 \, d\theta$ ...(1)

→ **Using standard formula in equation (1)**

$$\ldots\left[\sin 2\theta = 2\sin\theta\cos\theta \Rightarrow 2\sin\theta = \sin\frac{\theta}{2}\cos\frac{\theta}{2}, \quad \frac{1+\cos 2\theta}{2} = \cos^2\theta \Rightarrow 1 + \cos 2\theta = 2\cos^2\theta \Rightarrow 1 + \cos\theta = 2\cos^2\frac{\theta}{2} \right]$$

Step II :

$$\therefore I = \int_0^{2\pi} \left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2 \left(2\cos^2\frac{\theta}{2}\right)^4 d\theta = \int_0^{2\pi} 2^2\left(\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\right) 2^4\left(\cos^{2\times 4}\frac{\theta}{2}\right) d\theta$$

$$= 2^2 \cdot 2^4 \int_0^{2\pi} \sin^2\frac{\theta}{2} \cos^2\frac{\theta}{2} \cos^8\frac{\theta}{2} \, d\theta = 2^6 \int_0^{2\pi} \sin^2\frac{\theta}{2} \cos^{10}\frac{\theta}{2} \, d\theta \quad ...(2)$$

Put $\frac{\theta}{2} = t$ in equation (2)

$\therefore \theta = 2t \quad \therefore d\theta = 2dt$

When $\theta = 0$, $t = 0$ and when $\theta = 2\pi$, $t = \pi$

$\therefore t : 0 \to \pi$

$$\therefore I = 64 \int_0^{\pi} (\sin^2 t \cos^{10} t)(2dt) = 64 \times 2 \int_0^{\pi} \sin^2 t \cos^{10} t \, dt \quad ...(3)$$

→ **Using standard formula in equation (3)**

$$\left[\begin{array}{ll} \int_0^{\pi} \sin^m x \cos^n x \, dx = 2 \int_0^{\pi/2} \sin^m x \cos^n x \, dx & \text{if } n = \text{even integer} \\ = 0 & \text{if } n = \text{odd integer} \end{array} \right\} \text{m , may be even or may be odd integer}$$

Here m = 2 , n = 10

Since n is an even i.e. power of cosine term is even

Step V : $\therefore I = 64 \times 2 \times 2 \int_0^{\pi/2} \sin^2 t \cos^{10} t \, dt$ ...(4)

→ **Using standard reduction formula in equation (3)**

$$\ldots\left[\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{[(m-1)(m-3)(m-5) \ldots 2 \text{ or } 1]\,[(n-1)(n-3)(n-5) \ldots 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4) \ldots 2 \text{ or } 1} \times k \right.$$

Where $k = \pi/2$ if m and n both are even integers = 1 for all other values of m and n]

Here m = 2 , n = 10

Step VI : $\therefore I = 256 \times \frac{[(2-1)]\,[(10-1)(10-3)(10-5)(10-7)(10-9)]}{(2+10)(2+10-2)(2+10-4)(2+10-6)(2+10-8)(2+10-10)} \times k$

Here both m and n are even integers $\Rightarrow k = \frac{\pi}{2}$

$$\therefore I = 256 \left[\frac{(1)\ (9 \cdot 7 \cdot 5 \cdot 3 \cdot 1)}{12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}\right] = \frac{21}{8}\pi$$

Example 6.7.41

Evaluate $\int_0^{\pi} \sin^2 \theta \, (1 + \cos \theta)^4 \, d\theta$

Solution :

Step I : Let $I = \int_0^{\pi} \sin^2 \theta \, (1 + \cos \theta)^4 \, d\theta$ ...(1)

→ **Using standard formula in equation (1)**

$$\ldots\left[\sin 2\theta = 2\sin\theta\cos\theta \Rightarrow 2\sin\theta = \sin\frac{\theta}{2}\cos\frac{\theta}{2}, \quad \frac{1+\cos 2\theta}{2} = \cos^2\theta \Rightarrow 1 + \cos 2\theta = 2\cos^2\theta \Rightarrow 1 + \cos\theta = 2\cos^2\frac{\theta}{2} \right]$$

Step II : $\therefore I = \int_0^{\pi} \left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2 \left(2\cos^2\frac{\theta}{2}\right)^4 d\theta = \int_0^{\pi} 2^2\left(\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\right) 2^4\left(\cos^{2\times 4}\frac{\theta}{2}\right) d\theta$

$$= 2^2 \cdot 2^4 \int_0^{\pi} \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\cos^8\frac{\theta}{2}\,d\theta = 2^6 \int_0^{\pi} \sin^2\frac{\theta}{2}\cos^{10}\frac{\theta}{2}\,d\theta \quad \ldots(2)$$

Put $\frac{\theta}{2} = t$ in equation (2)

$\therefore \theta = 2t \quad \therefore d\theta = 2dt$

When $\theta = 0$, $t = 0$ and when $\theta = \pi$, $t = \frac{\pi}{2}$

$$\therefore t : 0 \to \frac{\pi}{2}$$

$$\therefore I = 64 \int_0^{\pi/2} (\sin^2 t\cos^{10} t)(2dt) = 64 \times 2 \int_0^{\pi/2} \sin^2 t\cos^{10} t\,dt \quad \ldots(3)$$

→ **Using standard reduction formula in equation (3)**

$$\ldots\left[\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \frac{[(m-1)(m-3)(m-5)\ldots 2 \text{ or } 1]\,[(n-1)(n-3)(n-5)\ldots 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1} \times k\right.$$

$$\left.\text{Where } k = \pi/2 \text{ if m and n both are even integers} = 1 \text{ for all other values of m and n}\right]$$

Here $m = 2$, $n = 10$

Step VI : $$\therefore I = 128 \times \frac{[(2-1)]\,[(10-1)(10-3)(10-5)(10-7)(10-9)]}{(2+10)(2+10-2)(2+10-4)(2+10-6)(2+10-8)(2+10-10)} \times k$$

Here both m and n are even integers $\Rightarrow k = \frac{\pi}{2}$

$$\therefore I = 128\left[\frac{(1)\,(9\cdot 7\cdot 5\cdot 3\cdot 1)}{12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2}\right] = \frac{21\pi}{16}$$

Exercise 6.1

Ex. 1 : show that $\int_0^{\pi} x\sin^5 x\,dx = \frac{8\pi}{15}$

Ex. 2 : show that $\int_0^{2\pi} \sin^4 x\cos^6 x\,dx = \frac{3\pi}{128}$

Ex. 3 : show that $\int_{-\pi}^{\pi} \sin^4 x\cos^2 x\,dx = \frac{\pi}{8}$

Ex. 4 : show that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x\cos^2 x\,dx = \frac{\pi}{16}$

Ex. 5 Evaluate $\int_0^{\pi/2} \sin^5 x\cos^2 x\,dx$ **Ans. :** $\frac{1}{24}$

Ex. 6 Evaluate $\int_0^{\pi} (1+\cos\theta)^3\,d\theta$ **Ans. :** $\frac{5\pi}{2}$

Ex. 7 Evaluate $\int_0^{\pi/2} \cos^3 x\sin^4 4x\,dx$ **Ans. :** 0

Ex. 8 Evaluate $\int_0^{\pi/6} \sin^6 3\theta\,d\theta$ **Ans. :** $\frac{5\pi}{36}$

Ex. 9 Evaluate $\int_{-\pi/2}^{\pi/2} \sin^4 x\cos^2 x\,dx$ **Ans. :** $\frac{\pi}{16}$

Ex. 10 Evaluate $\int_{-\pi}^{\pi} \sin^4 x\cos^2 x\,dx$ **Ans. :** $\frac{\pi}{8}$

Ex. 11 Evaluate $\int_0^{2\pi} \sin^4 x\cos^2 x\,dx$ **Ans. :** $\frac{\pi}{8}$

Ex. 12 Evaluate $\int_0^{\pi} \sin^2 x(1+\cos x)^4\,dx$ **Ans. :** $\frac{21\pi}{16}$

Ex. 13 Evaluate $\int_0^{6} \sin^4 \pi x\cdot\cos^2 2\pi x\,dx$ **Ans. :** $\frac{7}{16}$

(B)

Ex. 1 Evaluate $\int_0^{\pi} x\cos^6 x\,dx$ **Ans. :** $\frac{5\pi^2}{32}$

Ex. 2 Evaluate $\int_0^{\pi/2} x\sin^6 x\cos^4 x\,dx$ **Ans. :** $\frac{3\pi^2}{152}$

Ex. 3 Evaluate $\int_0^{\pi/2} x\sin^5 x\,dx$

Ans. : $\frac{149}{225}$ **Hint : Use result of** $\int_0^{\pi/2} x\sin x\,dx$

Ex. 4 Evaluate $\int_0^{\pi/2} x\cos^4 x\,dx$ **Ans. :** $\frac{3\pi^2}{64} - \frac{1}{4}$

Ex. 5 Evaluate $\int_0^{1} x^6\sin^{-1} x\,dx$ **Ans. :** $\frac{\pi}{14} - \frac{16}{245}$

Ex. 6 Evaluate $\int_0^{\pi} x\sin^5 x\cos^3 x\,dx$ **Ans. :** $\frac{8\pi}{1287}$

Ex. 7 Evaluate $\int_0^{\infty} \frac{1}{(x^2+1)^{n+1}}\,dx$ **Ans. :** $\frac{(2n)!}{2^{2n}(n!)^2}\cdot\frac{\pi}{2}$

Ex. 8 : Evaluate $\int_0^{\infty} \frac{dx}{(1+x^2)^{9/2}}$ **Ans. :** $\frac{16}{35}$

Ex. 9 : Evaluate $\int_0^{2} x^2\sqrt{2-x}\,dx$ **Ans. :** $\frac{128\sqrt{2}}{105}$

6.8 Important Examples :

Example 6.8.1

Find the reduction formula for $\int_0^{\frac{\pi}{4}} \sin^n x\,dx$ and hence evaluate $\int_0^{\frac{\pi}{4}} \sin^6 x\,dx$.

Solution :

Step I :

Let $$I_n = \int_0^{\frac{\pi}{4}} \sin^n x\,dx = \int_0^{\frac{\pi}{4}} \sin^{n-1} x\sin x\,dx \quad \ldots(1)$$

→ **Using standard integrating by parts rule in equation(1)**

$$\ldots\left[\int u\,v\,dx = u\int v\,dx - \int\left(\frac{d}{dx}u\right)\left(\int v\,dx\right)dx\right]$$

Here taking $u = \sin^{n-1} x$, $v = \sin x$

$$\therefore I_n = \sin^{n-1} x\int_0^{\frac{\pi}{4}}\sin x\,dx - \int_0^{\frac{\pi}{4}}\left(\frac{d}{dx}\sin^{n-1} x\right)\left(\int \sin x\,dx\right)dx \quad \ldots(2)$$

Step II :

→ **Using standard formula in equation (2)**

$$\ldots\left[\frac{d}{dx}(\sin^{n-1} x) = (n-1)\sin^{n-2} x\frac{d}{dx}(\sin x)\right.$$

$$\left.= (n-1)\sin^{n-2} x(\cos x) \quad \int \sin x\,dx = -\cos x\right]$$

$$\therefore I_n = [\sin^{n-1} x(-\cos x)]_0^{\pi/4} - \int_0^{\frac{\pi}{4}}[(n-1)\sin^{n-2} x(\cos x)](-\cos x)\,dx)$$

$$= -[\sin^{n-1} x\cos x]_0^{\pi/4} - (n-1)(-1)\int_0^{\frac{\pi}{4}}[\sin^{n-2} x(\cos x)](\cos x)\,dx)$$

$$= -[\sin^{n-1} x\cos x]_0^{\pi/4} + (n-1)\int_0^{\frac{\pi}{4}}\sin^{n-2} x\cos^2 x\,dx$$

$$= -\left[\left(\sin\frac{\pi}{4}\right)^{n-1}\left(\cos\frac{\pi}{4}\right) - (\sin 0)^{n-1}(\cos 0)\right] + (n-1)\int_0^{\frac{\pi}{4}}\sin^{n-2} x\cos^2 x\,dx \quad \ldots(3)$$

Step III :

→ **Using standard values of trigonometric functions in equation (3)**

$$\ldots\left[\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}},\ \cos\frac{\pi}{4}=\frac{1}{\sqrt{2}},\ \sin 0=0,\ \cos 0=1,\ \cos^2 x=1-\sin^2 x\right]$$

$$I_n=-\left[\left(\sin\frac{\pi}{4}\right)^{n-1}\left(\cos\frac{\pi}{4}\right)-(0)(1)\right]+(n-1)\int_0^{\frac{\pi}{4}}\sin^{n-2}x\,(1-\sin^2 x)\,dx$$

$$=-\left[\left(\frac{1}{\sqrt{2}}\right)^{n-1}\left(\frac{1}{\sqrt{2}}\right)-(0)\right]+(n-1)\int_0^{\frac{\pi}{4}}\sin^{n-2}x\,(1-\sin^2 x)\,dx \quad \ldots(4)$$

Step IV

→ **Using standard index rule in equation (4)**

$$\ldots\left[a^m\times a^n=a^{m+n},\ \text{Here } a=\frac{1}{\sqrt{2}},\ m=n-1,\ n=1\right]$$

$$\left(\frac{1}{\sqrt{2}}\right)^{n-1}\left(\frac{1}{\sqrt{2}}\right)=\left(\frac{1}{\sqrt{2}}\right)^{n}$$

$$I_n=-\left(\frac{1}{\sqrt{2}}\right)^{n}+(n-1)\left[\int_0^{\frac{\pi}{4}}\sin^{n-2}x\,dx-\int_0^{\frac{\pi}{4}}(\sin^{n-2}x)(\sin^2 x)\,dx\right]$$

$$=-\left(\frac{1}{\sqrt{2}}\right)^{n}+(n-1)\left[\int_0^{\frac{\pi}{4}}\sin^{r-2}x\,dx-\int_0^{\frac{\pi}{4}}\sin^{n}x\,dx\right] \quad \ldots(5)$$

Step V : From equation (1),

$$\int_0^{\frac{\pi}{4}}\sin^n x\,dx=I_n \Rightarrow \int_0^{\frac{\pi}{4}}\sin^{n-2}x\,dx=I_{n-2}$$

$$I_n=-\left(\frac{1}{\sqrt{2}}\right)^{n}+(n-1)\,[I_{n-2}-I_n]=-\left(\frac{1}{\sqrt{2}}\right)^{n}+(n-1)\,I_{n-2}-(n-1)\,I_n$$

$$I_n+(n-1)\,I_n = -\left(\frac{1}{\sqrt{2}}\right)^{n}+(n-1)\,I_{n-2}$$

$$\Rightarrow\quad [1+(n-1)]\,I_n = -\left(\frac{1}{\sqrt{2}}\right)^{n}+(n-1)\,I_{n-2}$$

$$\Rightarrow\quad n\,I_n = -\left(\frac{1}{\sqrt{2}}\right)^{n}+(n-1)\,I_{n-2}$$

$$I_n = -\frac{1}{n}\left(\frac{1}{\sqrt{2}}\right)^{n}+\frac{n-1}{n}\,I_{n-2} \quad \ldots(6)$$

Step VI : Now, $\int_0^{\frac{\pi}{4}}\sin^6 x\,dx=I_6$

$\therefore$ Put $n=6$ in Equation (6),

$$I_6=-\frac{1}{6}\left(\frac{1}{\sqrt{2}}\right)^{6}+\frac{6-1}{6}\,I_{6-2}=-\frac{1}{6}\left(\frac{1}{\sqrt{2}}\right)^{6}+\frac{5}{6}\,I_4$$

$$=-\frac{1}{6}\left(\frac{1}{2}\right)^{6\times 1/2}+\frac{5}{6}\,I_4=-\frac{1}{6}\left(\frac{1}{2}\right)^{3}+\frac{5}{6}\,I_4$$

$$=-\frac{1}{6}\left(\frac{1}{8}\right)+\frac{5}{6}\,I_4=-\frac{1}{48}+\frac{5}{6}\,I_4 \quad \ldots(7)$$

Step VII :

Put $n=4$ in Equation (6)

$$I_4=-\frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^{4}+\frac{4-1}{4}\,I_{4-2}=-\frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^{4}+\frac{3}{4}\,I_2$$

$$=-\frac{1}{4}\left(\frac{1}{2}\right)^{4\times 1/2}+\frac{3}{4}\,I_2=-\frac{1}{4}\left(\frac{1}{2}\right)^{2}+\frac{3}{4}\,I_2$$

$$=-\frac{1}{4}\left(\frac{1}{4}\right)+\frac{3}{4}\,I_2=-\frac{1}{16}+\frac{3}{4}\,I_2 \quad \ldots(8)$$

Step VIII : Put $n=2$ in Equation (6),

$$I_2=-\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^{2}+\frac{2-1}{2}\,I_{2-2}=-\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^{2}+\frac{1}{2}\,I_0$$

$$=-\frac{1}{2}\left(\frac{1}{2}\right)^{2\times 1/2}+\frac{1}{2}\,I_0=-\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{2}\,I_0$$

$$=-\frac{1}{4}+\frac{1}{2}\,I_0 \quad \ldots(9)$$

But from equation (1), $I_0=\int_0^{\frac{\pi}{4}}\sin^0 x\,dx=\int_0^{\frac{\pi}{4}}1\,dx=\frac{\pi}{4}$

Step IX : Using value of I_0 in equation (9)

$$\therefore\ \text{Equation (9)} \Rightarrow I_2=-\frac{1}{4}+\frac{1}{2}\left(\frac{\pi}{4}\right)=-\frac{1}{4}+\frac{\pi}{8}$$

Step X : Using value of I_2 in equation (8)

$$\text{Equation (8)} \Rightarrow I_4=-\frac{1}{16}+\frac{3}{4}\left(-\frac{1}{4}+\frac{\pi}{8}\right)$$

$$=-\frac{1}{16}+\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)+\left(\frac{3}{4}\right)\left(\frac{\pi}{8}\right)$$

$$=-\frac{1}{16}-\frac{3}{16}+\frac{3\pi}{32}=\frac{-1-3}{16}+\frac{3\pi}{32}=-\frac{4}{16}+\frac{3\pi}{32}=-\frac{1}{4}+\frac{3\pi}{32}$$

Step XI : Using value of I_4 in equation (7)

$$\text{Equation (7)} \Rightarrow I_6=-\frac{1}{48}+\frac{5}{6}\left(-\frac{1}{4}+\frac{3\pi}{32}\right)$$

$$=-\frac{1}{48}+\left(\frac{5}{6}\right)\left(-\frac{1}{4}\right)+\left(\frac{5}{6}\right)\left(\frac{3\pi}{32}\right)$$

$$=-\frac{1}{48}-\frac{5}{24}+\frac{15\pi}{6\times 32}=-\frac{1}{48}-\frac{5\times 2}{24\times 2}+\frac{15\pi}{6\times 32}$$

$$=-\frac{1}{48}-\frac{10}{48}+\frac{15\pi}{6\times 32}$$

$$=\frac{-1-10}{48}+\frac{5\pi}{64}=-\frac{11}{48}+\frac{5\pi}{64}$$

$$\therefore\quad \int_0^{\frac{\pi}{4}}\sin^6 x\,dx = -\frac{\mathbf{11}}{\mathbf{48}}+\frac{\mathbf{5\pi}}{\mathbf{64}}$$

Example 6.8.2

If $I_n=\int_0^{\pi/4}\cos^{2n}x\,dx$, then prove that

$I_n=\frac{1}{n\cdot 2^{n+1}}+\frac{2n-1}{2n}\,I_{n-1}$. Hence evaluate $\int_0^{\pi/4}\cos^6 x\,dx$.

Solution :

Step I

$$I_n=\int_0^{\pi/4}\cos^{2n}x\,dx=\int_0^{\pi/4}\cos^{2n-1}x\cos x\,dx \quad \ldots(1)$$

→ **Using standard integrating by parts rule in equation(1)**

$$\ldots\left[\int u\,v\,dx=u\int v\,dx-\int\left(\frac{d}{dx}u\right)\left(\int v\,dx\right)dx\right]$$

Here taking $u=\cos^{2n-1}x$, $v=\cos x$

$$I_n=\cos^{2n-1}x\int\cos x\,dx-\int_0^{\pi/4}\left[\frac{d}{dx}\cos^{2n-1}x\right][\sin x\,dx]\,dx \quad \ldots(2)$$

Step II

→ **Using standard formula in equation (2)**

$$\ldots\left[\frac{d}{dx}(\cos^{2n-1}x)=(2n-1)\cos^{2n-2}x\,\frac{d}{dx}(\cos x)=(2n-1)\cos^{2n-2}x\,(-\sin x)\ \int\cos x\,dx=\sin x\right]$$

$$I_n=\left[\cos^{2n-1}x\sin x\right]_0^{\pi/4}-\int_0^{\pi/4}[(2n-1)\cos^{2n-2}x\,(-\sin x)(\sin x)]\,dx \quad \ldots(3)$$

Step III

→ **Using standard values of trigonometric functions in equation (3)**

$\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$, $\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$, $\sin 0=0$, $\cos 0=1$, $\sin^2 x=1-\cos^2 x$

$$I_n=\left[\left(\frac{1}{\sqrt{2}}\right)^{2n-1}\left(\frac{1}{\sqrt{2}}\right)-(1)(0)\right]+(2n-1)\int_0^{\pi/4}\cos^{2n-2}x\sin^2 x\,dx$$

$$=\left[\left(\frac{1}{\sqrt{2}}\right)^{2n-1}\left(\frac{1}{\sqrt{2}}\right)-(0)\right]+(2n-1)\int_0^{\frac{\pi}{4}}\cos^{2n-2}x\,(1-\cos^2 x)\,dx \quad \ldots(4)$$

Step IV

→ **Using standard index rule in equation (4)**

$$\ldots\left[a^m\times a^n=a^{m+n},\ \text{Here } a=\frac{1}{\sqrt{2}},\ m=2n-1,\ n=1\right]$$

$$\left(\frac{1}{\sqrt{2}}\right)^{2n-1}\left(\frac{1}{\sqrt{2}}\right)=\left(\frac{1}{\sqrt{2}}\right)^{2n}$$

$$I_n=\left(\frac{1}{\sqrt{2}}\right)^{2n}+(2n-1)\left[\int_0^{\frac{\pi}{4}}\cos^{2n-2}x\,dx-\int_0^{\frac{\pi}{4}}(\cos^{2n-2}x)(\cos^2 x)\,dx\right]$$

$$=\left(\frac{1}{\sqrt{2}}\right)^{2n}+(2n-1)\left[\int_0^{\frac{\pi}{4}}\cos^{2n-2}x\,dx-\int_0^{\frac{\pi}{4}}\cos^{2n}x\,dx\right] \quad \ldots(5)$$

Step V : From equation (1),

$$\int_0^{\frac{\pi}{4}}\cos^{2n}x\,dx=I_n \Rightarrow \int_0^{\frac{\pi}{4}}\cos^{2n-2}x\,dx=\int_0^{\frac{\pi}{4}}\cos^{2(n-1)}x\,dx=I_{n-1}$$

$$I_n=\left[\left(\frac{1}{2}\right)^{1/2}\right]^{2n}+(2n-1)\left[\int_0^{\pi/4}\cos^{2n-2}x\,dx-\int_0^{\pi/4}\cos^{2n}x\,dx\right]$$

$$I_n=\left(\frac{1}{2}\right)^{n}+(2n-1)\,[I_{n-1}-I_n]$$

$$\Rightarrow I_n=\left(\frac{1}{2}\right)^{n}+(2n-1)\,I_{n-1}-(2n-1)\,I_n$$

$$I_n+(2n-1)\,I_n=\frac{1}{2^n}+(2n-1)\,I_{n-1}$$

$$\Rightarrow [1+(2n-1)]\,I_n=\frac{1}{2^n}+(2n-1)\,I_{n-1}$$

$$\Rightarrow 2n\,I_n=\frac{1}{2^n}+(2n-1)\,I_{n-1}$$

$$I_n=\frac{1}{(2n)}\,\frac{1}{2^n}+\frac{2n-1}{(2n)}\,I_{n-1}=\frac{1}{n\cdot 2^{n+1}}+\frac{2n-1}{(2n)}\,I_{n-1} \quad \ldots(6)$$

Hence Proof

To evaluate $\int_0^{\pi/4}\cos^6 x\,dx$

Step VI : Since $\int_0^{\pi/4} \cos^{2n} x\, dx = I_n$

$\therefore \int_0^{\pi/4} \cos^6 x\, dx = I_3$

$\therefore$ Put n = 3 in Equation (6)

$$I_3 = \frac{1}{3 \cdot 2^{3+1}} + \frac{2(3)-1}{2(3)} I_{4-2} = \frac{1}{3 \cdot 2^4} + \frac{6-1}{6} I_2$$

$$= \frac{1}{(3)\cdot(16)} + \frac{5}{6} I_2 = \frac{1}{48} + \frac{5}{6} I_2 \quad \ldots(7)$$

Step VII : Put n = 2 in Equation (6), $I_2 = \frac{1}{2 \cdot 2^{2+1}} + \frac{2(2)-1}{2(2)}$

$$I_{2-1} = \frac{1}{2 \cdot 2^3} + \frac{4-1}{4} I_1 = \frac{1}{(2)\cdot(8)} + \frac{3}{4} I_2$$

$$= \frac{1}{16} + \frac{3}{4} I_1 \quad \ldots(8)$$

Step VIII : Put n = 1 in Equation (6)

$$I_1 = \frac{1}{1 \cdot 2^{1+1}} + \frac{2(1)-1}{2(1)} I_{1-1} = \frac{1}{1 \cdot 2^2} + \frac{2-1}{2} I_0$$

$$= \frac{1}{(1)\cdot(4)} + \frac{1}{2} I_0 = \frac{1}{4} + \frac{1}{2} I_0 \quad \ldots(9)$$

But from equation (1),

$$I_0 = \int_0^{\pi/4} \cos^0 x\, dx = \int_0^{\pi/4} 1\, dx = (x)_0^{\pi/4} = \frac{\pi}{4}$$

Step IX : $\therefore$ From Equation (9),

$$I_1 = \frac{1}{4} + \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{1}{4} + \frac{\pi}{8}$$

Step X : Put value of I_1 in Equation (8)

$$I_2 = \frac{1}{16} + \frac{3}{4}\left(\frac{1}{4} + \frac{\pi}{8}\right) = \frac{1}{16} + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{\pi}{8}\right) = \frac{1}{16} + \left(\frac{3}{16}\right) + \left(\frac{3\pi}{32}\right)$$

$$I_2 = \frac{1+3}{16} + \frac{3\pi}{32} = \frac{4}{16} + \frac{3\pi}{32} = \frac{1}{4} + \frac{3\pi}{32}$$

Step XI : Put value of I_2 in Equation (7)

$$I_3 = \frac{1}{48} + \frac{5}{6}\left(\frac{1}{4} + \frac{3\pi}{32}\right) = \frac{1}{48} + \frac{5}{6} \times \frac{1}{4} + \frac{5}{6} \times \frac{3\pi}{32}$$

$$= \frac{1}{48} + \frac{5}{24} + \frac{5}{2} \times \frac{\pi}{32}$$

$$= \frac{1}{48} + \frac{5 \times 2}{24 \times 2} + \frac{5\pi}{64} = \frac{1}{48} + \frac{10}{48} + \frac{5\pi}{64} = \frac{1+10}{48} + \frac{5\pi}{64}$$

$$I_3 = \frac{11}{48} + \frac{5\pi}{64}$$

$$\therefore \int_0^{\pi/4} \cos^6 x\, dx = \mathbf{\frac{11}{48} + \frac{5\pi}{64}}$$

Example 6.8.3

If $U_n = \int x^n e^x\, dx$, then prove that $U_n = x^n e^x - n\, U_{n-1}$, Hence evaluate U_4.

Solution :

Step I : $U_n = \int x^n e^x\, dx \quad \ldots(1)$

➔ **Using standard integrating by parts rule in equation(1)**

$$\ldots\left[\int u\, v\, dx = u \int v\, dx - \int\left(\frac{d}{dx} u\right)\left(\int v\, dx\right) dx\right]$$

Here taking $u = x^n$, $v = e^x$

$$U_n = x^n \int e^x\, dx - \int\left(\frac{d}{dx} x^n\right)\left(\int e^x\right) dx \quad \ldots(2)$$

Step II

➔ **Using standard formula of derivative and integration in equation(2)**

$$\ldots\left[\int e^x\, dx = e^x, \frac{d}{dx} x^n = n\, x^{n-1}\right]$$

$$U_n = x^n e^x - \int n\, x^{n-1} e^x\, dx = x^n e^x - n \int x^{n-1} e\, dx \quad \ldots(9) \quad \ldots(3)$$

Step III :

But from equation (1),

$$U_n = \int x^n e^x\, dx \Rightarrow \int x^{n-1} e^x\, dx = U_{n-1}$$

Equation (3) can be written as, $U_n = x^n e^x - n\, U_{n-1} \quad \ldots(4)$

Hence proof

Step IV : To find U_4, put n = 4 in Equation (4)

$$U_4 = x^4 e^x - 4\, U_3 \quad \ldots(5)$$

Step V : Put n = 3 in Equation (4) we get,

$$U_3 = x^3 e^x - 3\, U_2 \quad \ldots(6)$$

Step VI : Put n = 2 in Equation (4) we get,

$$U_2 = x^2 e^x - 2\, U_1 \quad \ldots(7)$$

But from equation (1), $U_1 = \int x\, e^x = x\, e^x - \int e^x\, dx$

$$= x\, e^x - e^x = (x-1)\, e^x$$

Step VII :

Equation (7) $\Rightarrow U_2 = x^2 e^x - 2U_1 = x^2 e^x - 2(x-1)\, e^x$

Step VIII :

Equation (6) $\Rightarrow U_3 = x^3 e^x - 3U_2$

$$U_3 = x^3 e^x - 3[x^2 e^x - 2(x-1)\, e^x]$$

$$= x^3 e^x - 3x^2 e^x + 6(x-1)\, e^x]$$

$\therefore$ Equation (5) $\Rightarrow$

$$U_4 = x^4 e^x - 4[x^3 e^x - 3x^2 e^x + 6(x-1)\, e^x]$$

$$= \mathbf{e^x [x^4 - 4x^3 + 12x^2 - 24x + 24]}$$

Example 6.8.4

$U_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx$, then show that $U_n = \frac{1}{n-1} - U_{n-2}$.

Hence evaluate $\int_0^{\frac{\pi}{4}} \tan^6 x\, dx$.

Solution :

Step I : $U_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx \quad \ldots(1)$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \tan^2 x\, dx \quad \ldots(2)$$

Step II

➔ **Using standard trigonometric identity in equation (2)**

$$\ldots[1 + \tan^2 x = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1]$$

$$U_n = \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\sec^2 x - 1)\, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x\, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, dx \quad \ldots(3)$$

In first integral, put $\tan x = t$ $\therefore \sec^2 x\, dx = dt$

When $x = 0$, $t = \tan 0 = 0$

When $x = \frac{\pi}{4}$, $t = \tan\frac{\pi}{4} = 1$

And

x	0	$\frac{\pi}{4}$
t	$\tan 0 = 0$	$\tan\frac{\pi}{4} = 1$

In second integral, $\int_0^{\frac{\pi}{4}} \tan^{n-2} x\, dx = U_{n-2}$

$$\therefore \quad U_n = \int_0^1 t^{n-2}\, dt - U_{n-2} \quad \ldots(4)$$

Step III :

➔ **Using standard formula of integration in equation (4)**

$$\ldots\left[\int x^m\, dx = \frac{x^{m+1}}{m+1}, \text{ Here } x = t,\ m = n-2\right]$$

$$U_n = \left[\frac{t^{n-1}}{n-1}\right]_0^1 - U_{n-2} = \frac{1}{n-1}[1-0] - U_{n-2}$$

$$= \frac{1}{n-1} - U_{n-2} \quad \ldots(5)$$

Hence Proof

Step IV : Now, Put n = 6 in Equation (5)

$$U_6 = \frac{1}{6-1} - U_{6-2} = \frac{1}{5} - U_4 \quad \ldots(6)$$

Step V : Put n = 4 in Equation (5)

$$U_4 = \frac{1}{4-1} - U_{4-2} = \frac{1}{3} - U_2 \quad \ldots(7)$$

Step VI : Put n = 2 in Equation (5)

$$U_2 = \frac{1}{2-1} - U_{2-2} = \frac{1}{1} - U_0 \quad \ldots(8)$$

Step VII : But from equation (1),

$$U_0 = \int_0^{\frac{\pi}{4}} \tan^0 x\, dx = \int_0^{\frac{\pi}{4}} dx = [x]_0^{\pi/4} = \left[\frac{\pi}{4} - 0\right] = \frac{\pi}{4}$$

From Equation (8), $U_2 = 1 - \frac{\pi}{4}$

$$\text{Equation (7)} \Rightarrow U_4 = \frac{1}{3} - \left(1 - \frac{\pi}{4}\right) = \frac{1}{3} - 1 + \frac{\pi}{4}$$

$$= \frac{1}{3} - \frac{3}{3} + \frac{\pi}{4} = \frac{1-3}{3} + \frac{\pi}{4}$$

$$U_4 = -\frac{2}{3} + \frac{\pi}{4}$$

$$\text{Equation (6)} \Rightarrow U_6 = \frac{1}{5} - \left(-\frac{2}{3} + \frac{\pi}{4}\right)$$

$$= \frac{1}{5} + \frac{2}{3} - \frac{\pi}{4} = \frac{1 \times 3}{5 \times 3} + \frac{2 \times 5}{3 \times 5} - \frac{\pi}{4}$$

$$U_6 = \frac{3}{15} + \frac{10}{15} - \frac{\pi}{4} = \frac{3+10}{15} - \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{4}} \tan^6 x\, dx = \frac{13}{15} - \frac{\pi}{4}$$

Example 6.8.5

$I_n = \int_{\pi/4}^{\pi/2} \cot^n \theta\, d\theta$, prove that $I_n = \frac{1}{n-1} - I_{n-2}$.

Hence evaluate $\int_{\pi/4}^{\pi/2} \cot^6 \theta\, d\theta$.

Solution :

Step I : $I_n = \int_{\pi/4}^{\pi/2} \cot^n \theta\, d\theta \quad \ldots(1)$

$$= \int_{\pi/4}^{\pi/2} \cot^{n-2}\theta \cot^2\theta \, d\theta \quad ...(2)$$

Step II :

→ **Using standard trigonometric identity in equation (2)**

$$...[1+\cot^2\theta = \operatorname{cosec}^2\theta \Rightarrow \cot^2\theta = \operatorname{cosec}^2\theta - 1]$$

$$I_n = \int_{\pi/4}^{\pi/2} \cot^{n-2}\theta (\operatorname{cosec}^2\theta - 1)\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \cot^{n-2}\theta \operatorname{cosec}^2\theta \, d\theta - \int_{\pi/4}^{\pi/2} \cot^{n-2}\theta \, d\theta \quad ...(3)$$

In first integral, put $\cot\theta = t$

$\therefore -\operatorname{cosec}^2\theta \, d\theta = dt \Rightarrow \operatorname{cosec}^2\theta\, d\theta = -dt$

When $\theta = \frac{\pi}{2}$, $\quad t = \cot\frac{\pi}{2} = 1$

When $\theta = \frac{\pi}{4}$, $\quad t = \cot\frac{\pi}{4} = 0$

And

θ	$\frac{\pi}{2}$	$\frac{\pi}{4}$
t	$\cot\frac{\pi}{2} = 1$	$\cot\frac{\pi}{4} = 0$

In second integral, $\int_{\pi/4}^{\pi/2} \cot^{n-2}\theta \, d\theta = I_{n-2}$

$$\therefore \quad I_n = -\int_1^0 t^{n-2} dt - I_{n-2} \quad ...(4)$$

Step III

→ **Using standard formula of integration in equation (4)**

$$...\left[\int x^m dx = \frac{x^{m+1}}{m+1} \right.$$

$$\left. \text{Here } x = t, \ m = n-2 \right]$$

$$I_n = -\left[\frac{t^{n-1}}{n-1}\right]_1^0 - I_{n-2} \quad = -\left[0 - \frac{1}{n-1}\right] - I_{n-2}$$

$$I_n = \frac{1}{n-1} - I_{n-2} \quad ...(5)$$

Hence Proof

Step IV :

Now from equation (1) $\int_{\pi/4}^{\pi/2} \cot^n\theta \, d\theta = I_n$

$$\therefore \int_{\pi/4}^{\pi/2} \cot^6\theta \, d\theta = I_6$$

Step V :

Put n = 6 in Equation (1), $I_6 = \int_{\pi/4}^{\pi/2} \cot^6\theta \, d\theta = \frac{1}{5} - I_4 \quad ...(6)$

Put n = 4 in Equation (1), $I_4 = \frac{1}{3} - I_2 \quad ...(7)$

Put n = 2 in Equation (1), $I_2 = 1 - I_0 \quad ...(8)$

But again from equation (10),

$$I_0 = \int_{\pi/4}^{\pi/2} \cot^0\theta d\theta = \int_{\pi/4}^{\pi/2} (1) d\theta = [\theta]_{\pi/4}^{\pi/2} = \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{2\pi}{4} - \frac{\pi}{4} = \frac{2\pi - \pi}{4} = \frac{\pi}{4}$$

$\therefore$ From equation (8), $I_2 = 1 - \frac{\pi}{4}$

Step VI : Using value of I_2 in equation (7),

$$I_4 = \frac{1}{3} - I_2 = \frac{1}{3} - \left(1 - \frac{\pi}{4}\right) = \frac{1}{3} - 1 + \frac{\pi}{4} = \frac{1}{3} - \frac{3}{3} + \frac{\pi}{4}$$

$$= \frac{1-3}{3} + \frac{\pi}{4} = -\frac{2}{3} + \frac{\pi}{4}$$

Using value of I_4 in equation (6),

$$\Rightarrow I_6 = \frac{1}{5} - \left(-\frac{2}{3} + \frac{\pi}{4}\right) = \frac{1}{5} + \frac{2}{3} - \frac{\pi}{4} = \frac{1\times3}{5\times3} + \frac{2\times5}{3\times5} - \frac{\pi}{4}$$

$$I_6 = \frac{3}{15} + \frac{10}{15} - \frac{\pi}{4} = \frac{3+10}{15} - \frac{\pi}{4} = \mathbf{\frac{13}{15} - \frac{\pi}{4}}$$

Example 6.8.6

If $I_{(m,n)} = \int x^m (\log x)^n dx$, prove that

$$\mathbf{I_{(m,n)} = \frac{x^{m+1}}{(m+1)} (\log x)^n - \frac{n}{m+1} I_{(m,n-1)}}.$$

Solution :

Step I

$$I_{(m,n)} = \int x^m (\log x)^n dx \quad ...(1)$$

→ **Using standard Integrating by parts rule in equation(1)**

$$...\left[\int u v \, dx = u \int v \, dx - \int \left(\frac{d}{dx} u\right) \left(\int v \, dx\right) dx \right]$$

Here taking $u = (\log x)^n$, $v = x^m$

$$I_{(m,n)} = (\log x)^n \int x^m dx - \int \left(\frac{d}{dx} (\log x)^n\right) \left(\int x^m\right) dx \quad ...(2)$$

Step II

→ **Using standard formula of derivative and Integration in equation(2)**

$$...\left[\int x^m dx = \frac{x^{m+1}}{m+1}, \ \frac{d}{dx}(\log x)^n = n(\log x)^{n-1} \frac{d}{dx}(\log x) \right]$$

$$I_{(m,n)} = (\log x)^n \frac{x^{m+1}}{(m+1)} - \int n (\log x)^{n-1} \frac{1}{x} \frac{x^{m+1}}{(m+1)} dx$$

$$= \frac{x^{m+1}}{(m+1)} (\log x)^n - \frac{n}{(m+1)} \int x^m (\log x)^{n-1} dx \quad ...(3)$$

From equation (1), $\int x^m (\log x)^n dx = I_{(m,n)}$

Changing only n to (n − 1) we get,

$$\int x^m (\log x)^{n-1} dx = I_{(m,n-1)}$$

$$I_{(m,n)} = \frac{x^{m+1}}{(m+1)} (\log x)^n - \frac{n}{(m+1)} I_{(m,n-1)}$$

Example 6.8.7

If $f(m,n) = \int x^m (1-x)^n dx$,

show that $f(m,n) = \frac{x^{m+1}(1-x)^n}{m+n+1} + \frac{n}{m+n+1} f(m, n-1)$

Hence show that $\int_0^1 x^m (1-x)^n dx = \frac{m!\ n!}{(m+n+1)!}$

Solution :

Step I : Consider, $f(m,n) = \int x^m (1-x)^n dx \quad ...(1)$

→ **Using standard Integrating by parts rule in equation(1)**

$$...\left[\int u v \, dx = u \int v \, dx - \int \left(\frac{d}{dx} u\right) \left(\int v \, dx\right) dx \right.$$

$$\left. \text{Here taking } u = (1-x)^n, \ v = x^m \right]$$

$$I_{(m,n)} = (1-x)^n \int x^m dx - \int \left(\frac{d}{dx}(1-x)^n\right) \left(\int x^m\right) dx \quad ...(2)$$

Step II

→ **Using standard formula of derivative and Integration in equation(2)**

$$...\left[\int x^m dx = \frac{x^{m+1}}{m+1}, \right.$$

$$\left. \frac{d}{dx}(1-x)^n = n(1-x)^{n-1} \frac{d}{dx}(1-x) \right]$$

$$f(m,n) = \frac{(1-x)^n x^{m+1}}{m+1} - \int n (1-x)^{n-1} (-1) \frac{x^{m+1}}{m+1} dx$$

$$= \frac{(1-x)^n x^{m+1}}{m+1} + \frac{n}{m+1} \int (1-x)^{n-1} x^m \cdot x \, dx \quad ...(3)$$

Step III

Adjusting the term x in the form of

$(1-x)$ as $x = 1 - (1-x) = -(1-x) + 1$

$$f(m,n) = \frac{(1-x)^n x^{m+1}}{m+1} + \frac{n}{m+1} \left[\int (1-x)^{n-1} x^m [-(1-x) + 1] dx \right]$$

$$= \frac{(1-x)^n x^{m+1}}{m+1}$$

$$+ \frac{n}{m+1} \left[\int (1-x)^{n-1} x^m (-)(1-x) + (1-x)^{n-1} x^m (1) \, dx \right]$$

Step IV

Using law of index we write $(1-x)^{n-1}(1-x) = (1-x)^n$

$$f(m,n) = \frac{(1-x)^n x^{m+1}}{m+1}$$

$$+ \frac{n}{m+1} \left[-\int (1-x)^n x^m dx + \int (1-x)^{n-1} x^m dx \right]$$

$$= \frac{(1-x)^n x^{m+1}}{m+1}$$

$$+ \left[-\frac{n}{m+1} \int (1-x)^n x^m dx + \frac{n}{m+1} \int (1-x)^{n-1} x^m dx \right] \quad ...(4)$$

Step V : From equation (1), $\int x^m (1-x)^n dx = f(m,n)$

$$\Rightarrow \int x^m (1-x)^{n-1} dx = f(m, n-1)$$

$$f(m,n) = \frac{(1-x)^n x^{m+1}}{m+1} - \frac{n}{m+1} f(m,n) + \frac{n}{m+1} f(m, n-1)$$

$$f(m,n) + \frac{n}{m+1} f(m,n) = \frac{(1-x)^n x^{m+1}}{m+1} + \frac{n}{m+1} f(m, n-1)$$

$$\left(1 + \frac{n}{m+1}\right) f(m,n) = \frac{(1-x)^n x^{m+1}}{m+1} + \frac{n}{m+1} f(m, n-1)$$

$$\left(\frac{m+1+n}{m+1}\right) f(m,n) = \frac{(1-x)^n x^{m+1}}{m+1} + \frac{n}{m+1} f(m, n-1)$$

Multiplying each term by (m+1) we get,

$$f(m,n) = \frac{(1-x)^n x^{m+1}}{m+n+1} + \frac{n}{m+n+1} f(m, n-1) \quad ...(5)$$

Hence proof

Step VI : From equation (5),

$$f(m,n) = \int_0^1 x^m (1-x)^n dx = \frac{(1-x)^n x^{m+1}}{m+n+1} + \frac{n}{m+n+1} f(m, n-1)$$

Taking limits of integration on right hand side as 0 to 1,

$$\text{Now, } \int_0^1 x^m (1-x)^n dx = \left[\frac{(1-x)^n x^{m+1}}{m+n+1}\right]_0^1 + \frac{n}{m+n+1} f(m, n-1)$$

$$= \left[\frac{(1-1)^n (1)^{m+1}}{m+n+1} - \frac{(1-0)^n (0)^{m+1}}{m+n+1}\right] + \frac{n}{m+n+1} f(m, n-1)$$

$$= [0-0] + \frac{n}{m+n+1} f(m, n-1)$$

$$f(m, n) = \frac{n}{m+n+1} f(m, n-1) \quad \ldots(6)$$

Step VII : Applying formula (6) to

$$f(m, n-1) = \frac{n-1}{m+(n-1)+1} f(m, (n-1)-1)$$

$$= \frac{n-1}{m+n} f(m, n-2)$$

Similarly , $f(m, n-2) = \frac{n-2}{m+(n-2)+1} f(m, (n-2)-1)$

$$= \frac{n-2}{m+n-1} f(m, n-3)$$

Continuing in this way,equation (6) can be written as ,

$$f(m, n) = \frac{n}{m+n+1} \cdot \frac{n-1}{m+n} \cdot \frac{n-2}{m+n-1} \ldots f(m,0) \quad \ldots(7)$$

But, $f(m, 0) = \int_0^1 x^m \, dx = \left[\frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{m+1}$

$\therefore$ From equation (7) ,

$$\int_0^1 x^m (1-x)^n \, dx = \frac{n}{m+n+1} \cdot \frac{n-1}{m+n} \cdot \frac{n-2}{m+n-1} \cdots\cdots \frac{1}{m+1}$$

Step VIII : Multiplying and dividing by 1.2.3.....m = m!

$$\int_0^1 x^m (1-x)^n \, dx = \frac{(1 \cdot 2 \cdot 3 \cdots m)\,(n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots 3 \cdot 2 \cdot 1)}{(m+n+1)(m+n)(m+n-1)(m+n-2) \cdots (1 \cdot 2 \cdot 3 \cdots m)} \quad \ldots(8)$$

In equation (8) we observe that

$1 \cdot 2 \cdot 3 \cdots m = m!$, $1 \cdot 2 \cdot 3 \cdots (n-3)(n-2)(n-1)(n) = n!$,

$1 \cdot 2 \cdot 3 \cdots m \cdot (m+n-2)(m+n-1)(m+n-1)(m+n)(m+n+1) = (m+n+1)!$

Using these values in equation (8) ,

$$\int_0^1 x^m (1-x)^n \, dx = \frac{m! \; n!}{(m+n+1)!}$$

Example 6.8.8

If $I_n = \int_0^{\frac{\pi}{2}} x \cos^n x \, dx$ then show that $I_n = -\frac{1}{n^2} + \frac{n-1}{n} I_{n-2}$. Hence find I_3.

Solution : Step I : $I_n = \int_0^{\frac{\pi}{2}} x \cos^n x \, dx \quad \ldots(1)$

$$I_n = \int_0^{\frac{\pi}{2}} x \cos^{n-1} x \cos x \, dx \quad \ldots(2)$$

→ **Using standard integrating by parts rule in equation(2)**

$$\ldots\left[\int u\,v\,dx = u \int v\,dx - \int \left(\frac{d}{dx} u\right) \left(\int v\,dx\right) dx\right.$$

$$\left.\text{Here taking } u = x\cos^{n-1}x\,,\ v = \cos x\right]$$

$$I_n = [x(\cos^{n-1} x)(\int \cos x dx)] - \int_0^{\frac{\pi}{2}} \left(\frac{d}{dx} x \cos^{n-1} x\right) \left(\int \cos x \, dx\right) dx \quad \ldots(3)$$

Step II

→ **Using standard formula of derivative and integration in equation(3)**

$$\ldots\left[\int \cos x \, dx = \sin x\,,\right.$$

$$\frac{d}{dx}(x \cos^{n-1} x) = x[(n-1)\cos^{n-2} x] \frac{d}{dx}(\cos x)$$

$$+ (\cos^{n-1} x) \frac{d}{dx}(x) \text{ (By product rule of derivative)}$$

$$\left.\text{Here } \frac{d}{dx}(\cos x) = -\sin x\,,\ \frac{d}{dx}(x) = 1\right]$$

$$I_n = [x(\cos^{n-1} x)(\sin x)]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} [x(n-1)\cos^{n-2} x(-\sin x) + \cos^{n-1} x (1)] \sin x \, dx$$

$$= [\frac{\pi}{2} (\cos^{n-1} \frac{\pi}{2})(\sin \frac{\pi}{2}) - (0)(\cos 0)(\sin 0)]$$

$$- \int_0^{\frac{\pi}{2}} [-(n-1) x\cos^{n-2} x (\sin x) + \cos^{n-1} x (1)] \sin x \, dx \quad \ldots(4)$$

$$= (n-1) \int_0^{\frac{\pi}{2}} x \cos^{n-2} x \sin^2 x \, dx - \int_0^{\frac{\pi}{2}} \cos^{n-1} x \sin x \, dx \quad \ldots(5)$$

Step III

→ **Using standard formula in equation (5)**

$$\ldots\left[\sin^2 x = 1 - \cos^2 x\,,\ \int f^n(x) f'(x) \, dx = \frac{f^{n+1}(x)}{n+1}\right.$$

$$\left.\text{Here } f(x) = \cos x\,,\ f'(x) = -\sin x\right]$$

$$\therefore I_n = (n-1) \int_0^{\frac{\pi}{2}} x \cos^{n-2} x (1 - \cos^2 x) \, dx - \left[-\frac{\cos^n x}{n}\right]_0^{\frac{\pi}{2}}$$

$$= (n-1) \int_0^{\frac{\pi}{2}} x \cos^{n-2} x \, dx - (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x \cos^2 x \, dx - \left[-\frac{\cos^n x}{n}\right]_0^{\frac{\pi}{2}}$$

$$= (n-1) \int_0^{\frac{\pi}{2}} x \cos^{n-2} x \, dx - (n-1) \int_0^{\frac{\pi}{2}} x \cos^n x \, dx - \frac{1}{n}[-(0-1)] \quad \ldots(6)$$

From equation (1),

$$\int_0^{\frac{\pi}{2}} x \cos^n x \, dx = I_n \Rightarrow \int_0^{\frac{\pi}{2}} x \cos^{n-2} x \, dx = I_{n-2}$$

Equation (6) $\Rightarrow \quad I_n = (n-1) I_{n-2} - (n-1) I_n - \frac{1}{n}$

$$\therefore I_n + (n-1) I_n = (n-1) I_{n-2} - \frac{1}{n}$$

$$\Rightarrow [1 + (n-1)] I_n = (n-1) I_{n-2} - \frac{1}{n}$$

$$\therefore n I_n = (n-1) I_{n-2} - \frac{1}{n}$$

$$\therefore I_n = -\frac{1}{n^2} + \frac{n-1}{n} I_{n-2} \quad \ldots(7)$$

Hence proof

Step IV : Put n = 3 in Equation (7)

$$I_3 = -\frac{1}{3^2} + \frac{3-1}{3} I_{3-2}$$

$$\therefore I_3 = -\frac{1}{9} + \frac{2}{3} I_1 \quad \ldots(8)$$

Step V

$$\text{But } I_1 = \int_0^{\frac{\pi}{2}} x \cos x \, dx$$

Integrating by parts,

$$I_1 = x \int \cos x \, dx - \int [\frac{d}{dx}(x) \int \cos x \, dx] \, dx$$

$$= (x \sin x)_0^{\pi/2} - \int_0^{\frac{\pi}{2}} (1) \sin x \, dx$$

$$= (\frac{\pi}{2} \sin \frac{\pi}{2} - 0 \sin 0) - \int_0^{\frac{\pi}{2}} \sin x \, dx$$

$$= (\frac{\pi}{2} - 0) - (-\cos x)_0^{\pi/2} = \frac{\pi}{2} + (\cos x)_0^{\pi/2}$$

$$= \frac{\pi}{2} + (0-1) = \frac{\pi}{2} - 1$$

Step VI : $\therefore$ Using value of I_1 in Equation (8)

$$I_3 = -\frac{1}{9} + \frac{2}{3}\left(\frac{\pi}{2} - 1\right)$$

$$= -\frac{1}{9} + \left(\frac{2}{3}\right)\left(\frac{\pi}{2}\right) - \left(\frac{2}{3}\right)(1)$$

$$= -\frac{1}{9} + \left(\frac{\pi}{3}\right) - \left(\frac{2}{3}\right) = -\frac{1}{9} - \frac{2}{3} + \frac{\pi}{3}$$

$$= -\frac{1}{9} - \left(\frac{2 \times 3}{3 \times 3}\right) + \frac{\pi}{3}$$

$$= -\frac{1}{9} - \frac{6}{9} + \frac{\pi}{3} = \frac{-1-6}{9} + \frac{\pi}{3}$$

$$I_3 = -\frac{7}{9} + \frac{\pi}{3}$$

Example 6.8.9

If $I_n = \int_0^{\frac{\pi}{2}} \theta \sin^n \theta \, d\theta$ then, prove that $I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n^2}$.

Hence prove that $I_5 = \frac{149}{225}$.

Solution :

Step I : $\quad I_n = \int_0^{\frac{\pi}{2}} \theta \sin^n \theta \, d\theta \quad \ldots(1)$

$$I_n = \int_0^{\frac{\pi}{2}} \theta \sin^{n-1} \theta \sin \theta \, d\theta \quad \ldots(2)$$

Step II

→ **Using standard integrating by parts rule in equation(2)**

$$\ldots\left[\int u\,v\,dx = u \int v\,dx - \int \left(\frac{d}{dx} u\right) \left(\int v\,dx\right) dx\right.$$

$$\left.\text{Here taking } u = \theta \sin^{n-1} \theta\,,\ v = \sin\theta\right]$$

$$I_n = [(\theta \sin^{n-1} \theta)(\int \sin\theta \, d\theta)] - \int_0^{\frac{\pi}{2}} \left(\frac{d}{d\theta} \theta \sin^{n-1} \theta\right) \left(\int \sin\theta \, d\theta\right) d\theta \quad \ldots(3)$$

Step III

→ **Using standard formula of derivative and integration in equation(3)**

$$\cdots\left[\int \sin\theta\, d\theta = -\cos\theta,\right.$$

$$\frac{d}{d\theta}(\theta \sin^{n-1}\theta) = \theta\,[(n-1)\sin^{n-2}\theta]\frac{d}{d\theta}(\sin\theta)$$

$$+ (\sin^{n-1}\theta)\frac{d}{d\theta}(\theta) \text{ (By product rule of derivative)}$$

$$\left.\text{Here } \frac{d}{d\theta}(\cos\theta) = \cos\theta\ ,\ \frac{d}{d\theta}(\theta) = 1\right]$$

$$I_n = [\theta(\sin^{n-1}\theta)(-\cos\theta)]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} [\theta(n-1)\sin^{n-2}\theta(\cos\theta) + \sin^{n-1}\theta](-\cos\theta)\,d\theta$$

$$= \left(\frac{\pi}{2}\sin^{n-1}\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right)\left(-\cos\frac{\pi}{2}\right) - (0)(\sin^{n-1}0)(-\cos 0)]$$

$$+ \int_0^{\frac{\pi}{2}} [(n-1)\sin^{n-2}\theta(\cos\theta) + \sin^{n-1}\theta(1)]\cos\theta\,d\theta \quad \ldots(4)$$

$$= [\frac{\pi}{2}(1)(-0) - (0)(0)(-1)] + (n-1)\int_0^{\frac{\pi}{2}} \theta \sin^{n-2}\theta(\cos\theta)(\cos\theta)\,d\theta$$

$$+ \int_0^{\frac{\pi}{2}} \sin^{n-1}\theta\cos\theta\,d\theta$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \theta \sin^{n-2}\theta(\cos^2\theta)\,d\theta + \int_0^{\frac{\pi}{2}} \sin^{n-1}\theta\cos\theta\,d\theta$$

Step V

→ **Using standard formula in equation (5)**

$$\cdots\left[\cos^2\theta = 1 - \sin^2\theta,\right.$$

$$\left.\int f^n(\theta)\, f'(\theta)\,d\theta = \frac{f^{n+1}(\theta)}{n+1}\right]$$

Here $f(\theta) = \cos\theta$, $f'(\theta) = -\sin\theta$

$$\therefore I_n = (n-1)\int_0^{\frac{\pi}{2}} \theta \sin^{n-2}\theta(1-\sin^2\theta)\,d\theta + \left[\frac{\sin^n\theta}{n}\right]_0^{\frac{\pi}{2}}$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \theta\sin^{n-2}\theta\,d\theta - (n-1)\int_0^{\frac{\pi}{2}} \theta\sin^{n-2}\theta\sin^2\theta\,d\theta + \left[\frac{\sin^n\theta}{n}\right]_0^{\frac{\pi}{2}}$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \theta\sin^{n-2}\theta\,d\theta - (n-1)\int_0^{\frac{\pi}{2}} \theta\sin^n\theta\,d\theta + \frac{1}{n}[(1-0]$$

$$\ldots(5)$$

Step VI

From equation (1), $\int_0^{\frac{\pi}{2}} \theta\sin^n\theta\,d\theta = I_n \Rightarrow \int_0^{\frac{\pi}{2}} \theta\sin^{n-2}\theta\,d\theta = I_{n-2}$

Equation (5) $\Rightarrow$ $\quad I_n = (n-1)\,I_{n-2} - (n-1)\,I_n + \frac{1}{n}$

$\therefore I_n + (n-1)\,I_n = (n-1)\,I_{n-2} + \frac{1}{n} \quad \Rightarrow [1 + (n-1)]\,I_n$

$= (n-1)\,I_{n-2} + \frac{1}{n}$

$$\therefore\ n\,I_n = (n-1)\,I_{n-2} + \frac{1}{n}$$

$$\therefore\ I_n = \frac{1}{n^2} + \frac{n-1}{n}\,I_{n-2} \quad \ldots(6)$$

Hence proof

Step VII

Put $n = 5$ in Equation (6)

$$I_5 = \frac{1}{5^2} + \frac{5-1}{5}\,I_{5-2}$$

$$\therefore I_5 = \frac{1}{25} + \frac{4}{5}\,I_3 \quad \ldots(7)$$

Put $n = 3$ in Equation (6)

$$I_3 = \frac{1}{3^2} + \frac{3-1}{3}\,I_{3-2}$$

$$\therefore I_3 = \frac{1}{9} + \frac{2}{3}\,I_1 \quad \ldots(8)$$

Step VIII

But $\quad I_1 = \int_0^{\frac{\pi}{2}} \theta\sin\theta\,d\theta$

Integrating by parts,

$$I_1 = \theta\int \sin\theta\,d\theta - \int [\frac{d}{d\theta}(\theta)\int \sin\theta\,d\theta]\,d\theta$$

$$= (\theta(-\cos\theta))_0^{\pi/2} - \int_0^{\frac{\pi}{2}} (1)(-\cos\theta)\,d\theta$$

$$= (-\theta\cos\theta)_0^{\pi/2} + \int_0^{\frac{\pi}{2}} \cos\theta\,d\theta$$

$$= -(\frac{\pi}{2}\cos\frac{\pi}{2} - 0\cos 0) + (\sin\theta)_0^{\pi/2}$$

$$= -(0-0) + (\sin\frac{\pi}{2} - \sin 0) = -(0) + (1-0) = 1$$

$\therefore I_1 = 1$

Step IX :

Using value of I_1 in Equation (8)

$$I_3 = \frac{1}{9} + \frac{2}{3}(1) = \frac{1}{9} + \left(\frac{2}{3}\right)$$

$$= \frac{1}{9} + \frac{2\times 3}{3\times 3} = \frac{1}{9} + \frac{6}{9} = \frac{1+6}{9} = \frac{7}{9}$$

Using value of I_3 in Equation (7),

$$I_5 = \frac{1}{25} + \frac{4}{5}\left(\frac{7}{9}\right) = \frac{1}{25} + \frac{28}{45} = \frac{140+9}{225} = \mathbf{\frac{149}{225}}$$

Example 6.8.10

If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$ then

prove that $I_n = \left(\frac{\pi}{2}\right)^n - n\,(n-1)\,I_{n-2}$

Solution :

Step I

Consider $\quad I_n = \int_0^{\frac{\pi}{2}} x^n\cos x\,dx \quad \ldots(1)$

→ **Using standard Integrating by parts rule in equation(1)**

$$\cdots\left[\int u\,v\,dx = u\int v\,dx - \int\left(\frac{d}{dx}u\right)\left(\int v\,dx\right)dx\right]$$

Here taking $u = x^n$, $v = \cos x$

$$I_n = [(x^n)(\int_0^{\frac{\pi}{2}}\cos x\,dx)] - \int_0^{\frac{\pi}{2}}\left(\frac{d}{dx}x^n\right)\left(\int \cos x\,dx\right)dx \quad \ldots(2)$$

Step II

→ **Using standard formula of derivative and Integration in equation(2)**

$$\cdots\left[\int \cos x\,dx = \sin x,\ \frac{d}{dx}(x^n) = n\,x^{n-1}\right]$$

$$I_n = (x^n\sin x)_0^{\pi/2} - \int_0^{\frac{\pi}{2}} (n\,x^{n-1})(\sin x)\,dx$$

$$= \left[\left(\frac{\pi}{2}\right)^n(\sin\frac{\pi}{2}) - (0)(\sin 0)\right] - n\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\,dx$$

$$= \left[\left(\frac{\pi}{2}\right)^n(1) - 0\right] - n\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\,dx$$

Again integrating by parts

$$I_n = \left(\frac{\pi}{2}\right)^n - n\left\{\left[\left[x^{n-1}\int_0^{\frac{\pi}{2}}\sin x\,dx\right]\right] - \int_0^{\frac{\pi}{2}}\left(\frac{d}{dx}x^{n-1}\right)\int_0^{\frac{\pi}{2}}\sin x)\,dx)\right\}$$

$$= \left(\frac{\pi}{2}\right)^n - n\left\{[x^{n-1}(-\cos x)]_0^{\pi/2} - \int_0^{\frac{\pi}{2}}(n-1)\,x^{n-2}(-\cos x)\,dx\right\}$$

$$= \left(\frac{\pi}{2}\right)^n - n\,[-(x^{n-1}\cos x)_0^{\pi/2} + (n-1)\int_0^{\frac{\pi}{2}}(x^{n-2})\cos x\,dx$$

$$= \left(\frac{\pi}{2}\right)^n + n\left[\left(\frac{\pi}{2}\right)^{n-1}\cos\frac{\pi}{2} - (0)(\cos 0)\right] - n\,(n-1)\int_0^{\frac{\pi}{2}}(x^{n-2})\cos x\,dx$$

$$= \left(\frac{\pi}{2}\right)^n + n\,[0-0] - -n\,(n-1)\int_0^{\frac{\pi}{2}}(x^{n-2})\cos x\,dx$$

Step III

From equation (1) ,

$$\int_0^{\frac{\pi}{2}} x^n\cos x\,dx = \mathbf{I_n} \Rightarrow \int_0^{\frac{\pi}{2}}(x^{n-2})\cos x\,dx = I_{n-2} \quad \ldots(3)$$

Equation (3) $\Rightarrow$

$$I_n = \left(\frac{\pi}{2}\right)^n - n\,(n-1)\,I_{n-2}$$

Example 6.8.11

If $I_n = \int_0^{\pi/2} x^n\cos ax\,dx$ then show that $I_n = \frac{1}{a}\left(\frac{\pi}{2}\right)^n$

$\left[\sin\left(\frac{a\pi}{2}\right) + \frac{2n}{a\pi}\cos\left(\frac{a\pi}{2}\right)\right] - \frac{n\,(n-1)}{a^2}\,I_{n-2}$.

Solution :

Step I: $\quad I_n = \int_0^{\pi/2} x^n\cos ax\,dx \quad \ldots(1)$

→ **Using standard Integrating by parts rule in equation(1)**

$$\cdots\left[\int u\,v\,dx = u\int v\,dx - \int\left(\frac{d}{dx}u\right)\left(\int v\,dx\right)dx\right.$$

$$\left.\text{Here taking } u = x^n,\ v = \cos ax\right]$$

$$I_n = [(x^n)(\int_0^{\frac{\pi}{2}}\cos ax\,dx)] - \int_0^{\frac{\pi}{2}}\left(\frac{d}{dx}x^n\right)\left(\int \cos ax\,dx\right)dx \quad \ldots(2)$$

Step II :

→ **Using standard formula of derivative and integration in equation(2)**

$$\therefore \left[\int \cos ax\,dx = \frac{\sin ax}{a},\ \frac{d}{dx}(x^n) = n\,x^{n-1} \right]$$

$$I_n = x^n \left(\frac{\sin ax}{a}\right)_0^{\pi/2} - \int_0^{\pi/2} n\,x^{n-1}\left(\frac{\sin ax}{a}\right)dx$$

$$= \frac{1}{a}\left[\left(\frac{\pi}{2}\right)^n \sin\left(\frac{a\pi}{2}\right) - (0)^n \sin(0)\right] - n\int_0^{\pi/2} x^{n-1}\left(\frac{\sin ax}{a}\right)dx$$

Step III : Again integrating by parts,

$$I_n = \frac{1}{a}\left[\left(\frac{\pi}{2}\right)^n \sin\left(\frac{a\pi}{2}\right) - 0\right]$$

$$-n\left[x^{n-1}\int_0^{\pi/2}\left(\frac{\sin ax}{a}\right)dx - \int_0^{\pi/2}\frac{d}{dx}(x^{n-1})\int_0^{\pi/2}\left(\frac{\sin ax}{a}dx\right)dx\right]$$

$$= \frac{1}{a}\left[\left(\frac{\pi}{2}\right)^n \sin\left(\frac{a\pi}{2}\right)\right]$$

$$-n\left[x^{n-1}\left(\frac{-\cos ax}{a^2}\right)_0^{\pi/2} - \int_0^{\pi/2}(n-1)\,x^{n-2}\left(\frac{-\cos ax}{a^2}\right)dx\right]$$

$$= \frac{1}{a}\left(\frac{\pi}{2}\right)^n \sin\left(\frac{a\pi}{2}\right)$$

$$-n\left[x^{n-1}\left(\frac{-\cos ax}{a^2}\right)_0^{\pi/2} - \int_0^{\pi/2}(n-1)\,x^{n-2}\left(\frac{-\cos ax}{a^2}\right)dx\right]$$

$$= \frac{1}{a}\left(\frac{\pi}{2}\right)^n \sin\left(\frac{a\pi}{2}\right)$$

$$-n\left[-\frac{1}{a^2}(x^{n-1}\cos ax)_0^{\pi/2} + \frac{n-1}{a^2}\int_0^{\pi/2} x^{n-2}(\cos ax\,dx)\right]$$

$$= \frac{1}{a}\left(\frac{\pi}{2}\right)^n \sin\left(\frac{a\pi}{2}\right) + \left(\frac{n}{a^2}\right)$$

$$(x^{n-1}\cos ax)_0^{\pi/2} - \frac{n(n-1)}{a^2}\int_0^{\pi/2} x^{n-2}(\cos ax\,dx) \quad \ldots(3)$$

Step IV : From equation (1),

$$\int_0^{\pi/2} x^n \cos ax\,dx = .\,I_n \Rightarrow \int_0^{\pi/2} x^{n-2}\cos ax\,dx = .\,I_{n-2}$$

$$I_n = \frac{1}{a}\left(\frac{\pi}{2}\right)^n \sin\left(\frac{a\pi}{2}\right) + \frac{n}{a^2}\left[\left(\frac{\pi}{2}\right)^{n-1}\cos\left(\frac{a\pi}{2}\right)\right.$$

$$\left. -(0)^{n-1}\cos(0)\right] - \frac{n(n-1)}{a^2}I_{n-2}$$

$$= \frac{1}{a}\left(\frac{\pi}{2}\right)^n \sin\left(\frac{a\pi}{2}\right) + \frac{n}{a^2}\left[\left(\frac{\pi}{2}\right)^{n-1}\cos\left(\frac{a\pi}{2}\right) - 0\right]$$

$$-\frac{n(n-1)}{a^2}I_{n-2}$$

$$= \frac{1}{a}\left(\frac{\pi}{2}\right)^n \sin\left(\frac{a\pi}{2}\right) + \left(\frac{n}{a}\right)\left(\frac{1}{a}\right)\left[\left(\frac{\pi}{2}\right)^n\left(\frac{\pi}{2}\right)^{-1}\cos\left(\frac{a\pi}{2}\right)\right] - \frac{n(n-1)}{a^2}I_{n-2}$$

Step V :

Taking $\frac{1}{a}\left(\frac{\pi}{2}\right)^n$ common from two terms,

$$I_n = \frac{1}{a}\left(\frac{\pi}{2}\right)^n\left[\sin\left(\frac{a\pi}{2}\right) + \left(\frac{n}{a}\right)\left(\frac{2}{\pi}\right)\cos\left(\frac{a\pi}{2}\right)\right]$$

$$-\frac{n(n-1)}{a^2}I_{n-2}\,\frac{1}{a}\left(\frac{\pi}{2}\right)^n\left[\sin\left(\frac{a\pi}{2}\right) + \frac{2n}{a\pi}\cos\left(\frac{a\pi}{2}\right)\right]$$

$$-\frac{n(n-1)}{a^2}I_{n-2}$$

Example 6.8.12

If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx\,dx$ then prove that $I_n = \frac{1}{2}I_{n-1} = \frac{\pi}{2^{n+1}}$

Solution :

Step I : $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx\,dx \quad \ldots(1)$

→ **Using standard Integrating by parts rule in equation(1)**

$$\therefore \left[\int u\,v\,dx = u\int v\,dx - \int\left(\frac{d}{dx}u\right)\left(\int v\,dx\right)dx\right.$$

$$\left.\text{Here taking } u = \cos^n x,\ v = \cos nx\right]$$

$$I_n = \left[\cos^n x\left(\int_0^{\frac{\pi}{2}} \cos nx\,dx\right)\right] - \int_0^{\frac{\pi}{2}}\left(\frac{d}{dx}\cos^n x\right)\left(\int \cos nx\,dx\right)dx \quad \ldots(2)$$

$$I_n = \left[\cos^n x\left(\frac{\sin nx}{n}\right)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}[n\cos^{n-1}x\,(-\sin x)]\left(\frac{\sin nx}{n}\right)dx$$

$$= \left[\cos^n\left(\frac{\pi}{2}\right)\left(\frac{\sin n\frac{\pi}{2}}{n}\right) - \cos^n(0)\left(\frac{\sin n(0)}{n}\right)\right]$$

$$+\frac{n}{n}\int_0^{\frac{\pi}{2}}(\cos^{n-1}x\,\sin x)\sin nx\,dx \quad \ldots(3)$$

Step II :

→ **Using standard values of trigonometric terms in equation (3)**

$$\therefore \left[\cos\left(\frac{\pi}{2}\right) = 1,\ \sin 0 = 0\right]$$

$$I_n = \left[0 - 0\right] + \int_0^{\frac{\pi}{2}}\cos^{n-1}x\,\sin x\,\sin nx\,dx \quad \ldots(4)$$

→ **Using standard trigonometric formulae in equation (3)**

$$\ldots[\cos(n-1)x = \cos nx\cos x + \sin nx\sin x$$

$$\therefore \sin nx\sin x = \cos(n-1)x - \cos nx\cos x]$$

$\therefore$ from Equation (4)

$$I_n = \int_0^{\frac{\pi}{2}}\cos^{n-1}x\,[\cos(n-1)x - \cos nx\cos x]\,dx$$

$$= \int_0^{\frac{\pi}{2}}[\cos^{n-1}x\cos(n-1)x - \cos^{n-1}x\,\cos nx\cos x]\,dx$$

$$= \int_0^{\frac{\pi}{2}}\cos^{n-1}x\cos(n-1)x\,dx - \int_0^{\frac{\pi}{2}}\cos^n x\cos nx\,dx \quad \ldots(5)$$

From equation (1),

$$\int_0^{\frac{\pi}{2}}\cos^n x\cos nx\,dx = I_n \Rightarrow \int_0^{\frac{\pi}{2}}\cos^n x\cos(n-1)x\,dx = I_{n-1}$$

Equation (5) $\Rightarrow$

$$\therefore\ I_n = I_{n-1} - I_n = I_1 + I_n = I_{n-1} \Rightarrow 2\,I_n = I_{n-1}$$

$$\therefore\ I_n = \frac{1}{2}I_{n-1} \quad \ldots(6)$$

Step III :

Applying formula (6) repeatedly on R.H.S.

$$I_n = \left(\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\ldots n\text{ times}\right)I_0 \quad \ldots(7)$$

$$\text{But}\quad I_0 = \int_0^{\frac{\pi}{2}}dx = \frac{\pi}{2}$$

$(\because$ from Equation (1))

$\therefore$ From equation (7)

$$I_n = \frac{1}{2^n}\cdot\frac{\pi}{2} = \frac{\pi}{2^{n+1}}$$

Example 6.8.13

If $I_n = \int_0^{\frac{\pi}{2}} x^n \sin(2P+1)\,x\,dx$ then prove that $(2P+1)^2 I_n + n(n-1)I_{n-2} = (-1)^n\,n\left(\frac{\pi}{2}\right)^{n-1}$

Where n and P are positive integers.

Solution :

Step I :

$$\text{Let,}\quad I_n = \int_0^{\frac{\pi}{2}} x^n\sin(2P+1)\,x\,dx$$

Integrating by parts,

$$= \left\{x^n\left[\frac{-\cos(2P+1)\,x}{(2P+1)}\right]\right\}_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n\,x^{n-1}\left[\frac{-\cos(2P+1)\,x}{2P+1}\right]dx$$

$$= 0 + \frac{n}{2P+1}\int_0^{\frac{\pi}{2}} x^{n-1}\cos(2P+1)\,x\,dx$$

$$\left(\because 2p+1 \text{ is odd integer and } \cos\left(\frac{n\pi}{2}\right) = 0 \text{ if n is odd integer}\right)$$

$$= \frac{n}{2P+1}\left\{x^{n-1}\left[\frac{\sin(2P+1)\,x}{(2P+1)}\right]\right\}_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}(n-1)\,x^{n-2}\frac{\sin(2P+1)\,x}{2P+1}dx$$

$$= \frac{n}{2P+1}\left[\frac{1}{2P+1}\left(\frac{\pi}{2}\right)^{n-1}\sin(2P+1)\frac{\pi}{2} - \frac{n-1}{2P+1}\int_0^{\frac{\pi}{2}} x^{n-2}\sin(2P+1)\,x\,dx\right]$$

Step II :

$$\text{Since, } \sin(2P+1)\frac{\pi}{2} = \sin\left(P\pi + \frac{\pi}{2}\right) = (-1)^P$$

$$\therefore\ I_n = \frac{n}{2P+1}\left\{\frac{1}{2P+1}\left(\frac{\pi}{2}\right)^{n-1}(-1)^P - \frac{n-1}{2P+1}I_{n-2}\right.$$

$$= \frac{n}{(2P+1)^2}\left(\frac{\pi}{2}\right)^{n-1}(-1)^P - \frac{n(n-1)}{(2P+1)^2}I_{n-2}$$

$$\therefore\ (2P+1)^2 I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}(-1)^P$$

Example 6.8.14

If $I_n = \int_0^{\infty} e^{-px}\sin^n x\,dx\ (n \geq 2, p > 0)$ then prove that $(p^2 + n^2)\,I_n = n(n-1)\,I_{n-2}$.

Hence evaluate $\int_0^{\infty} e^{-2x}\sin^4 x\,dx$.

Solution :

Step I

Consider $I_n = \int_0^\infty e^{-px} \sin^n x \, dx$

Integrating by parts,

$$= \left[(\sin^n x)\left(\frac{e^{-px}}{-p}\right) \right]_0^\infty - \int_0^\infty (n \sin^{n-1} x)(\cos x)\left(\frac{e^{-px}}{-p}\right) dx$$

$$I_n = 0 + \frac{n}{p} \int_0^\infty (\sin^{n-1} x \cos x)\, e^{-px}\, dx$$

Step II

Again integrating by parts

$$I_n = \frac{n}{p}\left\{ \left[(\sin^{n-1} x \cos x)\left(\frac{e^{-px}}{-p}\right) \right]_0^\infty - \int_0^\infty \left[\frac{d}{dx}(\sin^{n-1} x \cos x) \int e^{-px} dx \right] dx \right.$$

$$= \frac{n}{p}\left\{ \left[0 - \int_0^\infty (\sin^{n-1} x(-\sin x)) + ((n-1)\sin^{n-2} x \cos x (\cos x)) \right] \frac{e^{-px}}{-p} \right.$$

$$= \frac{n}{p}\left[\int_0^\infty e^{-px} \sin^n x \, dx + \frac{n-1}{p} \int_0^\infty e^{-px} \sin^{n-2} x \cos^2 x \, dx \right]$$

$$= \frac{n}{p}\left[I_n + \frac{n-1}{p} \int_0^\infty e^{-px} \sin^{n-2} x (1 - \cos^2 x)\, dx \right]$$

$$= \frac{n}{p}\left[I_n + \frac{n-1}{p} \int_0^\infty e^{-px} \sin^{n-2} x \, dx - \frac{n-1}{p} \int_0^\infty e^{-px} \cos^n x \, dx \right]$$

$$= \frac{n}{p^2} [I_n + (n-1) I_{n-2} - (n-1) I_n]$$

$$= \frac{n}{p^2} [-n I_n + (n-1) I_{n-2}]$$

$$I_n = -\frac{n^2}{p^2} I_n + \frac{n(n-1)}{p^2} I_{n-2}$$

$$\left(1 + \frac{n^2}{p^2}\right) I_n = \frac{n(n-1)}{p^2} I_{n-2} \left(\frac{n^2+p^2}{p^2}\right) I_n = \frac{n(n-1)}{p^2} I_{n-2}$$

$$\therefore (p^2 + n^2) I_n = n(n-1) I_{n-2} \quad \ldots(1)$$

Hence proof

Step III

To evaluate $\int_0^\infty e^{-2x} \sin^4 x \, dx$

Put $p = 2$ and $n = 4$ in Equation (1)

we get,

$$(4 + 16) I_4 = 4(3) I_2$$

$$\Rightarrow I_4 = \frac{3}{5} I_2 \quad \ldots(2)$$

Put $p = 2$, $n = 2$ in Equation (1)

$$(4 + 4) I_2 = 2(1) I_0 \Rightarrow 8 I_2 = 2 I_0$$

$$I_2 = \frac{1}{4} I_0 \quad \ldots(3)$$

But $I_0 = \int_0^\infty e^{-2x} \, dx$ ($\because$ put $n = 0$ in Equation (1))

$$= \left[\frac{e^{-2x}}{-2} \right]_0^\infty = \frac{1}{2}$$

Step IV

From Equation (3), $I_2 = \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{8}$

From Equation (2), $I_4 = \frac{3}{5}\left(\frac{1}{8}\right) = \frac{3}{40}$

$$\therefore \int_0^\infty e^{-2x} \sin^4 x \, dx = \mathbf{\frac{3}{40}}$$

Example 6.8.15

If $I_n = \int_0^\infty e^{-x} \sin^n x \, dx$ $(n > 2)$ then prove that $(1 + n^2) I_n = n(n-1) I_{n-2}$ and hence evaluate I_4.

Solution :

Step I

$$I_n = \int_0^\infty e^{-x} \sin^n x \, dx$$

Integrating by parts

$$= [\sin^n x(-e^{-x})]_0^\infty - \int_0^\infty n \sin^{n-1} x \cos x (-e^{-x}) \, dx = 0 + n \int_0^\infty \sin^{n-1} x \cos x \, e^{-x} \, dx$$

$$= n\left\{ [\sin^{n-1} x \cos x(-e^{-x})]_0^\infty - \int_0^\infty [\sin^{n-1} x(-\sin x) + \cos x (n-1) \sin^{n-2} x \cos x](-e^{-x}) \, dx \right\}$$

$$= n\left[-\int_0^\infty \sin^n x \, e^{-x} dx + (n-1) \int_0^\infty \sin^{n-2} x \cos^2 x \, e^{-x} dx \right] = -n I_n + n(n-1) \int_0^\infty \sin^{n-2} x (1 - \sin^2 x) e^{-x} dx$$

$$= -n I_n + n(n-1) \left[\int_0^\infty \sin^{n-2} x \, e^{-x} dx - \int_0^\infty \sin^n x \, e^{-x} dx \right]$$

$$I_n = -n I_n + n(n-1) I_{n-2} - n(n-1) I_n$$

$$= n(n-1) I_{n-2} - n^2 I_n$$

$$(1 + n^2) I_n = n(n-1) I_{n-2}$$

$$I_n = \mathbf{\frac{n(n-1)}{1+n^2} I_{n-2}} \quad \ldots(1)$$

Hence proof.

Step II

In Equation (1), put $n = 4$,

$$I_4 = \frac{4(3)}{1+16}, \quad I_2 = \frac{12}{17} I_2 \quad \ldots(2)$$

In Equation (1), put $n = 2$, $I_2 = \frac{(2)(1)}{1+4} I_0 = \frac{2}{5} I_0$ ...(3)

$$\text{But } I_0 = \int_0^\infty e^{-x} dx = (-e^{-x})_0^\infty = [-e^{-\infty} - (-1)] = 1$$

Step III

$\therefore$ From Equation (3) we get $\therefore$ $I_2 = \frac{2}{5}$

From Equation (2) we get $\therefore$ $I_4 = \frac{12}{17} \cdot \frac{2}{5} = \mathbf{\frac{24}{85}}$ $\therefore \int_0^\infty e^{-x} \sin^4 x \, dx = \frac{24}{85}$

Example 6.8.16

If $I_n = \int_0^{\pi/4} \sec^n \theta \, d\theta$, prove that $I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$.

Hence evaluate $\int_0^{\pi/4} \sec^6 \theta \, d\theta$

Solution : $I_n = \int_0^{\pi/4} \sec^n \theta \, d\theta = \int_0^{\pi/4} \sec^{n-2} \theta \sec^2 \theta \, d\theta$

$$= [\sec^{n-2} \theta \tan \theta]_0^{\pi/4} - \int_0^{\pi/4} (n-2) \sec^{n-3} \theta (\sec\theta \tan\theta) \tan \theta \, d\theta$$

$$= (\sqrt{2})^{n-2} - (n-2) \int_0^{\pi/4} \sec^{n-2} \theta \tan^2 \theta \, d\theta \quad \left(\because \sec \frac{\pi}{4} = \sqrt{2}\right)$$

$$= (\sqrt{2})^{n-2} - (n-2) \left[\int_0^{\pi/4} \sec^{n-2} \theta (\sec^2 \theta - 1) \, d\theta \right]$$

$$I_n = (\sqrt{2})^{n-2} - (n-2) I_n + (n-2) I_{n-2}$$

$$(1 + n - 2) I_n = (\sqrt{2})^{n-2} + (n-2) I_{n-2}$$

$$\Rightarrow (n-1) I_n = (\sqrt{2})^{n-2} + (n-2) I_{n-2}$$

$$\mathbf{I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}} \quad \ldots(1)$$

Hence proof

Now, put $n = 6$ in Equation (1)

$$I_6 = \frac{(\sqrt{2})^4}{5} + \frac{4}{5} I_4 = \frac{4}{5} + \frac{4}{5} I_4 \quad \ldots(2)$$

Put $n = 4$ in Equation (1),

$$I_4 = \frac{(\sqrt{2})^2}{3} + \frac{2}{3} I_2 = \frac{2}{3} + \frac{2}{3} I_2 \quad \ldots(3)$$

Put $n = 2$ in Equation (1)

$$I_2 = \frac{1}{1} + 0 = 1 \qquad \ldots(4)$$

$$\therefore\ I_4 = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \text{ (From Equation (3))}$$

$$I_6 = \frac{4}{5} + \frac{4}{5} \times \frac{4}{3} = \frac{4}{5} + \frac{16}{15}$$

$$= \frac{12+16}{15} = \mathbf{\frac{28}{15}}$$

Exercise 6.2

Ex. 1 Prove that the reduction formula for

$$\int_0^{\pi/2} x \cos^n x\, dx = -\frac{1}{n^2} + \left(\frac{n-1}{n}\right) \int_0^{\pi/2} x \cos^{n-2} dx \text{ hence find } \int_0^{\pi/2} x \cos^3 x\, dx$$

Ex. 2 If $U_n = \int_0^{\pi/2} \theta \cos^n \theta\, dQ$ then

show that $U_n = \frac{-1}{n^2} + \frac{n-1}{n}\, U_n$

hence evaluate U_4.

Ex. 3 If $I_n = \int_0^{\pi/2} x^n \sin x\, dx\ (n > 1)$ then

prove that $I_n + n(n-1)\, I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$

Hint : Refer Example 6.8.10

Ex. 4 If $I_n = \int_0^{\frac{\pi}{2}} x^n (\sin x + \cos x)\, dx$ then

show that $I_n = \left[\left(\frac{\pi}{2}\right)^n + n\left(\frac{\pi}{2}\right)^{n-1}\right] - n(n-1)\, I_{n-2}$.

Hint : $I_n = \int_0^{\frac{\pi}{2}} x^n (\sin x + \cos x)\, dx$

$$= \int_0^{\frac{\pi}{2}} x^n \sin x\, dx + \int_0^{\frac{\pi}{2}} x^n \cos x\, dx$$

Refer Example 6.8.10

Ex. 5 If $I_n = \int_0^{\frac{\pi}{4}} \frac{\sin(2n-1)x}{\sin x}\, dx$ then

prove that $I_{n+1} - I_n = \frac{1}{n} \sin \frac{n\pi}{2}$. Hence find I_3.

Ans. : $I_n = \int_0^{\frac{\pi}{4}} \frac{\sin(2n-1)x}{\sin x}\, dx$

Ex. 6 If $I_n = \int_0^{\pi/2} x^n \sin ax\, dx$ show that

Ans. : $a^2 I_n = -ax^n \cos ax + n\, x^{n-1} \sin ax - n(n-1)\, I_{n-2}$

Chapter Ends

□□□

CHAPTER 7

Curve Tracing and Rectification of Curves

UNIT IV

Syllabus

Curve Tracing : Tracing of the Curves given in Cartesian, Parametric & Polar forms.

7.1 Introduction

For integral calculus, we often require to draw the region of the given curve for finding areas, surface area of revolution, volume of revolution and many more. Curve tracing is a procedure or method to find approximate shape of the given curve without the labour of plotting a large number of points on it.

By tracing the curves, we get better idea about the function. The main aim of this chapter is to introduce the general properties of curve tracing, So that mathematical equations can be understood in a better way.

7.2 Some Basic Definitions

Let, y = f (x) be the curve and P be any point on it.

1. Concave upward (convex downward) :

If on both sides of point P, portion of the curve lies above the tangent at P, then the curve is called concave upward (convex downward).

The point P is called as point of concavity. (Fig. 7.2.1)

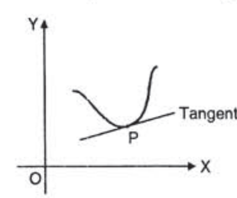


Fig. 7.2.1

2. Concave downward (convex upwards)

If on both sides of point P, portion of the curve lies below the tangent at P, then the curve is called concave downward or convex upwards. (Fig. 7.2.2)

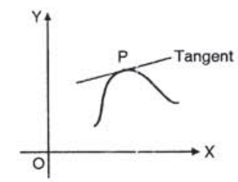


Fig. 7.2.2

3. Point of inflexion

The point which separate concave upward and concave downward of the curve is called point of inflexion.

At the point of inflexion, two portion of the curves lies on different sides of the tangent at P i.e. the curve crosses the tangents at p. (Fig. 7.2.3)

Inflexional tangent

The tangent at a point of inflexion is called a inflexional tangent.

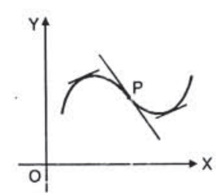


Fig. 7.2.3

4. Multiple point

If number of branches of the curve passes through a point, then that point is called multiple point. Through multiple point, more than one branches of the curve passes.

If n branches of the curve pass through a point, then such point is callec a multiple point of n^{th} order. (Fig. 7.2.4)

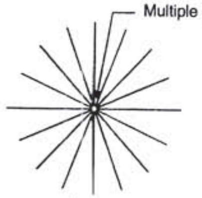


Fig. 7.2.4

5. Double point

A point, through which only two branches of the curve passes, is called double point.

6. Node

A double point P is called node, if the branches of the curve pass, through it are real and the tangents at the point of intersection are real and different.

(Fig. 7.2.5)

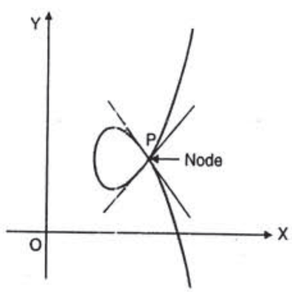


Fig. 7.2.5

7. Cusp

A double point P is called cusp if the tangents at the point of intersections are real and coincides to each other. (Fig. 7.2.6)

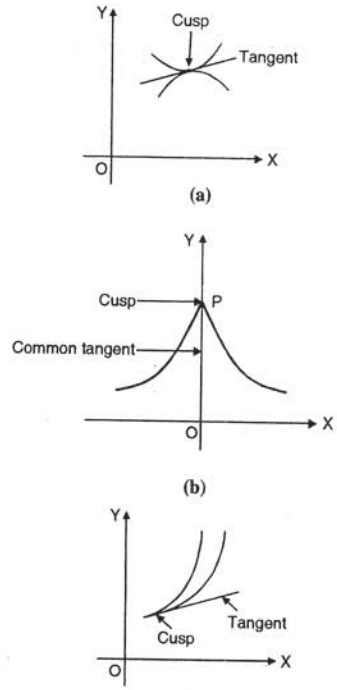


Fig. 7.2.6

8. Conjugate point

The point P is called conjugate or isolated point if there are no any real point in its neighborhood.

Conjugate point satisfies the equation of curve but it does not lies on curve. (Fig. 7.2.7)

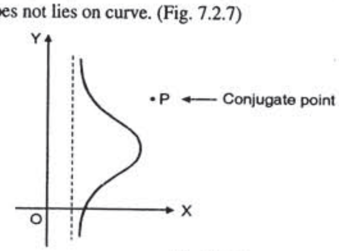


Fig. 7.2.7

9. Singular points

The point of inflexion, node, cusp, isolated point, multiple point etc. are called singular points.

10. Even function / odd function

If f (– x) = f (x) then f (x) is called an **even function.**

If f (– x) = f (x) then f (x) is called an **odd function.**

In this chapter we will study the tracing of :

(a) Cartesian curves (Explicit form)

(b) Polar curves (r = f (θ) form)

(c) Parametric curves (x = f (t), y = g (t) form)

(d) Cartesian curves (Implicit form)

(e) Some spiral curves

7.3 Cartesian Curves (Explicit Form)

Cartesian curves are in the form of

(i) y = f (x) (ii) x = f (y) (iii) f (x, y) = 0

7.3.1 Rules for tracing Cartesian Equations (Explicit Form)

We can trace the cartesian curves by using following rules.

(I) Symmetry : (Symmetry means mirror image)

(i) Symmetric about X-axis

In the given equation, if all powers of y are even then the curve is symmetric about X-axis. (Fig. 7.3.1)

e.g. $y^2 = 4ax$

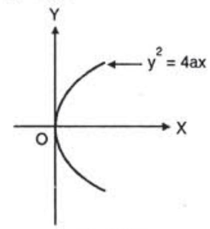


Fig. 7.3.1

(ii) Symmetric about Y-axis

In the given equation, if all powers of x are even then the curve is symmetric about Y-axis. (Fig. 7.3.2)

e.g. $x^2 = 4ay$

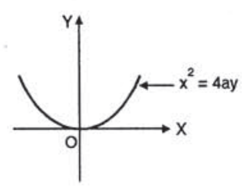


Fig. 7.3.2

(iii) Symmetric to both X and Y-axis

In the given equation, if all powers of both, x and y are even then the curve is symmetric about both X and Y-axis. (Fig. 7.3.3)

e.g. $x^2 + y^2 = a^2$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

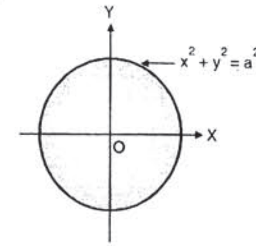


Fig. 7.3.3

(iv) Symmetric about y = x line

By interchanging x and y i.e.

put x = y and y = x, if the given equation is remain unchanged then the given curve is symmetric about y = x line. (Fig. 7.3.4)

e.g. $xy = c^2$

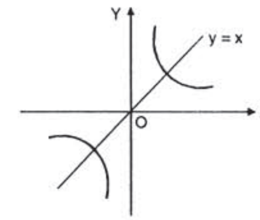


Fig. 7.3.4

(v) Symmetric about y = – x line

By interchanging x by – y and y by – x, if the given equation remains unchanged then the given curve is symmetric about y = – x line. (Fig. 7.3.5)

e.g. xy = – c

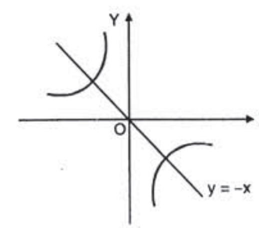


Fig. 7.3.5

(vi) Symmetric in opposite quadrants (or at origin)

By interchanging x by – x and y by – y, if the given equation remain unchanged, then the curve is symmetrical in opposite quadrants. (at origin). (Fig. 7.3.6)

e.g. $y = x^3$

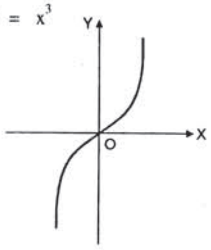


Fig. 7.3.6

(II) Points of Intersection

(i) With x-axis : Put y = 0 in the given equation and find the values of x; we get $(a_1, 0)$, $(b_1, 0)$... are the points of intersection with X-axis.

(ii) With Y-axis : Put x = 0 in the given equation and find the values of y, we get $(0, a_2)$, $(0, b_2)$ ... are the points of intersection with Y-axis.

(iii) Origin : Put x = 0, y = 0 in the given equation if we get, 0 = 0 (or if the given equation free from absolute constant term) then the curve passes through origin.

(III) Tangent at origin

If the curve passes through origin, then find tangent at origin by using Newton's method.

"By equating the lowest degree term or terms with zero," we get equation of tangent at origin.

e.g.,

$$y^2(2a - x) = x^3$$

$$\Rightarrow y^2 \cdot 2a - y^2 x - x^3 = 0$$

$$\text{Degree } \uparrow_2 \ldots \quad \uparrow_3 \ldots \uparrow_3$$

$$\text{Lowest degree term : } y^2 \, 2a = 0$$

$$\Rightarrow y^2 = 0$$

i.e. X-axis is tangent at origin.

(IV) Special point

By using the following rules, we can find the tangent other than origin point. Let (a, b) be any point (except origin).

1. If $\left(\frac{dy}{dx}\right)_{(a,b)} = 0$, then at (a, b), tangent is parallel to X-axis.
2. If $\left(\frac{dy}{dx}\right)_{(a,b)} = \pm\infty$, then at (a, b), tangent is parallel to Y-axis.
3. The curve increasing strictly in the interval (a, b) where $\frac{dy}{dx} > 0$.
4. The curve decreasing strictly in the internal (a, b) where $\frac{dy}{dx} < 0$.
5. The curve attains its maximum or minimum values at the points where $\frac{dy}{dx} = 0$.

(V) Asymptote

Asymptote means the tangent to the curve at infinity.

A curve of degree 'n' have n or less than n asymptotes.

(i) Asymptote parallel to X-axis

By equating, the co-efficient of highest power of x to zero, we get the equation of asymptote parallel to X-axis.

Or

For the given equation, As $x \to \infty$ for $y \to a$ then, y = a is asymptote parallel to X-axis.

i.e. for x = f (y) curve, put denominator equal to zero, we get the values of y where, curve have tangent parallel to X-axis.

(Sometimes, if the curve is symmetric to Y-axis then find asymptote parallel to X-axis).

(ii) Asymptote parallel to Y-axis

By equating the co-efficient of highest power of y to zero, we get equation of asymptote parallel to Y-axis.

Or

As $y \to \infty$ for $x \to a$ then x = a is a asymptote parallel to Y-axis. i.e. for y = f (x) ; put denominator terms equal to zero, we get the values of x where, the curve have tangent parallel to Y-axis. (Sometimes, if the curve is symmetrical to X-axis, then find asymptote parallel to Y-axis).

(iii) Oblique asymptote

(Oblique asymptote means asymptote neither parallel to X-axis nor parallel to Y-axis). Also, if the given curve is not symmetric about any axis, then find oblique asymptote.

Steps : To find oblique asymptote :

(a) Let y = mx + c be the equation of oblique asymptote.

(b) Let $\phi_2(x, y)$ - be the second degree term equation.

(c) Let $\phi_3(x, y)$ - be the third degree term equation.

(d) Put x = 1 and y = m in $\phi_2(x, y)$ and $\phi_3(x, y)$, then solve $\phi_3(m) = 0$

(e) Find the values of m,

(f) Find c by using : $c = -\left[\frac{\phi_2(m)}{\phi_3'(m)}\right]$

(g) Substitute the values of m and c in y = mx + c, we get the required equation of oblique asymptote to the curve.

Or

Oblique asymptote is also obtained by putting the value, y = mx + c in the given equation and then equate the coefficients of the two highest powers of x to zero, we get the values of m and c.

(VI) Region of absence

(i) If the given curve is symmetric about X – axis then express the equation in the explicit form (say) y = f (x) and check how y varies as x varies. If $y^2 < 0$ (y is imaginary) for some value of x > a (say) then the curve does not exists for x > a.

(ii) If the given curve is symmetric about Y – axis then express the given equation as x = f (y) and check how x varies as y varies continuously.

If $x^2 < 0$ (x is imaginary) for some value of y > b (say) then the curve does not exists for y > b.

Or

In general

(i) If the curve is symmetrical about X-axis or in the opposite quadrants then only consider positive values of y.

(ii) If there are two points of the curve on any axis and if the curve does not exist on either side of these two points, then there is always a **loop** (closed curve) between these two points.

(iii) The circle; ellipse etc. curves does not have asymptote.

(iv) x = 0 means, Y-axis

(v) y = 0 means X-axis.

Note :

1. If Asymptote parallel to X-axis, then its slope must be zero.
2. If the coefficient of the highest degree term of x is constant or if its linear factors are all imaginary then there is no asymptote parallel to the X-axis.
3. If the co-efficient of the highest degree term in y is a constant or if its linear factor are all imaginary, then there will be no asymptote parallel to Y-axis.

Working Rule to trace the curve

Step I : Find symmetry of the curve.

Step II : Find the points of integration.

Step III : Find equation of tangent at origin, at points of intersection.

Step IV : Find the equation of asymptotes.

Step V : Find the region of absence of the curve.

Step VI : Trace the rough sketch of the given curve.

7.4 Cartesian Curve (Explicit form i.e. y = f (x) or x = f (y)

Example 7.4.1

Trace the curve : $xy^2 = 4a^2(a - x)$.

Solution :

Step I : Given equation of curve, $xy^2 = 4a^2(a - x)$

Symmetry : Symmetry about X-axis, since, all powers of y are even.

Step II : Points of intersection :

(i) With X-axis : Put y = 0 in the given equation, we get

$0 = 4a^2(a - x) \Rightarrow a - x = 0 \# \Rightarrow x = a$

$\therefore$ The point of intersection with X-axis is (a, 0).

(ii) With Y-axis : Put x = 0 in the given equation, we get

$0 = 4a^3$, not getting any value of y.

Not intersection with Y-axis.

This curve does not passes through origin (because by putting x = 0, and y = 0, we are not getting 0 = 0).

Step III : Equation of tangent :

(i) At origin : No tangent at origin.

(curve does not passes through origin).

(ii) At (a, 0)

Since, $y^2 = \frac{4a^2(a-x)}{x}$

Differentiate w.r.t. x

$$2y\frac{dy}{dx} = 4a^2\left(\frac{x(-1)-(a-x)\cdot 1}{x^2}\right)$$

$$\left(\frac{dy}{dx}\right)_{p(a,0)} = 4a^2\left(\frac{-\cancel{x}-a+\cancel{x}}{x^2\cdot 2y}\right)_{(a,0)} = \infty$$

Therefore, at (a, 0) the curve have tangent parallel to Y-axis.

Step IV : Asymptote : **Since, $y^2x = 4a^2(a-x)$.**

$\Rightarrow \quad y^2x - 4a^3 + 4a^3x = 0$

(i) Asymptote parallel to X-axis : Equate the coefficient of highest degree term in x to zero.

$\therefore \quad y^2 + 4a^2 = 0$; which gives imaginary values

$(\because y^2 = -4a^2 \Rightarrow y = \pm 2ai)$

$\therefore$ No asymptote parallel to X-axis

(ii) Asymptote parallel to Y-axis

Equate the co-efficient of highest power in y to zero.

i.e. **x = 0** $\rightarrow$ means **y-axis** is a asymptote.

Or Since, $y^2x = 4a^2(a-x)$

$$y^2 = \frac{4a^2(a-x)}{x}$$

By equating denominator to zero i.e. x = 0 means Y-axis, is the equation of asymptote.

Step V : Region of absence :

Since, curve is symmetrical about X-axis

So, solve for y (i.e. write as y = f(x) form)

$$\therefore y^2 = \frac{4a^2(a-x)}{x}$$

For $y^2 < 0$ if $x < 0$

and $y^2 < 0$ if $x > a$

$(\because$ y is imaginary for $x < 0)$

Hence, curve does **not exists for x < 0 and x > a.**

i.e. curve does exist in $0 \le x \le a$

The rough sketch of the curve is Fig. P. 7.4.1.

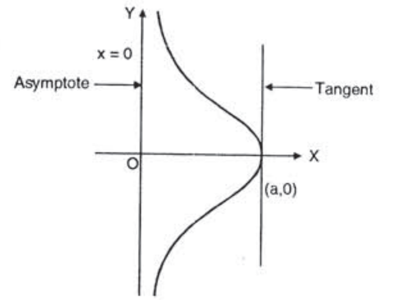


Fig. P. 7.4.1

Example 7.4.2

Trace the curve : $y^2(2a-x) = x^3$.

Solution :

Step I : Given equation of the curve is $y^2(2a-x) = x^3$

Symmetry : Symmetric about X-axis (since, all powers of y are even)

Step II : Points of intersection :

(i) With X-axis : Put y = 0 in the given equation, we get, $0 = x^3 \Rightarrow x = 0$

Hence, (0, 0) is the point.

(ii) With Y-axis : Put x = 0 in the given equation, we get $y^2 = 0 \Rightarrow y = 0$. Hence, (0, 0) be the point. This, curve intersects X-axis, Y-axis at origin only.

(iii) Origin : By putting x = 0; y = 0 in the given equation.

We get, 0 = 0. Hence, the given curve passes through origin.

Step III : Equation of tangent at origin

Since; $y^2(2a-x) = x^3$

$\Rightarrow \quad 2a\,y^2 - xy^2 = x^3$

By equating the lowest degree term to zero, we get equation of tangent at origin.

$\therefore \quad 2ay^2 = 0 \quad \Rightarrow \quad y^2 = 0$

$\Rightarrow \quad y = 0\ ;\ y = 0$

Means, X-axis is the tangent at origin.

(y = 0, y = 0; two tangents are coincides, i.e. have common tangents, origin becomes cusp).

Step IV : Asymptote

$\therefore$ Since, $y^2(2a-x) = x^3$ 2

$$2ay^2 - y^2x = x^3$$

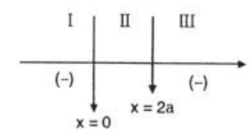
(i) Parallel to X-axis : By equating the co-efficient of highest power of x to zero.

$\therefore$ 1 = 0 ; meaningless

$\therefore$ No asymptote parallel to X-axis.

(ii) Parallel to Y-axis : By equating the co-efficient of highest powers of y to zero.

$\therefore \quad 2a - x = 0 \Rightarrow \boxed{x = 2a}$ is the equation of **asymptote parallel to Y-axis**



Step V : Region of Absence

Since, the curve is symmetric about X-axis, so, solve for y

$$y^2 = \frac{x^3}{(2a-x)}$$

Find the values of x for which y^2 is negative (i.e. y is imaginary)

$y^2 < 0$ for : (i) $x^3 < 0$ i.e. for $x < 0$

(ii) $2a - x < 0$ i.e. $2a < x \Rightarrow x > 2a$

i.e. curve does not exists for x < 0 and x > 2a

Hence, curve exist between $\mathbf{0 \le x \le 2a}$

The rough sketch of the curve is (Fig. P. 7.4.2)

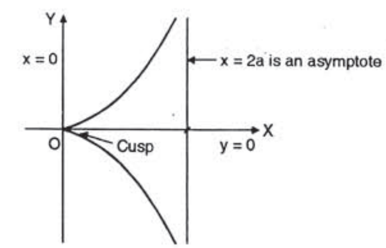


Fig. P. 7.4.2

Example 7.4.3

Trace the curve : $xy^2 = a(x^2 - a^2)$.

Solution :

Step I : Given curve is : $xy^2 = a(x^2 - a^2)$

Symmetry : Symmetric about X-axis (since, all y have even powers)

Step II : Points of intersection

(i) With X-axis : Put y = 0, we get $0 = a(x^2 - a^2)$

$\Rightarrow x^2 = a^2 \quad \Rightarrow \quad x = \pm a$

Hence, points are (a, 0), (– a, 0). Curve intersects X-axis at (a, 0), (– a, 0).

(ii) With Y-axis : Put x = 0, we get, 0 = 0, Curve intersect Y-axis at (0, 0)

(iii) Origin : Put x = 0, y = 0 in the given equation, we get, 0 = 0. Hence, curve passes through origin.

Step III : Equation of tangent

(i) At origin : By equating the lowest degree term to zero, we get the equation of tangent.

Since, $xy^2 = ax^2 - a^3$

$\uparrow \quad \uparrow \quad \uparrow$

degree 3 degree 2 degree 0

$-a^3 = 0$ (meaning less)

$\therefore$ No tangent at origin.

(ii) Tangents At (a, 0), (– a, 0) :

Since, $y^2 = \frac{a(x^2-a^2)}{x} = \left(\frac{ax^2-a^3}{x^3}\right)$

Differentiate w.r.t. x.

$$2y\frac{dy}{dx} = \frac{x(2ax) - a(x^2-a^2)\cdot 1}{x^2}$$

$$\frac{dy}{dx} = \left[\frac{ax^2+a^3}{x^3\cdot 2y}\right]$$

1. **At (a, 0),** $\left(\frac{dy}{dx}\right)_{(a,0)} = \infty$ **:** At (a, 0) the curve have tangents parallel to Y-axis.
2. **At (– a, 0),** $\left(\frac{dy}{dx}\right)_{(-a,0)} = -\infty$ **:** Tangent parallel to Y-axis at (– a, 0)

Step IV : Asymptote

(i) Parallel to X-axis : Equating the co-efficient of highest degree term in x to zero.

Since, $xy^2 - ax^2 + a^3 = 0$

i.e. a = 0 (meaningless), No asymptote parallel to X-axis.

(ii) Parallel to Y-axis : Equating the co-efficient of highest degree term in Y to zero.

i.e. x = 0 ; Means, Y-axis itself is an asymptote.

Step V : Region of absence

Since, the curve is symmetric about X-axis, so, solve the equation for y.

$$\therefore \quad y^2 = \frac{a(x^2-a^2)}{x}$$

Find the values of x, for which $y^2 < 0$ (i.e. y is imaginary)

$0 < x < a$; $y^2 < 0$

$x = 0$; $y^2 = \infty$

$-a < x < a$; $y^2 > 0$ but

$x < -a$; $y^2 < 0$

Curve exists in $-a < n < 0$ and $x > a$

A rough sketch of the curve is as shown below (Fig. P.7.4.3)

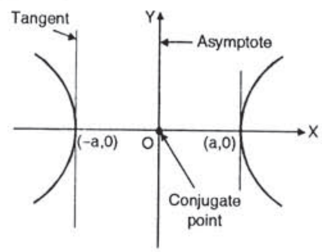


Fig. P. 7.4.3

Example 7.4.4

Trace the curve : $y^2(4-x) = x(x-2)^2$.

Solution :

Step I : Symmetry : The curve is symmetric about X-axis. (Since, all Y have even powers)

Step II : Points of intersection :

(i) **With X-axis :** Put y = 0; we get, $0 = x(x-2)^2$

$\Rightarrow x = 0, 2$. This curve intersects X-axis at (0, 0) (2, 0)

(ii) **With Y-axis :** Put x = 0, we get

$4y^2 = 0 \Rightarrow (0, 0)$. This curve intersects Y-axis at (0, 0)

(iii) **Origin :** Put x = 0, y = 0, we get, 0 = 0; Hence, the curve passes through origin.

Step III : Equation of tangents :

(i) **At origin :** By equating the lowest degree term or terms to zero.

$$\therefore \quad \underset{\text{deg 2}}{\underset{\uparrow}{4y^2}} - \underset{\text{deg 3}}{\underset{\uparrow}{xy^2}} = x(x^2-2x+4) = \underset{\text{deg 3}}{\underset{\uparrow}{x^3}} - \underset{\text{deg 2}}{\underset{\uparrow}{2x^2}} + \underset{\text{deg 1}}{\underset{\uparrow}{4x}}$$

$$\therefore \quad 4x = 0 \Rightarrow x = 0$$

i.e. Y-axis itself is tangent at origin.

(ii) **Tangents At (2, 0) :**

$$\because \quad y^2(4-x) = x(x-2)^2$$

Differentiate w.r.t. x

$$2y\left(\frac{dy}{dx}\right)(4-x) + y^2(-1) = 2x(x-2) + (x-2)^2$$

$$\therefore \quad \frac{dy}{dx} = \left[\frac{2x(x-2)+(x-2)^2+y^2}{2y(4-x)}\right]$$

$$\left(\frac{dy}{dx}\right)_{(2,0)} = \infty$$

This means at (2, 0) tangent is parallel to Y-axis.

Step IV : Asymptote :

(i) **Parallel to X-axis :** Equating the co-efficient of highest degree term in x to zero, we get the equation of asymptote parallel to X-axis.

i.e. 1 = 0 (No equation)

No asymptote parallel to X-axis

(ii) **Parallel to Y-axis :** Equating the co-efficient of highest degree term in y to zero.

i.e. $(4-x) = 0 \Rightarrow x = 4$; at x = 4, asymptote parallel to Y-axis.

Step V : Region of absence :

Since, the curve is symmetrical about X-axis, so solve for y.

$$y^2 = \frac{x(x-2)^2}{(4-x)}$$

Find the values of x for which y^2 is negative

For $x > 4$; $y^2 < 0$

and $x < 0$; $y^2 < 0$. Hence, curve exists in $0 \le x \le 4$

A rough sketch of the curve is as shown in below. (Fig. P.7.4.4)

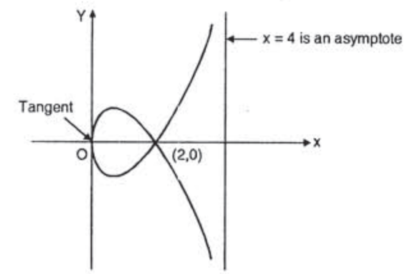


Fig. P. 7.4.4

Example 7.4.5

Trace the curve : $a^2y^2 = x^2(a^2-x^2)$.

Solution :

Step I : Symmetry : Symmetry about both X and Y-axis (Since all x and y have even powers)

Step II : Points of intersection ;

(i) **With X-axis :** Put y = 0, we get, $x^2(a^2-x^2) = 0$

$x = 0, x = \pm a$. Hence, intersection with X-axis at (0, 0), (a, 0), (– a, 0)

(ii) **With Y-axis :** Put x = 0, we get $y^2 = 0 \Rightarrow y = 0$

Hence, intersection with Y-axis at (0, 0)

(iii) **Origin :** Put x = 0, y = 0, we get 0 = 0, hence the given curve passes through **origin.**

Step III : Equation of tangent :

(i) **At origin :** Equating lowest degree term to zero.

$$\therefore \quad \underset{\text{deg 2}}{\underset{\uparrow}{a^2y^2}} = \underset{\text{deg 2}}{\underset{\uparrow}{x^2a^2}} - \underset{\text{deg 4}}{\underset{\uparrow}{x^4}}$$

Lowest degree term is, $a^2(y^2-x^2)$

$$\text{degree} \quad a^2(y^2-x^2) = 0$$

$$\Rightarrow y^2 = x^2$$

$$\Rightarrow y = \pm x$$

be the equation of tangent at origin

(ii) **Tangent At (a, 0) and (– a, 0) :**

$$\therefore \quad a^2y^2 = x^2(a^2-x^2)$$

Differentiate w.r.t. x

$$a^2 \cdot 2y\frac{dy}{dx} = x^2(-2x) + (a^2-x^2)\,2x$$

$$\frac{dy}{dx} = \frac{-x^3 + x(a^2-x^2)}{a^2y} \; ;$$

$$\left(\frac{dy}{dx}\right)_{(a,0)} = \left(\frac{dy}{dx}\right)_{(-a,0)} = \pm\infty$$

Hence, at (a, 0) and (– a, 0), the curve have tangent is parallel to Y-axis.

Step IV : Asymptote :

(i) **Parallel to X-axis :** Equating the co-efficient of highest degree term in x to zero.

$\therefore \; -1 = 0$ (No equation)

No asymptote parallel to X-axis.

(ii) **Parallel to Y-axis :** Equating the coefficients of highest degree term in Y to zero

i.e. $a^2 = 0$, (No equation)

No asymptote parallel to Y-axis.

Step V : Region of absence :

Since, the given curve is symmetrical about both axis.

$$y^2 = \frac{x^2(a^2-x^2)}{a^2}$$

Find the values of x for which y^2 is negative

at $x = 0$, $y^2 = 0 \Rightarrow y = 0$

$x = a$, $y^2 = 0 \Rightarrow y = 0$

$x > a$, y^2 is negative and $x < -a$; $y^2 < 0$

and $x = -a$, $y^2 = 0 \Rightarrow y = 0$

Hence, the given curve exists in $-a \le x \le a$

(Fig. P.7.4.5)

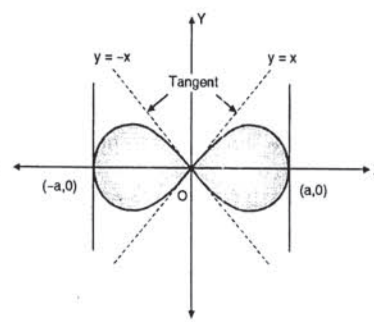


Fig. P. 7.4.5

Example 7.4.6

Trace the curve : $yx^2 = a^2 (a - y)$.

Solution :

Step I : The given curve is : $yx^2 = a^2 (a - y)$

Symmetry : Symmetric about Y-axis. (Since, powers of all x are even).

Step II : Points of intersection

(i) **With X-axis :** Put $y = 0$,

we get $0 = a^2 (a - 0) \Rightarrow a = 0$.

Means no intersection with X-axis,

(ii) **With Y-axis :** Put $x = 0$,

we get $0 = a^2 (a - y)$

$\Rightarrow y = a$, Points of intersection with Y-axis is $(0, a)$

(iii) **Origin :** Put $x = 0$, $y = 0$, not getting $0 = 0$.

Hence curve does not passes through origin.

Step III :Equation of tangent

(i) **Tangent at origin :** Equating lowest degree term to zero.

i.e. $a^3 = 0$ (meaningless) , No tangent at origin.

(ii) Tangent at $(0, a)$

$\therefore\ yx^2 = a^2 (a - y)$

Differentiate w.r.t. x

$$\frac{dy}{dx} x^2 + 2x\,y = a^2\left(0 - \frac{dy}{dx}\right);$$

$$(a^2 + x^2)\frac{dy}{dx} = -2xy$$

$$\frac{dy}{dx} = \frac{-2xy}{a^2 + x^2};$$

$$\left(\frac{dy}{dx}\right)_{(0,a)} = \frac{0}{a^2 + 0} = 0$$

Hence, at $(0, a)$, the curve have tangent parallel to X-axis.

Step IV :Asymptote

(i) **Parallel to X-axis** : Equating the co-efficient of highest of powers of x to zero.

$\therefore$ i.e. $y = 0$

i.e. X-axis itself is the asymptote.

(ii) **Parallel to Y-axis** : Equating the co-efficient of highest powers of Y to zero.

i.e. $(x^2 + a^2) = 0 \Rightarrow x^2 = -a^2$

$x = \pm\, ai$ (which is imaginary)

$\therefore$ No asymptote parallel to Y-axis.

Step V : Region of absence : Since, the curve is symmetrical to Y-axis, so solve it for x.

$$\therefore\ x^2 = \frac{a^2 (a - y)}{y}$$

Find the values of y for which x^2 is negative

at $y = 0$; $x^2 = \infty \Rightarrow x^2$ - is positive.

at $y = a$; $x^2 = 0 \Rightarrow x$ - is positive.

But for $y > a$; x^2 - also negative.

and

$y < 0$; x^2 also negative.

Hence the given curve exists in, $0 \le y \le a$

The rough sketch of the given curve is (Fig.P.7.4.6)

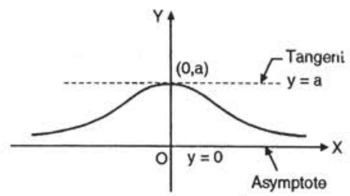


Fig. P. 7.4.6

Example 7.4.7

Trace the curve : $ay^2 = x^2 (a - x)$.

Solution :

Step I : The given curve is : $ay^2 = x^2 (a - x)$

Symmetry : This curve is symmetric about X-axis. (Since, ay have even powers)

Step II : Points of intersection

(i) **With X-axis :** Put $y = 0$, we get $0 = x^2 (a - x) \Rightarrow x = 0$, $x = a$.

Points of intersection with X-axis is (0, 0), (a, 0). So there is a loop between (0, 0) to (a, 0).

(ii) **With Y-axis :** Put $x = 0$, we get, $ay^2 = 0 \Rightarrow y = 0$

$\therefore$ The given curve intersects Y-axis at (0, 0)

(iii) **Origin :** Put $x = 0$, $y = 0$, we get $0 = 0$. This curve passes through origin.

Step III : Equation of tangent

(i) **At origin :** By equating the lowest degree term or terms to zero.

$$a(y^2 - x^2) = 0$$

$$\Rightarrow y = \pm x$$

$y = \pm x$ are a tangents at origin

(ii) **tangent At (a, 0) :**

$$\therefore\ ay^2 = x^2 (a - x)$$

Differentiate w.r.t. x.

$$a \cdot 2y \frac{dy}{dx} = 2ax - 3x^2$$

$$\frac{dy}{dx} = \frac{2ax - 3x^2}{2ay}$$

$$\left(\frac{dy}{dx}\right)_{(a,0)} = \infty$$

Thus at (a, 0) curve have tangent parallel to Y-axis.

Step IV : Asymptote

(i) **Parallel to X-axis :** Equating the co-efficient of highest degree term in x to zero. $-1 = 0$ (meaningless)

No asymptote parallel to X-axis.

(ii) **Parallel to Y-axis :** Equating the co-efficient of highest degree term in y to zero.

i.e. $a = 0$ (meaningless)

No asymptote parallel to Y-axis.

Step V : Region of absence

Since, the given curve is symmetric to X-axis so solve for Y.

$$y^2 = \frac{x^2 (a - x)}{a}$$

Find the values of x for which y^2 is negative

At	$x = 0$;	$y^2 = 0$, curve exists
At,	$x = a$;	$y^2 = 0$
But,	$x > a$;	y^2 – is negative
and	$x < 0$;	y^2 is always positive

Hence, curve exist in $x < 0$ and $0 \le x \le a$.

The rough sketch of the given curve is Fig. P.7.4.7.

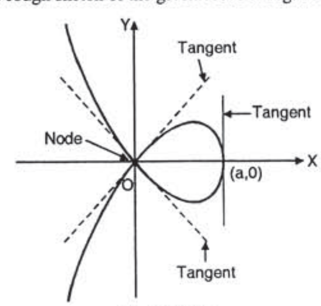


Fig. P. 7.4.7

Example 7.4.8

Trace the curve : $x (x^2 + y^2) = a (x^2 - y^2)$; $a > 0$.

Solution :

Step I : The given curve is : $x (x^2 + y^2) = a (x^2 - y^2)$

Symmetry : The given curve symmetric about X-axis. (since, all y have even powers)

Step II : Points of intersection :

(i) **With X-axis :** Put $y = 0$,

we get $x^3 = a (x^2) \Rightarrow x^2 (x - a) = 0 \Rightarrow x = 0, a$

This curve intersects X-axis at (0, 0) and (a, 0).

(ii) **With Y-axis :** Put $x = 0$, we get, $-ay^2 = 0 \Rightarrow y = 0$

This curve intersects Y-axis at (0, 0) only.

(iii) **Origin :** Put $x = 0$, $y = 0$, we get $0 = 0$.

Hence, the curve passes through origin.

Step III :Equation of tangents :

(i) **At origin :** By equating the lowest degree term or terms to zero.

i.e. $a (x^2 - y^2) = 0 \Rightarrow y = \pm x$ be the equation of tangent at origin.

(ii) **At (a, 0) :** Since $x (x^2 + y^2) = a (x^2 - y^2)$;

Differentiate w.r.t. x

$$x\left(2x + 2y\frac{dy}{dx}\right) + (x^2 + y^2) = a\left(2x - 2y\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}(2xy + 2ay) = 2ax - 2x^2 - x^2 - y^2 = 2ax - 3x^2 - y^2$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{2ax - 3x^2 - y^2}{2xy + 2xy}\right)$$

$$\left(\frac{dy}{dx}\right)_{(a,0)} = \infty,$$

Thus, at (a, 0) given curve have tangents parallel to Y-axis

Step IV : Asymptote

(i) **Parallel to X-axis :** Equating the co-efficient of highest degree term in x to zero.

$\therefore\ 1 = 0$ (meaningless)

$\therefore$ No asymptote parallel to X-axis.

(ii) **Parallel to Y-axis :** Equating the co-efficient of highest degree terms in y to zero.

$\therefore\ x + a = 0 \Rightarrow x = -a$

is the equation of asymptote parallel to Y-axis.

Step V : Region of absence :

Since, the curve is symmetric to X-axis, so solve for y,

Since, $x^3 + xy^2 - ax^2 + ay^2 = 0$

$$y^2 (x + a) = ax^2 - x^3$$

$$y^2 = \frac{x^2 (a - x)}{(a + x)}$$

Find the values or x for which y^2 is negative

at $x = 0$; $y^2 = 0$

$x = a$; $y^2 = 0$

$x > a$; y^2 – is negative

$x = -a$; $y^2 = \infty$

and $x < -a$; y^2 – is negative

Hence, curve exist in $\mathbf{-a \le x \le a}$.

The rough sketch of the given curve is Fig. P.7.4.8.

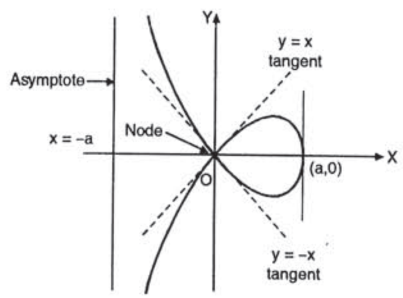


Fig. P. 7.4.8

Example 7.4.9

Trace the curve : $y^2 (x^2 + 4) = (x^2 + 2x)$

Solution :

Step I : The given curve is : $y^2 (x^2 + 4) = (x^2 + 2x)$

Symmetry : Sy6mmetric about X-axis ($\because$ since all y have even powers)

Step II : Points of intersection :

(i) With X-axis : Put y = 0, we get

$$0 = x^2 + 2x$$

$\Rightarrow x = 0, -2,$

this curve intersects X-axis at (0, 0) and (– 2, 0)

(ii) With Y-axis : Put x = 0 ; we get

$$y^2(4) = 0$$

$\Rightarrow y = 0,$ The curve intersects Y-axis at (0, 0)

(iii) Origin : Put x = 0, y = 0, we get

$$0 = 0 ;$$

Hence, the given curve passes through origin.

Step III : Equation of tangents

(i) At origin : Equating the lowest degree term to zero.

$\therefore \quad 2x = 0 \quad \Rightarrow \quad x = 0$

Means, Y-axis itself is a tangent at origin.

(ii) At (– 2, 0) :

$$y^2 (x^2 + 4) = x^2 + 2x$$

Differentiate w.r.t.x

$$2y \frac{dy}{dx} (x^2 + 4) + (x^2 + 4)\, 2y \frac{dy}{dx} = 2x + 2$$

$$\frac{dy}{dx} [\, 2y (x^2 + 4) + 2y (x^2 + 4)\,] = 2x + 2$$

$$\frac{dy}{dx} = \frac{2x + 2}{2y (x^2 + 4) + 2y (x^2 + 4)}$$

$$\left(\frac{dy}{dx}\right)_{(-2,0)} = \left(\frac{(x + 1)}{y (x^2 + 4) + y (x^2 + 4)}\right)_{(-2,0)} = \infty$$

Means at (– 2, 0), the curve have tangents parallel to Y-axis.

Step IV : Asymptote :

(i) Parallel to X-axis : By equating the coefficient of highest degree term in x to zero.

i.e. $(y^2 - 1) = 0 \quad \Rightarrow \quad y = \pm 1$ be the equation of asymptote parallel to X-axis.

(ii) Parallel to Y-axis : By equating the coefficient of highest degree term in y to zero.

i.e. $x^2 + 4 = 0 \Rightarrow x = \pm 2i$ (imaginary)

No asymptote parallel to Y-axis.

Step V : Region of absence : Since, the curve is symmetrical to X-axis, so solve it for y

$$\therefore \quad y^2 = \frac{x^2 + 2x}{x^2 + 4}$$

Find the values of x for which y^2 is negative

For $x = 0$; $y^2 = 0$; y^2 – is positive

$x = 1$; $y^2 = \text{Positive} = \frac{3}{5}$

$x = 2$; $y^2 = \text{Positive} = 1$

x-increasing, y also increasing.

and $x = -0.1, \ldots, -0.5, \ldots, -1$

y^2 is negative and for $x \le -2$,

y^2 is positive,

Hence, in $-2 < x < 0$; curve does not exists

and curve exists : $x \le -2$ and $x \ge 0$

The rough sketch of the given curve is as shown Fig. P. 7.4.9.

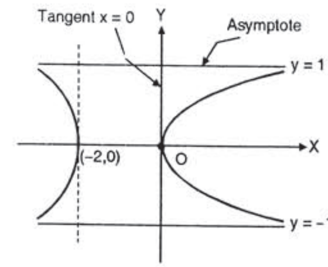


Fig. P. 7.4.9

Example 7.4.10 :

Trace the curve : $x^2y^2 = a^2 (y^2 - x^2)$

Solution :

Step I : The given curve : $x^2 y^2 = a^2 (y^2 - x^2)$

Symmetry : The given curve is symmetric about both axes. (since all y and x have even powers)

Step II : Points of Intersection

(i) With X-axis : Put y = 0, we get

$0 = a^2x^2 \quad \Rightarrow \quad x = 0$, This curve intersects X-axis at (0, 0) only.

(ii) With Y-axis : Put x = 0, we get,

$$0 = a^2y^2 \quad \Rightarrow \quad y = 0$$

Means this curve intersects Y-axis at origin (0, 0) only.

Step III :Equation of tangent

At origin : Equating the lowest degree term to zero.

$$(y^2 - x^2)\, a^2 = 0$$

$\Rightarrow \quad y^2 = x^2$ i.e.

$$y = \pm x$$

is the equation of tangents at origin

Step IV :Asymptote

(i) Parallel to X-axis : Equating the co-efficient of highest degree term in x to zero.

$\therefore \quad (y^2 + a^2) = 0 \quad \Rightarrow \quad y = \pm ai$ (Imaginary)

No asymptote parallel to X-axis.

(ii) Parallel to Y-axis : Equating the co-efficient of highest degree term in y to zero.

$$x^2 - a^2 = 0 \quad \Rightarrow \quad x = \pm a$$

be the equation of asymptote parallel to Y-axis.

Step V : Region of absence

Since, curve is symmetrical to both axis,

So, we can write for x or y

$$\because \quad x^2y^2 = a^2y^2 - a^2x^2$$

$$x^2 (y^2 + a^2) = a^2y^2$$

$$x^2 = \frac{a^2y^2}{y^2 + a^2}$$

Find the values of y for which x^2 is negative

for, $y = 0$ to ∞, x^2 - is positive

and $y = 0$ to $-\infty$, x^2 - is also positive

But, for $x < -a$ and $x > a$, curve does not exists

$\therefore$ The given curve exists in $-\infty \le y \le \infty$

and $-a \le x \le a$

The rough sketch of the given curve is Fig. P. 7.4.10.

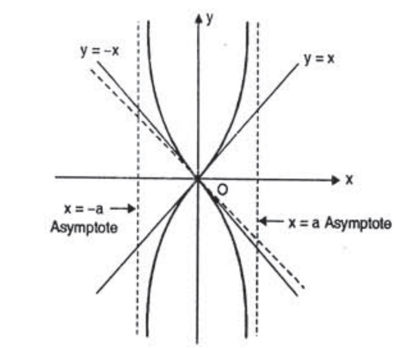


Fig. P. 7.4.10

Example 7.4.11

Trace the curve ; $y^2 = x^5 (2a - x)$

Solution :

Step I : The given curve is : $y^2 = x^5 (2a - x)$

Symmetry : The given curve is symmetrical about X-axis (since, all y have even powers)

Step II : Points of Intersection

(i) **With X-axis :** Put $y = 0$, we get

$\therefore$ origin is a cusp

$$0 = x^5(2a-x) \Rightarrow x = 0,\ x = 2a$$

Hence, this curve intersects X-axis at (0, 0), (2a, 0).

There is a loop between (0, 0) and (2a, 0)

(ii) **With Y-axis :** Put $x = 0$, we get,

$$y = 0$$

Hence, this curve intersects Y-axis at (0, 0).

(iii) **Origin :** Put $x = 0$; and $y = 0$. we get, $0 = 0$, Hence, this curve passes through origin.

Step III :Equation of tangents

(i) **At origin :** By equating the lowest degree term / terms with zero.

$$\therefore\ y^2 = 0 \Rightarrow y = 0,\ y = 0$$

(two tangents are coincides)

X-axis is a tangent at origin

Origin is a cusp

(ii) **At point (2a, 0) :**

$$\because\ y^2 = x^5(2a-x)\ ;$$

$$2y\frac{dy}{dx} = x^5(-1) + 5x^4(2a-x)$$

$$\frac{dy}{dx} = \frac{-x^5 + 5x^4(2a-x)}{2y}\ ;$$

$$\left(\frac{dy}{dx}\right)_{(2a,0)} = \infty$$

At (2a, 0), The curve have tangent parallel to Y-axis.

Step IV : Asymptote

(i) **Parallel to X-axis :** Equating the co-efficient of highest degree term in x to zero.

$$\therefore\ -1 = 0 \quad \text{(meaningless)}$$

No asymptote parallel to X-axis.

(ii) **Parallel to Y-axis :** Equating the co-efficient of highest degree term in y to zero.

i.e. $1 = 0$ (meaningless)

No asymptote parallel to Y-axis.

Step V : Region of absence

Since, the curve is symmetrical about X-axis, so solve it for y

$$\therefore\ y^2 = x^5(2a-x)$$

For, $x = 0$; $y^2 = 0$

$x = a$; $y^2 = \text{positive}$

$x = 2a$; $y^2 = 0$

$x > 2a$; y^2 – becomes negative

and $x < 0$, y^2 – becomes negative

Hence, curve exists in $0 \le x \le 2a$ interval.

The rough sketch of the curve is as Fig. P. 7.4.11.

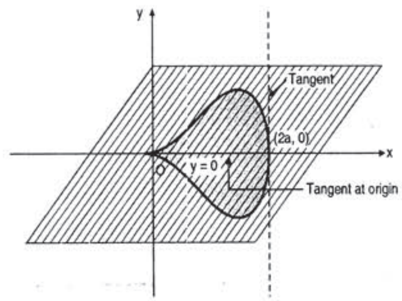


Fig. P. 7.4.11

Example 7.4.12

Trace the curve : $a^2x^2 = y^3(2a - y)$.

Solution :

Step I : The given curve is : $a^2x^2 = y^2(2a-y)$

Step I : Symmetry : The given curve is symmetric about Y-axis. (Since, all x have even powers)

Step II : Points of intersection

(i) **With X-axis :** Put, $y = 0$, we get,

$$x^2 = 0 \Rightarrow x = 0,\ x = 0$$

(tangents coincides, at origin, so origin becomes cusp).This curve intersects X-axis at (0, 0)

(ii) **With Y-axis :** Put $x = 0$, we get,

$$0 = y^3(2a-y) \Rightarrow y = 0,\ 2a$$

The point of intersection with Y-axis is (0, 0), (0 2a)

There is a loop between (0, 0) to (0, 2a)

(iii) **Origin :** Put $x = 0$; $y = 0$

We get, $0 = 0$; curve passes through origin.

Step III :Equation of tangents

(i) **At origin :** Equating the lowest degree term / terms to zero.

$$\therefore\ a^2x^2 = 0 \Rightarrow x = 0\ ;\ x = 0$$

Hence, Y-axis is the tangent at origin.

(ii) **At (0, 2a)**

Since, $x^2a^2 = y^2(2a-y)$;

$$2xa^2 = 2y(2a-y)\frac{dy}{dx} + y^2\left(-\frac{dy}{dx}\right)$$

$$2xa^2 = \frac{dy}{dx}[2y(2a-y) - y^2]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xa^2}{2y(2a-y)-y^2}$$

$$\left(\frac{dy}{dx}\right)_{(0,2a)} = 0$$

This shows that at (0, 2a), the curve have tangents parallel to X-axis.

Step IV : Asymptote

(i) **Parallel to X-axis :** Equating the co-efficient of highest degree term in x to zero.

$$\therefore\ a^2 = 0 \quad \text{(Not possible)}$$

No asymptote parallel to x – axis

(ii) **Parallel to Y-axis :** Equating the co-efficient of highest degree term in y to zero.

$$\therefore\ -1 = 0 \quad \text{(Meaningless)}$$

$\therefore$ No asymptote parallel to Y-axis.

Step V : Region of absence

Since, the curve is symmetrical to Y-axis, so solve it for x

$$\therefore\ x^2 = \frac{y^3}{a^2}(2a-y)$$

Find the values of y for which x^2 is negative

For, $y = 0$, $x^2 = 0$

$y = a$, $x^2 = \text{positive}$

$y = 2a$, $x^2 = 0$

$y > 2a$, x^2 - becomes negative

and $y < 0$, x^2 becomes negative.

Hence, curve does exist in $0 < y \le 2a$.

The rough sketch of the given curve is Fig. P.7.4.12.

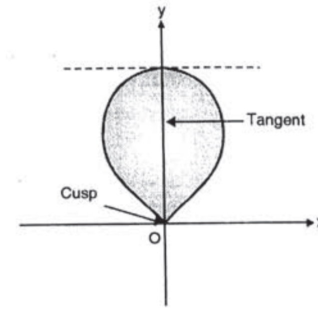


Fig. P. 7.4.12

Example 7.4.13

Trace the curve of : $y(1 + x^2) = x$.

Solution :

Step I : The given curve is : $y(1 + x^2) = x$

Symmetry : The given curve is symmetric in opposite quadrants.

(Since, by replacing $x = -x$ and $y = -y$, No change in the given equation)

Step II : Points of Intersections

(i) **With X-axis :** Put $y = 0$, we get

$0 = x \Rightarrow$ This curve intersects X-axis at (0, 0) only

(ii) **With Y-axis :** Put $x = 0$, we get,

$y = 0$. This curve intersects Y-axis at (0, 0). This curve passes through origin.

Step III :Equation of tangents

(i) **At origin :** Equating lowest degree term or terms to zero.

$$\therefore\ y - x = 0 \Rightarrow y = x,$$ is the equation of tangents at origin.

Step IV :Asymptote

(i) **Parallel to X-axis :** Equating the coefficient of highest degree term in x to zero.

$$\therefore\ y = 0 \Rightarrow \text{X-axis itself an asymptote}$$

(ii) **Parallel to Y-axis :** Equating the coefficient of highest degree term in y to zero.

$$\therefore\ 1 + x^2 = 0 \Rightarrow x^2 = \pm i \quad \text{(imaginary)}$$

No asymptote parallel to Y-axis.

Step V : Region of absence

Since, the curve is symmetrical in opposite quadrant

$\because\ y(1+x^2) = x \Rightarrow y = \dfrac{x}{1+x^2}$

For $x = 0;\ y = 0$

$x = 1;\ y = \frac{1}{2}$

$x \to \infty;\ y \to 0$

for $x < 0;\ y$ - is negative

$x \to -\infty;\ y \to 0$

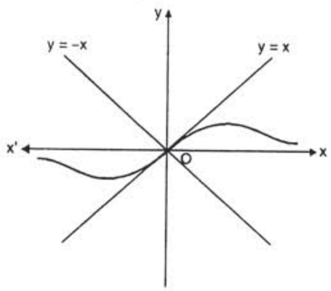


Fig. P. 7.4.13

The rough sketch of the given curve is as shown Fig. P.7.4.13.

Example 7.4.14

Trace the curve : $y^2 (x - a) = x^2 (2a - x)$.

Solution :

Step I : The given curve is : $y^2 (x - a) = x^2 (2a - x)$.

Symmetry : The given curve is symmetrical about X-axis. (Since, all y have even powers).

Step II : Points of Intersection

(i) With X-axis : Put $y = 0$, we get

$$0 = x^2(2a - x) \Rightarrow x = 0,\ x = 2a$$

This curve intersects with X-axis at (0, 0) and (2a, 0)

(ii) With Y-axis : Put $x = 0$; we get

$$y^2(0 - a) = 0$$

$$\Rightarrow y^2 = 0 \Rightarrow y = 0$$

Intersection with Y-axis at (0, 0) only ,

Origin : Put $x = 0$, $y = 0$ we get $0 = 0$

Hence, this curve passes through origin.

Step III :Equation of tangent

(i) At origin : Equating the lowest degree term / terms to zero

$$2ax^2 = -y^2 a \Rightarrow \text{No tangents of origin.}$$

(ii) At (2a, 0)

$$\because\ y^2(x - a) = x^2(2a - x)$$

Differentiate w.r.t. x.

$$2y(x - a)\frac{dy}{dx} + y^2 = 4ax - 3x^2$$

$$\therefore\ \frac{dy}{dx}(x - a)\cdot 2y = 4ax - 3x^2 - y^2$$

$$\frac{dy}{dx} = \left[\frac{4ax - 3x^2 - y^2}{(x - a)\cdot 2y}\right]$$

$$\left(\frac{dy}{dx}\right)_{(2a,\,0)} = \infty$$

The curve have tangents parallel to Y-axis at (2a, 0)

Step IV :Asymptote

(i) With X-axis : Equating the co-efficients of highest degree term in x to zero

$$-1 = 0 \quad \text{(meaningless)}$$

$\therefore$ No asymptote parallel to X-axis

(ii) With Y-axis : Equating the co-efficients of highest degree term in y to zero.

$\therefore\ x - a = 0 \quad \Rightarrow x = a$ is an asymptote parallel to Y-axis

Step V : Region of absence

Since the curve is symmetric about X-axis, so solve for y

$$\therefore\ y^2 = \frac{x^2(2a - x)}{x - a}$$

Find the values of x for which, y^2 is negative

For $x = 0;\ y^2 = 0$

$x = a;\ y^2 =$ positive

$x = 2a;\ y^2 = 0$

$x > 2a;\ y^2$ - is negative

For $x = 0$, and $x = a$ curve exists but $0 < x < a$, curve does not exists because, y^2 becomes negative.

Also, $x < 0$, y^2 - is negative

$\therefore$ Curve exist in $a \le x \le 2a$

Here, origin 0 is a conjugative point.

The rough sketch of the given curve is Fig. P. 7.4.14.

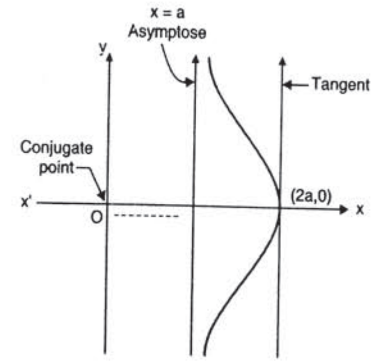


Fig. P. 7.4.14

Example 7.4.15

Trace the curve : $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

Solution :

Step I : The given curve is : $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Symmetry : The given curve is symmetric about $y = x$ line. (Since, by interchanging x and y, equation of the curve remains as it is)

Step II : Points of Intersection

(i) With X-axis : Put $y = 0$, we get

$$\sqrt{x} = \sqrt{a}$$

$\Rightarrow x = a$,Intersection with X-axis is at (a, 0)

(ii) With Y-axis : Put $x = 0$, we get

$$\sqrt{y} = \sqrt{a}$$

$\Rightarrow y = a$,Intersection with Y-axis is at (0, a)

(iii) Origin : Put $x = 0$, $y = 0$, does not getting $0 = 0$

Hence, curve does not pass through origin.

Step III :Equation of tangent

(i) At origin : No tangent at origin

(Since, this curve does not passes through origin)

(ii) At (a, 0) and (0, a)

Since, $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Differentiate w.r.t x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx} = 0$$

$$\therefore\ \frac{dy}{dx} = -\left(\frac{\sqrt{y}}{\sqrt{x}}\right) = -\sqrt{\frac{y}{x}}$$

$$\left(\frac{dy}{dx}\right)_{(a,\,0)} = 0 \quad \text{and} \quad \left(\frac{dy}{dx}\right)_{(0,\,a)} = -\infty$$

Hence, at (a, 0) the curve have tangent parallel to X-axis but at (0, a) tangent parallel to Y-axis.

Step IV :Asymptote

(i) Parallel to X-axis and (ii) Parallel to Y-axis

Here, the asymptote is not parallel to X and Y-axis.

Step V : Region of absence

$$\because\ \sqrt{x} + \sqrt{y} = \sqrt{a}$$

$$x^{1/2} + y^{1/2} = a^{1/2}$$

$$\Rightarrow y^{1/2} = \sqrt{a} - \sqrt{x}$$

$$\Rightarrow\ y = (\sqrt{a} - \sqrt{x})^2 = a + x - 2a^{1/2}x^{1/2}$$

For $x = 0;\ y = a$

$x = a;\ y = 0$

as $x \to \infty;\ y \to \infty$

But, for $x < 0$ and $y < 0$, curve does not exists

The rough sketch of the curve is shown Fig. P. 7..4.15.

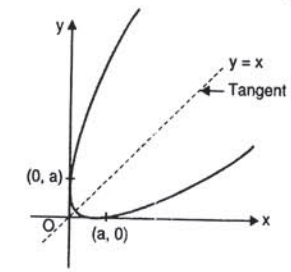


Fig. P. 7.4.15

Example 7.4.16

Trace the curve : $y^2 (a^2 + x^2) = a^2 x^2$.

Solution :

Step I : The given curve : $y^2 (a^2 + x^2) = a^2 x^2$

Symmetry :

The given curve is symmetric to both the axes. (Both x and y have even powers)

Step II : Points of Intersection

(i) Withy X-axis : Put $y = 0$; we get,

$$0 = a^2 x^2 \Rightarrow x^2 = 0$$

This curve intersects X-axis at (0, 0)

(ii) With Y-axis : Put $x = 0$; we get

$$y^2 = 0$$

$\Rightarrow$ (0, 0) be the point of intersection on Y-axis.

(iii) Origin : Put $x = 0, y = 0$; We get, $0 = 0$

Hence, the given curve passes through origin.

Step III : Equation of tangents

(i) At origin : Equating the lowest degree term or terms to zero

$$\therefore\ y^2(a^2) - a^2x^2 = 0$$

$$\therefore\ y^2 = x^2$$

$\Rightarrow\ y = \pm x$ be the equation of tangents at origin.

Step IV : Asymptote

(i) Parallel to X-axis : Equating the coefficient of highest degree term in x to zero.

$$y^2 - a^2 = 0$$

$$\Rightarrow\ y = \pm a$$

be the equation of asymptote parallel to X-axis

(ii) Parallel to Y-axis : Equating the co-efficient of highest degree term in y to zero.

$$(x^2 + a^2) = 0 \Rightarrow x = \pm ai \quad \text{(Imaginary)}$$

So, no asymptote parallel to Y-axis.

Step V : Region of absence :

Since, $y^2(a^2 + x^2) = a^2x^2$

$$\Rightarrow\ y^2 = \frac{a^2x^2}{a^2 + x^2}$$

For

x	0	a	2a	∞	– a	– 2a
y^2	0	$\frac{a^2}{2}$	+ ve	a^2	0	+ ve

The rough sketch of the given curve is as below Fig. P. 7.4.16.

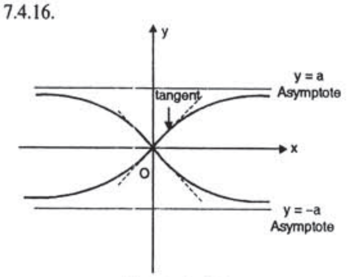


Fig. P. 7.4.16

Example 7.4.17

Trace the curve : $y^2 = x^2(1 - x)$.

Solution :

Step I : The given curve is : $y^2 = x^2(1 - x)$

Symmetry : The given curve is symmetrical to X-axis. (Since, all y have even powers)

Step II : Points of intersections

(i) With X-axis : Put $y = 0$, we get,

$$0 = x^2(1 - x)$$

$$\Rightarrow\ x = 0,\ x = 1,$$

Intersection with X-axis at (0, 0), (1, 0)

(ii) With Y-axis : Put $x = 0$; We get,

$y^2 = 0$. Intersection with Y-axis is at (0, 0).

(iii) Origin : put , $x = 0$ and $y = 0$, we get, $0 = 0$

Also, this curve passes through origin.

Step III : Equation of tangents

(i) At origin : Equating the lowest degree term to zero.

$$y^2 - x^2 = 0$$

$\Rightarrow\ y = \pm x$ be the equation of tangent at origin

(ii) At (1, 0)

$$\therefore\ y^2 = x^2(1 - x)$$

Differentiate w.r.t. x.

$$2y\frac{dy}{dx} = x^2(-1) + (1 - x)\cdot 2x$$

$$\frac{dy}{dx} = \frac{(-x^2 + 2x(1 - x))}{2y}$$

$$\left(\frac{dy}{dx}\right)_{(1,0)} = \infty$$

At (1, 0), the curve have tangent parallel to Y-axis.

Step IV : Asymptote

(i) Parallel to X-axis : Equating the coefficients of highest degree term in x to zero.

$$\therefore\ -1 = 0 \quad \text{(Meaningless)}$$

No asymptote parallel to X-axis.

(ii) Parallel to Y-axis : Equating the coefficient of highest degree term in y to zero.

$$\therefore\ 1 = 0 \quad \text{(Meaningless)}$$

No asymptote parallel to Y-axis.

Step V : Region of absence

Since, the curve is symmetric to X-axis, solve it for y.

$$\therefore\ y^2 = x^2(1 - x)$$

For

x	0	1	– 1	$x \to -\infty$
y^2	0	0	2	+ ve

$x > 0$, y^2 - Negative Fig. P. 7.4.17

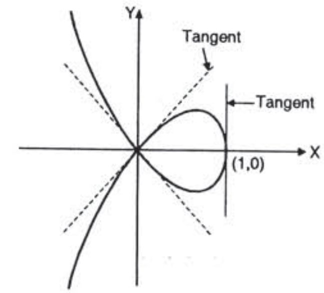


Fig. P. 7.4.17

Example 7.4.18

Trace the curve : $y^2 = (x - 1)(x - 2)(x - 3)$.

Solution :

Step I : The given curve : $y^2 = (x - 1)(x - 2)(x - 3)$

Symmetry

This curve is symmetrical about X-axis (Powers of y are even)

Step II : Points of intersection

(i) With X-axis : Put $y = 0$;

We get, $x = 1, 2, 3$

This curve intersects X-axis at (1, 0), (2, 0), (3, 0)

(ii) With Y-axis : Put $x = 0$, we get

$$y = (0 - 1)(0 - 2)(0 - 3)$$

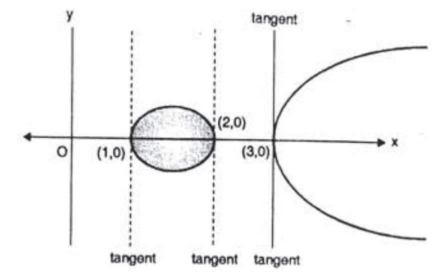


Fig. P. 7.4.18

$y^2 = -6$ (y becomes imaginary) No intersection with Y-axis.

(iii) Origin : Put $x = 0$; $y = 0$, we get

$0 = -6$, curve does not passes through origin.

Step III : Equation of tangents

(i) At origin :

This curve does not passes through origin, so no tangent at origin .

(ii) At (1, 0), (2, 0) (3, 0)

$$\therefore\ y^2 = (x - 1)(x - 2)(x - 3)$$

Differentiate w.r.t. x

$$2y\frac{dy}{dx} = (x - 2)(x - 3) + (x - 1)(x - 3) + (x - 1)(x - 3)$$

$$\frac{dy}{dx} = \frac{(x - 2)(x - 3) + (x - 1)(x - 3) + (x - 1)(x - 3)}{2y}$$

$$\therefore\ \frac{dy}{dx} = \infty \quad \text{at } (1, 0), (2, 0), (3, 0)$$

Hence, this curve have tangents parallel to Y-axis at (1, 0), (2, 0), (3, 0)

Step IV : Asymptote

(i) Parallel to X-axis

Equating the co-efficient of highest power of x to zero

$$\therefore\ 1 = 0 \quad \text{(Meaningless)}$$

No asymptote parallel to x – axis

(ii) Parallel to Y - axis

Equating the co-efficient of highest power of y to zero

$$\therefore\ 1 = 0 \quad \text{(Meaningless)}$$

No asymptote parallel to y – axis

Step V : Region of absence :

Since, $y^2 = (x-1)(x-2)(x-3)$

x	0	1	2	3	$x = 4$	$x > 3$
y^2	-6	0	0	0	6	+ve

The rough sketch of the given in Fig. P. 7.4.18.

Exercise 7.1

Q. Trace the following curves

1. $y^2(4-x) = x(x-2)^2$
2. $ay^2 = x(a^2 - x^2)$
3. $y^2 = (x^2 - 1) = x$
4. $y = x(x^2 - 1)$
5. $y = \frac{8a^3}{(x^2 - 4a^2)}$
6. $a^2 y^2 = x^2(2a - x)(x - a)$
7. $x = (y-1)(y-2)(y-3)$
8. $y = \frac{x-2}{2x+3}$
9. $y^2(x^2 + a^2) = x^2(a^2 - x^2)$
10. $y^2(a + x) = x^2(3a - x)$
11. $y = \frac{x}{1 - x^2}$
12. $27ay^2 = 4(x - 2a^3)$
13. $x^2y^2 = x^2 + 1$
14. $a^2y^2 = x^2(x + 2a)(x - a)$
15. $ay^2 = (x - a)(x - 5a)^2$
16. $a^2y^2 = x^2(a - x)(x - b)$; $a > b$
17. $x^2y^2 = (1 + y)^2(4 - y^2)$
18. $x^2 = y^2(x + 1)^2$
19. $y = (x-1)(x-2)(x-3)$
20. $x^2(x^2 + y^2) = a^2(y^2 - x^2)$

Answers

Ex. 1

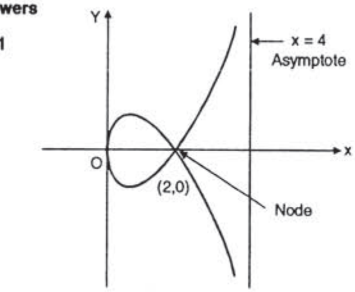


Fig. 1

Ex. 2

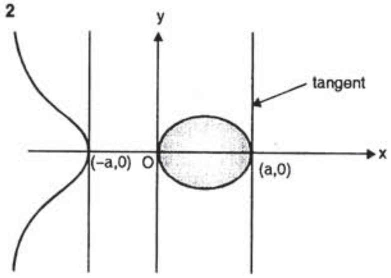


Fig. 2

Ex. 3

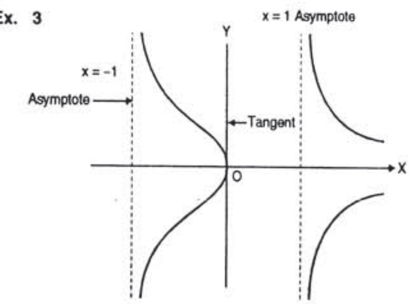


Fig. 3

Ex. 4

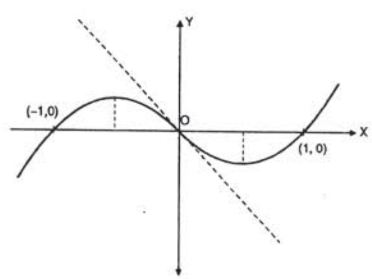


Fig. 4

Ex. 5

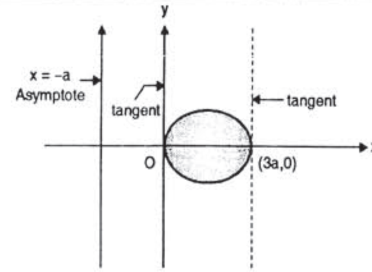


Fig. 5

Ex. 6

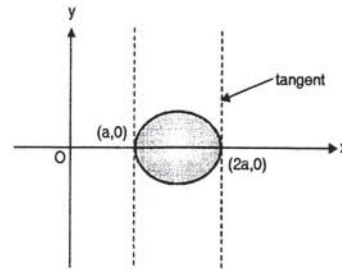


Fig. 6

Ex. 7

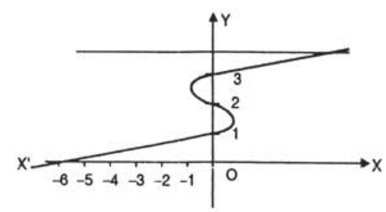


Fig. 7

Ex. 8

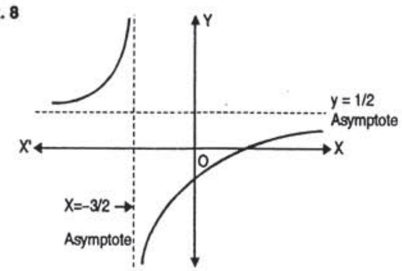


Fig. 8

Ex. 9

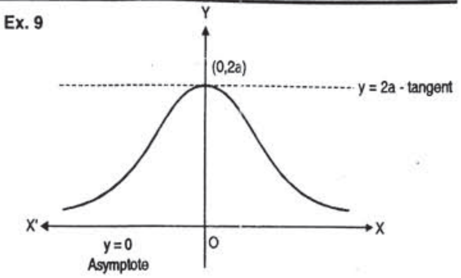


Fig. 9

Ex. 10

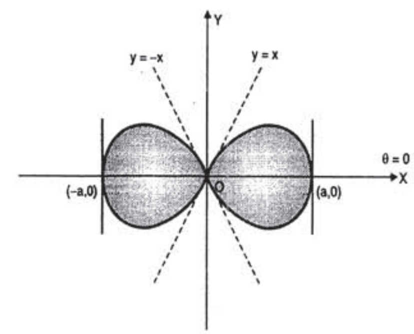


Fig. 10

Ex. 11

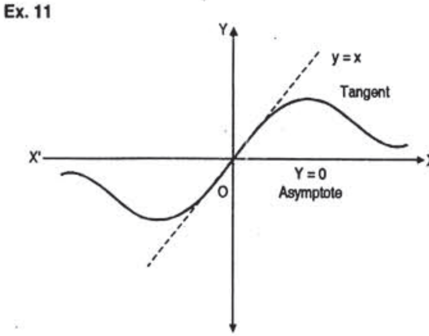


Fig. 11

Ex. 12

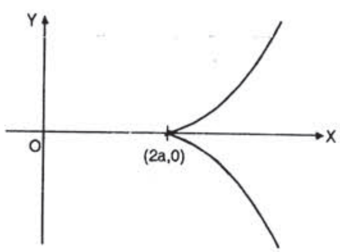


Fig. 12

Ex. 13

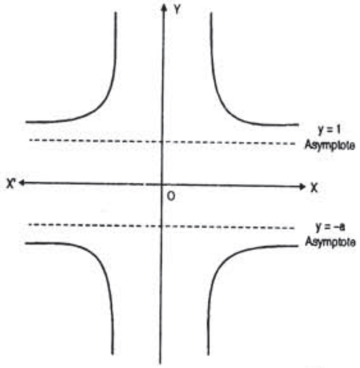


Fig. 13

Ex. 14

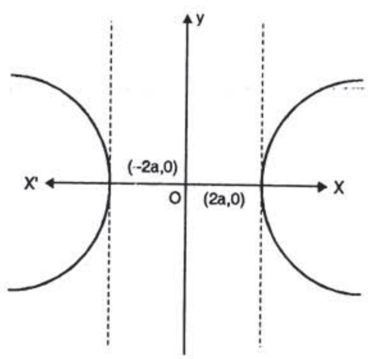


Fig. 14

Ex. 15

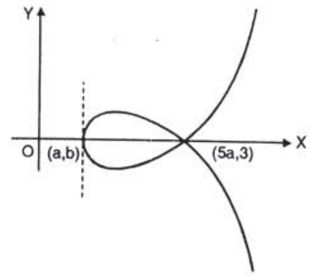


Fig. 15

Ex. 16

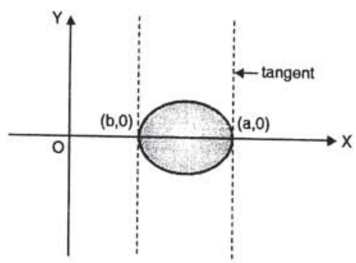


Fig. 16

Ex. 17

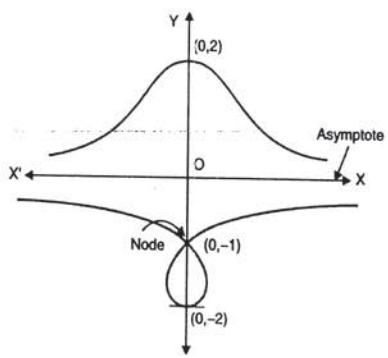


Fig. 17

Ex. 18

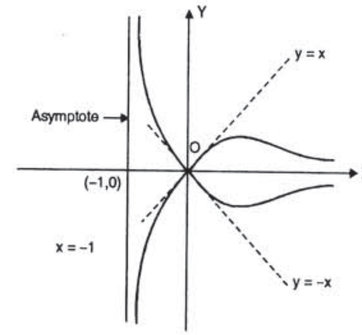


Fig. 18

Ex. 19

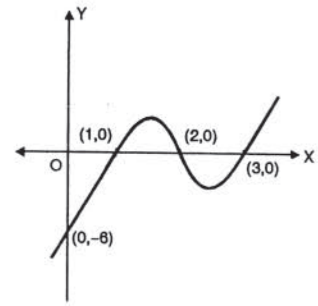


Fig. 19

Ex. 20

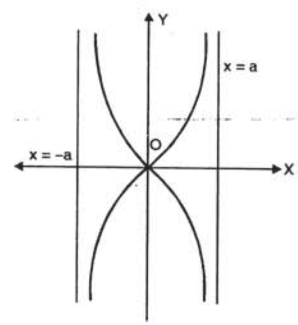


Fig. 20

7.5(A) Tracing of Curves in Polar Form

In general, the curves in polar form can be written as $r = f(\theta)$.

In polar system,

(i) **Pole :** The fixed point is called pole (Here, origin O is a pole).

(ii) **Initial line :** A fixed line (positive x-axis) is called initial line.

(iii) **Radius vector :** The distance of a point from pole is called radius vector. (Here, $OP = r$).

(iv) **Vectorial angle (θ) :** The angle made by radius vector with initial line is called vectorial angle. ($\angle XOP = \theta$)

And ϕ is the angle made by tangent with radius vector (r).

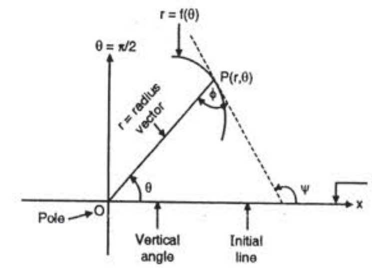


Fig. 7.5.1

(v) θ is measured positive if it rotate in anticlockwise and measured as negative if it rotate in clockwise direction.

(vi) θ is a rotation, so it starts from initial line (OX), (or initial line denoted as $\theta = 0$).

(vii) $\theta = \pi/2$, represent a straight line passes through pole and perpendicular to initial line (i.e. Y-axis)

(viii) If it is difficult to trace the curve in Cartesian form, we may convert it into polar form by using the transformation

$$x = r\cos\theta \,;\; y = r\sin\theta$$

$$r^2 = x^2 + y^2 \quad \text{and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

and then trace the curve by using the rules in polar form.

(ix) ψ - is the angle made by tangent with initial line.

(x) $r = a$ (constant)

It represent the family of concentric circles with radius a and centre at origin (pole).

and $\theta = a$ (constant)

It represents the family of straight lines passing through origin (pole).

Important note

(i) r is taken as positive, if measured away from pole (origin) along the line bounding the vectorial angle θ.

(ii) r is taken as negative for any value of θ, we proceed as follows :

First find the value of θ for which r is negative, then produce (draw) the line in opposite (backwards) side of θ measured through pole with distance | r |, we get the required point P (r, θ).

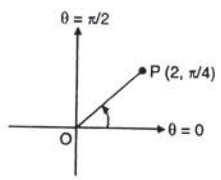


Fig. 7.5.1(i)

For example,

(i) Point P (2, π/4) : Fig. 7.5.1(i)

(ii) Point P (– 2, π/4) : Fig. 7.5.1(ii)

(iii) P (– 2, – π/4) : Fig. 7.5.1 (iii)

(iv) P (2, – π/4) : Fig. 7.5.1 (iv)

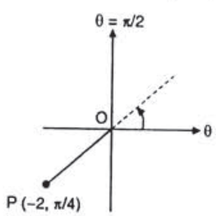


Fig. 7.5.1(ii)

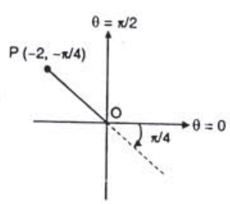


Fig. 7.5.1(iii)

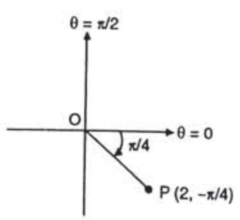


Fig. 7.5.1(iv)

7.5 (B) Rules for Tracing of Curves In Polar Form : (r = f(θ))

Generally, we use the following rules to trace the curve in polar form.

(I) Symmetry

(i) **About initial line :** By replacing θ by – θ, if the given equation remains unchanged, then the curve is symmetric about initial line (x-axis)

For example, r = a cos θ

(ii) **About pole :** If the powers of r are even, then the curve is symmetrical about pole.

(Also, by putting r by – r, if given equation remains same)

For example, $r^2 = a^2 \cos 2\theta$

(iii) **About θ = π/2 line :** If there is no change in the given equation by changing θ = – θ and r = – r at the same time, then the curve is symmetrical about the line θ = π/2 (Y axis)

(θ = π/2 is the line through pole perpendicular to the initial line)

For example, r = a sin 3θ

(iv) **About θ = π/2 line :** If the equation remains unchanged by replacing θ by π – θ , then also the curve is symmetrical about the line θ = π/2.

(v) **About pole (or opposite quadrants) :**

If the given equation remains unchanged by putting r = – r **or** θ = π + θ then the curve is symmetric in opposite quadrant (or pole).

(vi) **About the line θ = π/4 :** If the given equation remains unchanged by putting θ = π/2 – θ, then the curve is symmetric about the line θ = π/4.

(II) tan ϕ

Since, $\tan\phi = \dfrac{r\,d\theta}{dr} = \dfrac{r}{\left(\dfrac{dr}{d\theta}\right)}$

Where, ϕ - is the angle made by tangent with radius vector (r),

Find the values of θ where tan ϕ = 0 or ∞ (i.e. for, ϕ = 0 or π/2)

(i) If tan ϕ = 0 for θ = θ_1 then, at θ = θ_1 the curve have a tangent parallel to radius vector r (tangents coinside with radius vector)

(ii) If tan ϕ = ∞ for θ = θ_2 then, at θ = θ_2 the curve have tangent perpendicular to radius vector.

(For this use table)

Let Ψ = θ + ϕ - be the angle made by tangent with initial line.

(III) Table : Form a table for different values of r, θ, tan ϕ

θ	0	π/6	π/4	π/3	π/2	3π/4	π
r = f(θ)							
$\tan\phi = r\dfrac{d\theta}{dr}$							

(IV) Pole

From table find the values of θ for which r = 0. At that value of θ, curve passes through pole.

(In polar system, distance r is measured from pole (origin)), (use table)

(V) Equation of tangents at pole

Put, r = 0 in the given equation, the values of θ gives the equation of tangent at pole. (use table)

(VI) Region of absence : (use table)

(i) Determine the maximum and minimum values of r. If maximum value of r is **a**, then the whole curve lies, within the circle of radius **a** and centre at the pole. If minimum value of r is **b**, then no part of the curve lies inside the circle of radius **b**.

(ii) Determine the range of θ where $r^2 < 0$, i.e. r is imaginary, means in that range of θ, curve does not exists.

Observe the values of θ, how r varies, as θ increases or decreases.

Note : Most of the polar curves involves periodic functions, sin θ and cos θ with period 2π. Hence consider only 0 to 2π values. Remaining values of θ, no new curve.

(VII) Asymptote

Find asymptote by converting the equation to Cartesian form.

If r → ∞ for some θ, then the curve have asymptote. Also, the asymptote corresponding to θ = α is

$$r \sin(\theta - \alpha) = \frac{1}{f'(\alpha)} \qquad \dots (7.5.1)$$

where, $f'(\alpha) = [f'(\theta)]_{\theta=\alpha}$

Procedure

(i) Write the given equation of the form $\dfrac{1}{r} = f(\theta)$

(ii) Solve the equation f(θ) = 0

We get α, β, γ, ...

(iii) Find asymptote from Equation (7.5.1) for different values like α, β, γ.

Note :

(i) If $\dfrac{dr}{d\theta} > 0$, then r increases as θ increases

And

(ii) If $\dfrac{dr}{d\theta} < 0$, then r decreases as θ increasing.

(iii) If the curve meets the line of symmetry at two points, then there is a loop (closed curve) between these two points.

The polar curves are classified into the following categories:

(a) **Rose curves :** r = a sin nθ : r = a cos nθ

(b) **Leminiscate :** $r^n = a^n \cdot \sin n\theta$; $r^n = a^n \cos n\theta$

(c) **Cardiodes :** r = a ± b cos θ ; r = a ± b sin θ

(d) **Spiral.**

Remember

$$\sin n\theta = 0 \Rightarrow \theta = 0, \frac{\pi}{n}, \frac{2\pi}{n}, \dots ;$$

$$\cos n\theta = 0 \Rightarrow \theta = \frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{n}, \dots$$

7.5.1 Type (a) : Rose Curves : r = a sin nθ or r = a cos nθ

Use the rules of curve tracing in polar form.

Rose curves consists of ;

(i) n similar loops – if n is an odd

(ii) 2n similar loops – if n is an even

(iii) If n = 1, we get a circle

Draw the loops using following points :

(i) Divide each quadrant into n equal sector / parts. Measurement of each sector is $\dfrac{\pi}{2n}$

(ii) For r = a sin nθ ; first loop is drawn along $\theta = \dfrac{\pi}{2n}$

(iii) For r = a cos nθ ; first loop is drawn along θ = 0

(iv) If n is even, draw loops in two sectors consecutively from θ = 0 to 2π.

(v) If n is odd, draw loops into two sectors alternately, keeping two sectors vacunt between the loops.

(vii) If for the range $\alpha_1 < \theta < \alpha_2$, r is negative, then the branch of the curve exists in opposite sector.

(viii) The family of curves $r = a \cos n\theta$ are similar to the family of curves $r = a \sin n\theta$ which are obtained by rotating through $\theta = \frac{\pi}{2n}$.

Example 7.5.1

Trace the curve : r = a sin 2θ

Solution :

Step I : Given curve : $r = a \sin 2\theta$

Symmetry : The given curve is symmetrical about $\theta = \frac{\pi}{2}$ line.

($\because$ put $r = -r$ and $\theta = -\theta$ at the same time, no change in the given curve)

Step II : $\tan \phi = \frac{r\,d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}}$

Since, $r = a \sin 2\theta$;

$$\frac{dr}{d\theta} = 2a \cos 2\theta$$

$$\therefore \tan \phi = \frac{a \sin 2\theta}{2a.\cos 2\theta} = \frac{1}{2} \tan 2\theta$$

Step III : From a table for r, θ, ϕ

θ	0	π/4	π/2	3π/4	π
$r = a \sin 2\theta$	0	a	0	– a	0
$\tan\phi = \frac{1}{2} \tan 2\theta$	0	∞	0	∞	0

Step IV : Pole :

From table it is clear that $r = 0$ when $\theta = 0, \frac{\pi}{2}, \pi$.

Means, curve passes through a pole at $\theta = 0, \frac{\pi}{2}, \pi$.

Step V : Also, $\tan \phi = 0$ when $\theta = 0, \frac{\pi}{2}, \pi$.

This shows that, at $\theta = 0, \frac{\pi}{2}, \pi$. The curve have tangent coincide with radius vector.

Step VI : $\tan \phi = \infty$ at $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

Thus, at $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$, true curve have tangents, perpendicular to radius vector (see table)

Step VII : Asymptote : This curve does not have any asymptote because r have finite value from $\theta = 0$ to $\theta = 2\pi$.

Step VIII : From table it is clear that, the maximum absolute value of r is a, so the entire curve lies within a circle of radius a.

In short : Here, n = 2, we have four equal loops.

Divide each quadrant into 4 equal parts.

Each part is equals to : $\frac{\pi}{2n} = \frac{\pi}{2 \times 2} = \frac{\pi}{4}$

First loop is drawn along $\theta = \frac{\pi}{2n} = \frac{\pi}{2 \times 2} = \frac{\pi}{4}$

Draw loops in two sector consecutively ($\because$ n is even)

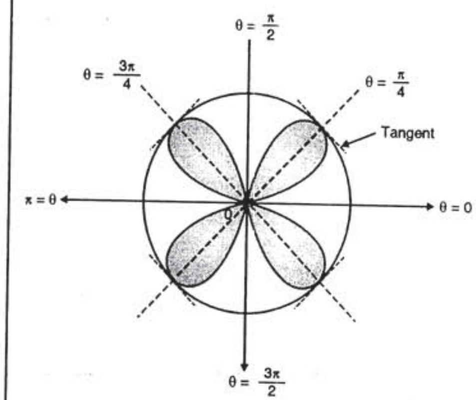


Fig. P. 7.5.1

Example 7.5.2

Trace the curve : r = a cos 2θ.

Solution :

Step I : The given curve is : $r = a \cos 2\theta$

Symmetry : This curve is symmetric about the initial line ($\theta = 0$) and the line $\theta = \pi/2$

(Since, by putting $\theta = -\theta$, in the given equation $r = a \cos 2\theta = a \cos 2(-\theta) = a \cos 2\theta$; equation of the curve unchanged.) And by putting $\theta = \pi - \theta$;

$r = a \cos 2\theta = a \cos 2(\pi - \theta) = a \cos (2\pi - 2\theta) = a \cos 2\theta$

Equation of the curve remains unchanged

Step II : $\tan \phi = r\frac{d\theta}{dr} = r\frac{1}{\left(\frac{d\theta}{dr}\right)}$

Since, $r = a \cos 2\theta$

$$\frac{dr}{d\theta} = -2a \sin 2\theta$$

$$\therefore \tan \phi = \frac{r}{\left(\frac{d\theta}{dr}\right)} = \frac{a \cos 2\theta}{-2a \sin 2\theta} = \frac{-1}{2} \cot 2\theta$$

Step III :

Table : From a table for different values of, r, tan ϕ, θ.

θ	0	π/6	π/4	π/3	π/2
$r = a \cos 2\theta$	a	$a\frac{1}{2}$	0	$-a \cdot \frac{1}{2}$	– a
$\tan \phi = \frac{-1}{2} \cot \theta$	∞	$-\frac{1}{2}\frac{1}{\sqrt{3}}$	0	$\frac{1}{2\sqrt{3}}$	∞

Step IV : Pole :

From table, $r = 0$ when $\theta = \pi/4$ means, curve passes through pole at $\theta = \pi/4$ and at $\theta = \pi/4$, the curve have tangent at pole. (See table)

Also, $\tan \phi = 0$ at $\theta = \pi/4$, means at $\theta = \pi/4$, the curve have tangent coincides with radius vector and $\tan \phi = \infty$ at $\theta = 0, \pi/2$. Thus at $\theta = 0, \pi/2$, the curve have tangents perpendicular to radius vector. (See table)

Step V : Asymptote :

This polar curve does not have any asymptote because r have finite value as $\theta \to 0$ to 2π. (See table)

Step VI : Region of Absence

From table it is observed that the maximum value of r = a. So entire curve lies within a circle of radius a.

As θ-increases from 0 to π/4, r goes on decreases from a to 0. and from $\theta = \pi/4$ to $\frac{\pi}{2}$, r is negative so we trace the curve in opposite quadrant for this range.

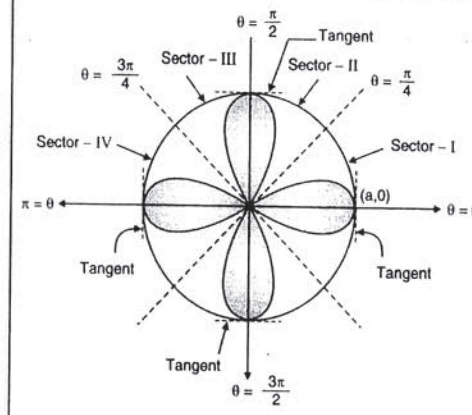


Fig. P. 7.5.2

This curve intersects initial line at (a, 0).

Step VIII : In short : Here, n = 2, we have four equal loops.

Divide each quadrant into 4 equal parts

Each part is equal to $\frac{\pi}{2.2} = \pi/4$

First loop is drawn along $\theta = 0$

Draw loops in two sector consecutively ($\because$ n is even).

See, Fig. P.7.5.2.

Example 7.5.3

Trace the curve : r = a sin 3θ.

Solution :

Step I : The given curve is : $r = a \sin 3\theta$

Symmetry : The given curve is symmetric about $\theta = \pi/2$ line.

(Since, by replacing θ by π – θ, the given equation of the curve remains unchanged.)

Step II : $\tan \phi = r\frac{d\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)}$

Since, $r = a \sin 3\theta$

$$\frac{dr}{d\theta} = 3a \cos 3\theta$$

$$\therefore \tan \phi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{a \sin 3\theta}{3a \cos 3\theta} = \frac{1}{3} \tan 3\theta$$

Step III : Table : Form table for θ, r and tan ϕ :

θ	0	π/6	π/3	π/2	4π/6	5π/6	π
r = a sin 3θ	0	a	0	– a	0	a	0
$\tan\phi = \frac{1}{3}\tan 3\theta$	0	∞	0	∞	0	∞	0

Step IV : Pole/equation of tangent at pole : Put r = 0 and find the value of θ.

From table; r = 0 for θ = 0, π/3, 4π/6, π. At these values of θ, curve passes through pole.

Also, at θ = 0, π/3, 4π/6, π, the curve have tangents at pole.

∵ tan ϕ = 0 at θ = 0, π/3, 4π/6 π, So at these values or θ, the tangents coinsides with radius vectors.

And tan ϕ = ∞ at θ = π/6, π/2, 5π/6,

So, at these values of θ, the curve have tangents perpendicular to radius vector.

Step V : Asymptote

This curve does not have any asymptote because r is finite for any value of θ. (See table).

Step VI : Region of absence

From table, it is observed that, the maximum value of r = a, so entire curve lies inside the circle of radius a. Also, r is negative from θ = π/3 to π/2, so the curve is drawn in opposite side.

Step VII : In short

Here, n = 3, so this curve have only 3 loops.

Divide each quadrant in 3 equal parts

Measurement of each sector is $\theta = \frac{\pi}{2\cdot 3} = \frac{\pi}{6}$

First loop is drawn along $\theta = \frac{\pi}{2\cdot 3} = \frac{\pi}{6}$

Draw loops in two sectors alternately, keeping two sectors vacant between two loops.

The rough sketch of the given curve is as Fig. P. 7.5.3.

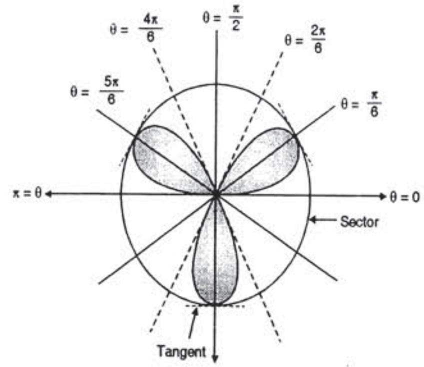


Fig. P.7.5.3

Example 7.5.4

Trace the curve : r = a cos 3θ.

Solution :

Step I : The given curve : r = a cos 3 θ

Symmetry : The curve is symmetrical about initial line.
(Since, by putting θ = – θ, the given equation remains unchanged.)

Step II : tan ϕ

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)}$$

Since, r = a cos 3θ, $\frac{dr}{d\theta} = -3a\sin 3\theta$

$$\therefore \quad \tan\phi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{a\cos 3\theta}{-3a\sin\theta} = -\frac{1}{3}\cot 3\theta$$

Step III : From a table for θ, r, ϕ

θ	0	π/6	π/3	π/2	4π/6	5π/6	π
r = a cos 3θ	a	0	– a	0	a	0	– a
$\tan\phi = -\frac{1}{3}\cot 3q$	∞	0	∞	0	∞	0	∞

Step IV : Pole / tangents at pole : From table, r = 0, for θ = π/6, π/2, 5π/6.

So, at this values of θ, curve passes through pole and, at θ = π/6, π/2, 5π/6, this curve have tangent at origin. Also, it is observed from the table that tan ϕ = 0 at θ = π/6, π/2, 5π/6,

So, the curve have tangents, coincide with radius vector

at θ = π/6, π/2, 5π/6,

And, tan ϕ = ∞ at θ = 0, 4π/6, π, means, the curve have tangents perpendicular to radius vector.

This curve intersects initial line at (a, 0)

Step V : Asymptote : This curve does not have any asymptote because, r is finite for all values of θ.

Step VI : Region of absence :

From table, it is observed that the maximum value of r is a, so the entire curve lies within a circle of radius a and r is negative from θ = π/6 to π/3 , π/2 and θ = 5π/6 to π, so it is drawn in opposite quadrants.

Step VII : In short

Here, n = 3, so, the curve have only three loops.

Divide each quadrant into 3 equal parts.

Measurement of each sector = $\frac{\pi}{2\cdot 3} = \frac{\pi}{6}$

First loop is drawn along θ = 0 in two sectors.

Draw loops into two sectors alternately keeping two sectors vacant between the loop.

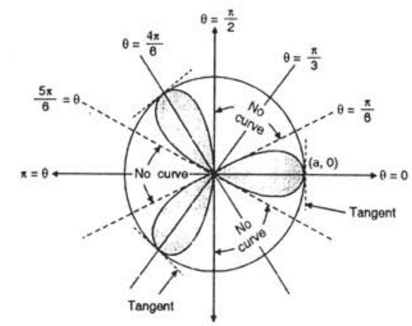


Fig. P. 7.5.4

The rough sketch of the curve is as follows : (Fig. P.7.5.4)

7.5.2 Type (b) : Leminscale $r^n = a^n \cos n\theta$ or $r^n = a^n \sin n\theta$

Example 7.5.5

Trace the curve : $r^2 = a^2 \cos 2\theta$

Solution :

Step I : The given curve is : $r^3 = a^2 \cos 2\theta$

Symmetry : The given curve is symmetric about the initial line θ = 0 as well as the line θ = π/2. (Since, the equation of the curve remain unchanged by putting θ = – θ and r = – r respectively.) Also, the given curve is symmetric about pole because power of r is even.

Step II : $\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)}$

Since, $r^2 = a^2\cos 2\theta$

$$2r\frac{d\theta}{dr} = -\cancel{2}a^2\sin\cancel{2}\theta$$

$$\therefore \quad \tan\phi = \frac{r}{-\left(\frac{a^2\sin 2\theta}{r}\right)} = \frac{r^2}{-a^2\sin 2\theta} = \frac{a^2\cos 2\theta}{-a^2\cos 2\theta}$$

$$\boxed{\tan\phi = -\cot 2\theta}$$

Step III : Table : Form a table for r, θ, tan ϕ

θ	0	π/8	π/6	π/4	π/2	3π/4	π
$r^2 = a^2\cos 2\theta$	a^2	$a/(\sqrt{2})^{1/2}$	$a/\sqrt{2}$	0	$-a^2$	0	a^2
$\tan\phi = -\cot 2\theta$	∞	– 1	–ve	0	–	0	∞

Step IV : Pole : From table, r = 0 for θ = π/4, 3π/4. Curve passes through pole for θ = π/4, 3π/4. Also the curve have tangents at pole for θ = π/4, 3π/4.

Also from table, tan ϕ = 0 for θ = π/4, 3π/4.

For these values of θ, the curve have tangents parallel to radius vector (coincide)

And, tan ϕ = ∞ for θ = 0, π. For these values of θ, the curves have tangents perpendicular to radius vector. This curve intersects initial line at (a, 0) and (a, π).

Step V : Asymptote

This curve does not exist any asymptote because, r is finite for all values of θ.

Step VI : Region of absence

From table, it is clear that, the maximum value of r = a, (∵ $r^2 = a^2 \Rightarrow r = \pm a$)

The entire curve lies within a circle of radius a.

Also observed that from θ = π/4. to 3π/4 ; r^2 = negative i.e. r = imaginary value means, curve does not exist in that range. The rough sketch of the given curve is drawn Fig. P.7.5.5.

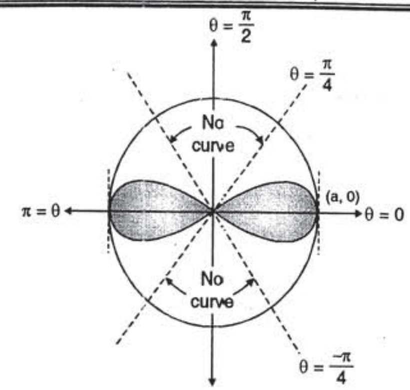


Fig. P.7.5.5

Example 7.5.6

Trace the curve $r^2 = a^2 \sin 2\theta$.

Solution :

Step I : Since, $r^2 = a^2 \cos 2\theta = a^2 \cos 2(\pi/4 - \theta)$

Hence, this curve is obtained by rotating the curve $r^2 = a^2 \cos 2\theta$ through $\pi/4$. We can write the points, we get the curve like Fig. P. 7.5.6.

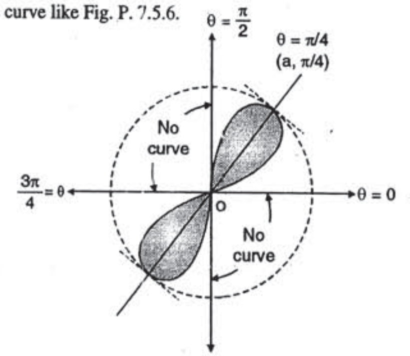


Fig. P. 7.5.6

7.5.3 Type (c) : Cardiode / Limacon

$r = a \pm b \cos\theta$ or $r = a \pm b \sin\theta$

Here, there are three cases (i) $a > b$ (ii) $a < b$ (iii) $a = b$

Case (I) : a > b

Example 7.5.7

Trace the curve : $r = a + b\cos\theta$; for $a > 0, b > 0$

Or $r = 3 + 2\cos\theta$ Or $r = \sqrt{2} + \cos\theta$

Solution :

Step I : The given curve : $r = a + b\cos\theta$; $a > b$

Symmetry : The curve is symmetrical about the initial line.

(Since, by replacing $\theta = -\theta$, equation of the curve remains unchanged).

Step II : $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{r}{\left(\dfrac{dr}{d\theta}\right)}$

Since, $r = a + b\cos\theta$

$\dfrac{dr}{d\theta} = -b\sin\theta$

$\therefore \quad \tan\phi = \dfrac{r}{(-b\sin\theta)} = \dfrac{(a + b\cos\theta)}{-b\sin\theta}$ ...($\because a > b$)

Step III : Table : Form a table for r, θ, ϕ

θ	0	π/4	π/2	3π/4	π
$r = a + b\cos\theta$ $\because (a > b)$	$a + b$	$a + \frac{b}{\sqrt{2}}$	a	$a - \frac{b}{\sqrt{2}}$	$a - b$
$\tan\phi = \frac{a + b\cos\theta}{-b\sin\theta}$	∞	–	$\frac{a}{-b}$	–	∞

Step IV : Pole / tangents at pole : Here, $a > b$, so r is never zero nor negative, r is always Positive.

The curve does not passes through pole

No tangent at pole

From table, it is clear that,

$\tan\phi = \infty$ at $\theta = 0$, and $\theta = \pi$

The curve have tangent perpendicular to radius vector.

Step V : Asymptote : No asymptote

Step VI : Region of absence :

From table, it is clear that r has maximum value $r = a + b$ for $\theta = 0$, and minimum value of $r = a - b$ at $\theta = \pi/2$ As θ increases from $\theta = 0$ to π, r also decreases from $a + b$ to $a - b$.

The rough sketch of the curve is in Fig. P. 7.5.

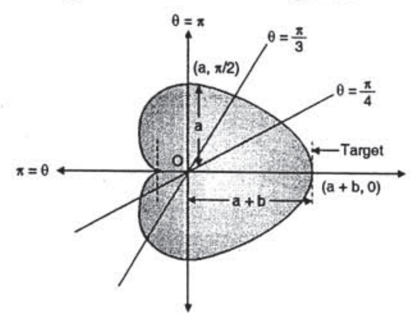


Fig. P. 7.5.7

Similarly, we can trace the polar curves for,

(i) $r = a - b\cos\theta$; $a > b$

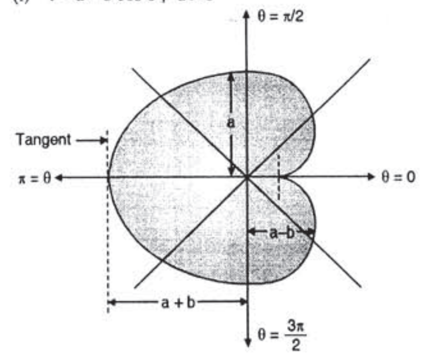


Fig. P.7.5.7(a)

(ii) $r = a + b\sin\theta$; $a > b$

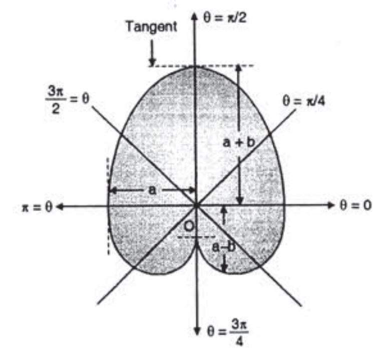


Fig. P.7.5.7(b)

(iii) $r = a - b\sin\theta$; $a > b$

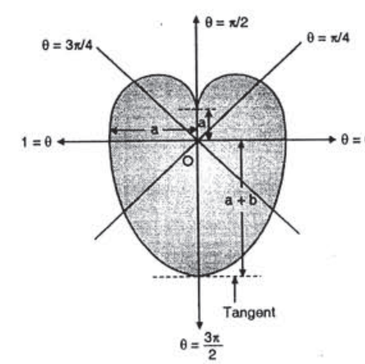


Fig. P.7.5.7(c)

Case (ii) a < b

Example 7.5.8

Trace the curve : $r = a + b\cos\theta$ Or $r = 2 + 3\cos\theta$

Solution :

Step I : The given curve is : $r = a + b\cos\theta$; $a < b$

Symmetry

The curve is symmetric about the initial line $\theta = 0$

($\because$ since, by replacing $\theta = -\theta$, equation is remain unchanged)

Step II :

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)}$$

Since $r = a + b\cos\theta$

$\dfrac{dr}{d\theta} = -b\sin\theta$

$\therefore \quad \tan\phi = \dfrac{r}{-b\sin\theta} = \dfrac{(a + b\cos\theta)}{-b\sin\theta}$

Step III : Table : Form a table for r, θ, tan ϕ

θ	0	π/4	π/2	3π/4	π
$r = a + b\cos 2\theta$; $(a < b)$	$a + b$	$a + \frac{b}{\sqrt{2}}$	a	$a - \frac{b}{\sqrt{2}}$	$a - b$
$\tan\phi = \frac{a + b\cos\theta}{-b\sin\theta}$	∞	–	$\frac{a}{-b}$	–	∞

Step IV : Pole / tangents at pole

Since, $a < b$,

$\because$ r is negative for some value of θ. So, the given curve passes through pole.

Here, the negative value of r reflects in opposite side, so we get inner loop.

($\because$ From $\theta = \frac{3\pi}{4}$ to $\theta = 5\frac{\pi}{4}$, r is negative) At $\theta = 0, \pi$, $\tan\phi = \infty$, so this curve have tangents perpendicular to radius vector at $\theta = 0, \pi$.

Step V : Region of absence

From table it is observed that, when θ increases from 0 to π, r goes on decreases

The maximum value of $r = a + b$ for $\theta = 0$ and minimum value of $r = a - b$ for $\theta = \pi$.

The rough sketch of the curve is as following Fig. P. 7.5.8.

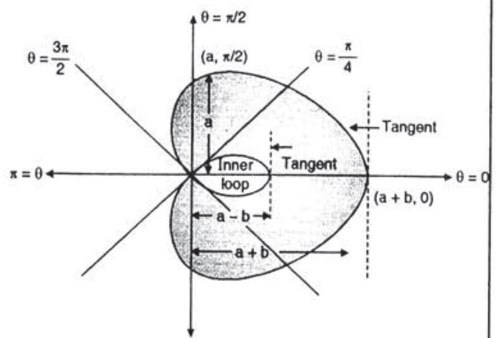


Fig. P.7.5.8

Similarly, we can trace the polar curves for

(i) $r = a - b \cos \theta$; $a < b$

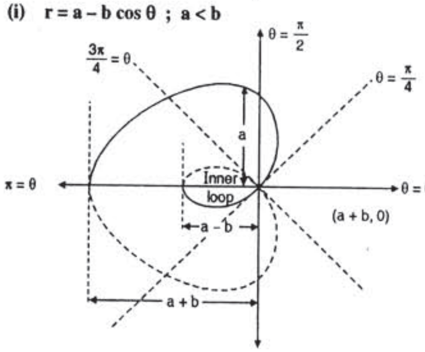


Fig. P.7.5.8(a)

(ii) $r = a + b \sin \theta$; $a < b$

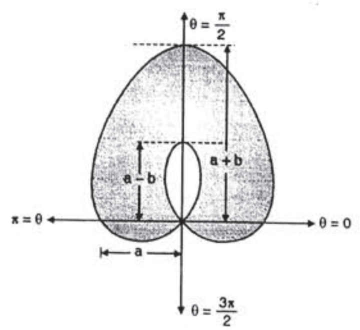


Fig. P.7.5.8(b)

(iii) $r = a - b \sin \theta$; $a < b$

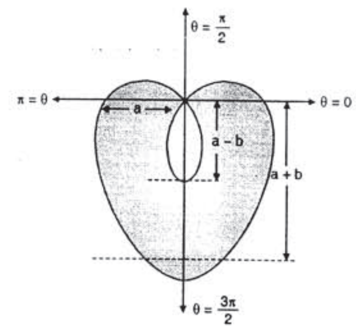


Fig. P. 7.5.8(c)

Case (iii) : a = b :

Example 7.5.9

Trace the curve : $r = a + b \cos \theta = a + a \cos \theta = a(1 + \cos \theta)$. This curve is called Cardioids (like heart shape)

$r = b(1 + \cos \theta)$; OR $r = \frac{a}{2}(1 + \cos \theta)$

Solution :

Step I : The given curve is : $r = a(1 + \cos \theta)$

Symmetry : This curve is symmetric about initial line $\theta = 0$

($\because$ By putting $\theta = -\theta$, the given equation reaming unchanged)

Step II : $\tan \phi = r \frac{d\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)}$

$$\text{Since} \quad r = a(1 + \cos \theta)$$

$$\frac{dr}{d\theta} = -a \sin \theta$$

$$\therefore \quad \tan \phi = \frac{a(1 + \cos q)}{-a \sin q} = \frac{(1 + \cos q)}{-\sin q}$$

Step III : Table : Form a table for r, θ, tan φ

θ	0	π/4	π/2	3π/4	π
$r = a(1 + \cos \theta)$	2a	$a\left(1 + \frac{1}{\sqrt{2}}\right)$	a	$a\left(1 - \frac{1}{\sqrt{2}}\right)$	0
$\tan \phi = \frac{1 + \cos \theta}{-\sin \theta}$	∞	$\frac{1 + 1/\sqrt{2}}{-1/\sqrt{2}}$	$\frac{-1}{2}$	$\frac{1 - 1/\sqrt{2}}{-1/\sqrt{2}}$	0

Step IV : Pole / tangent at pole

Here a = b

Therefore $r = 0$ for $\theta = \pi$. Hence, curve passes through pole at $\theta = \pi$.

The curve have tangent at pole at $\theta = \pi$

($\because$ where $r = 0$)

Also, $\tan \phi = 0$ at $\theta = \pi$

So the curve have tangent coincide with radius vector at $\theta = \pi$

And $\tan \phi = \infty$ at $\theta = 0$, so the curve have tangent perpendicular to radius vector at $\theta = 0$.

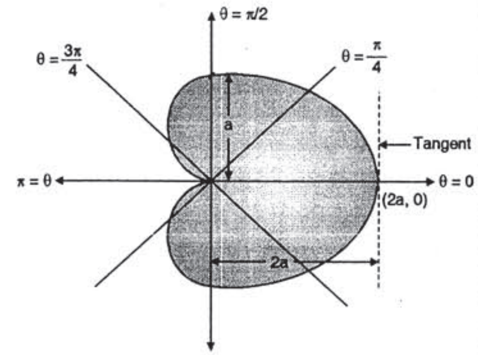


Fig. P.7.5.9

Step V : Asymptote : No asymptote because r is finite for all values of θ.

Step VI : Region of absence : Since, from table, the maximum value of $r = 2a$ for $\theta = 0$ and minimum value or $r = 0$ at $\theta = \pi$.

The rough sketch of the given curve is Fig. P.7.5.9.

Similarly, we can trace the polar curves, such as

(i) $r = a(1 - \cos \theta)$

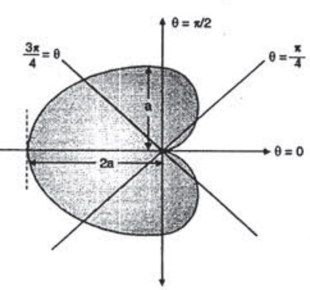


Fig. P.7.5.9(a)

(ii) $r = a(1 + \sin \theta)$

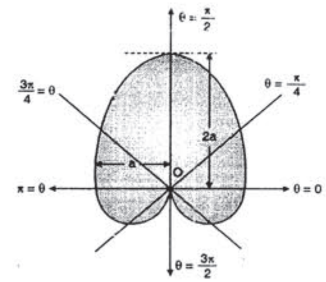


Fig. P.7.5.9(b)

(iii) $r = a(1 - \sin \theta)$

For case (i) a > b Similarly we can trace the curves

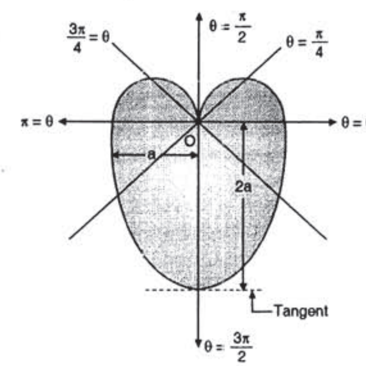


Fig. P.7.5.9(c)

Example 7.5.10

Trace the curve : $r = \frac{2a}{1 + \sin \theta}$

Solution :

Step I : The given curve is : $r = \frac{2a}{1 + \sin \theta}$

Symmetry : This curve is symmetric about $\theta = \pi/2$ line.

(Since, by putting $\theta = \pi - \theta$, equation of the given curve unchanged)

Step II : $\tan \phi = r \frac{d\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)}$

$$\text{Since} \quad r = \frac{2a}{1 + \sin \theta}$$

$$\frac{dr}{d\theta} = -\frac{2a}{(1 + \sin \theta)^2} \cdot (\cos \theta)$$

$$\therefore \quad \tan\phi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\left(\frac{2a}{1+\sin\theta}\right)}{\left[\frac{-2a(\cos\theta)}{(1+\sin\theta)^2}\right]}$$

$$= -\left(\frac{(1+\sin\theta)}{\cos\theta}\right)$$

Step III : Table : Form a table for r, θ, tan φ

θ	0	π/4	π/2	3π/4	π
$r = \frac{2a}{1+\sin\theta}$	2a	$\frac{2a}{1+1/\sqrt{2}}$	a	$\frac{2a}{1+1/\sqrt{2}}$	2a
$\tan\phi = -\left(\frac{1+\sin\theta}{\cos\theta}\right)$	– 1	$-\left(\frac{1+1/\sqrt{2}}{1/\sqrt{2}}\right)$	∞	$\left(\frac{1+1/\sqrt{2}}{1/\sqrt{2}}\right)$	1

Also as $\theta \to 3\pi/2$; $r \to \infty$

and same thing will happen when θ rotates in clockwise direction.

Step IV : Pole / tangents at pole : Here, r is never zero. Hence the curve does not passes through pole and there is no tangent at pole.

Step V : Asymptote : Since, $r \to \infty$ as $\theta \to 3\pi/2$ means this curve have asymptote.

The equation of asymptote is,

$$r\sin(\theta - \alpha) = \frac{1}{f'(\alpha)} \qquad \dots (1)$$

where, $f'(\alpha) = [f'(\theta)]_{\theta=\alpha}$

To find equation of asymptote;

$$\text{Since,} \quad r = \frac{2a}{1+\sin\theta} \; ;$$

$$\therefore \quad \frac{1}{r} = \frac{1+\sin\theta}{2a} = f(\theta)$$

$$\text{Solve,} \quad f(\theta) = 0 \Rightarrow \frac{1+\sin\theta}{2a} = 0 \Rightarrow 1+\sin\theta = 0$$

$$\sin\theta = -1$$

$$\boxed{\theta = (2n+1)\frac{\pi}{2} = \alpha}$$

∴ Equation of asymptote, from Equation (1)

$$f'(\alpha) = [f'(\theta)]_{\theta=\alpha=(2n+1)\frac{\pi}{2}}$$

$$= \left[\frac{\cos\theta}{2a}\right]_{\theta=(2n+1)\frac{\pi}{2}} = \frac{\cos(2n+1)\frac{\pi}{2}}{2a}$$

$$\therefore \quad r\sin\left(\theta - (2n+1)\frac{\pi}{2}\right) = \frac{1}{f'(\alpha)} = \frac{2a}{\cos(2n+1)\frac{\pi}{2}}$$

$$\Rightarrow \boxed{r\sin\left(\theta - (2n+1)\frac{\pi}{2}\right) = \frac{2a}{\cos(2n+1)\frac{\pi}{2}}}$$

be the equation of asymptote.

Step VI : Region of absence :

Since, r is never negative. Curves lies in the interval θ = 0 to θ = 3π/2 and θ = 0 to – 3π/2

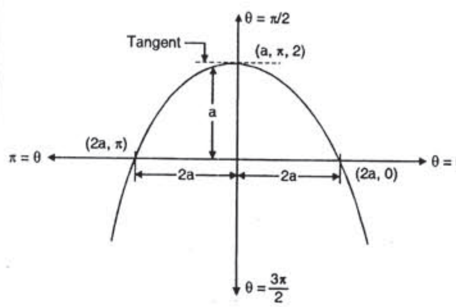


Fig. P. 7.5.10

The rough sketch of the curve is Fig. P. 7.5.10

Example 7.5.11

Trace the curve : $r = \frac{2a}{1+\cos\theta}$

Solution :

Step I : The given curve is : $r = \frac{2a}{1+\cos\theta}$

Symmetry : This curve is symmetric to initial line.

(Since, the equation of the curve remains unchanged by putting θ = – θ)

Step II : $\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)}$

$$\text{Since} \quad r = \frac{2a}{1+\cos\theta} = \frac{2a}{2\cos^2\frac{\theta}{2}} = a\sec^2\theta/2$$

$$\frac{dr}{d\theta} = 2a\sec\frac{\theta}{2}\cdot\sec\frac{\theta}{2}\cdot\tan\frac{\theta}{2}\cdot\frac{1}{2} = a\sec^2\frac{\theta}{2}\cdot\tan\frac{\theta}{2}$$

$$\therefore \tan\phi = \frac{a\sec^2\theta/2}{a\sec^2\frac{\theta}{2}\cdot\tan\frac{\theta}{2}} = \cot\theta/2$$

Step III : Table : Form a table for r, θ, tan φ

θ	0	π/2	π	3π/2	2π
$r = a\sin^2\frac{\theta}{2}$	a	2a	∞	2a	a
$\tan\phi = \cot\frac{\theta}{2}$	∞	1	0	1	∞

Step IV : Pole : Here, r is never zero, so curve does not pass through pole. Also, no tangent at pole.

tan φ = ∞ at θ = 0, 2π, means this curve have tangent perpendicular to radius vector.

And tan φ = 0 for θ = π, means the curve have tangent parallel to radius vector at θ = π.

Step V : Asymptote :

Here, $r \to \infty$ as $\theta \to \pi$

So, this curve have asymptote.

(We can find asymptote for this curve by using the procedure which is used in previous example.)

Step VI : Region of absence :

This curve exists from θ = 0 to 2π

The rough sketch of the curve is as Fig. P.7.5.11.

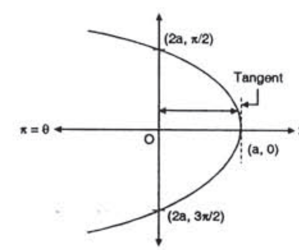


Fig. P. 7.5.11

Exercise 7.2

Trace the following curves :

1 r = 2a cos θ
2 r = a sin 2θ
3 r = a sin 4θ
4 r = a sin 3θ
5 r = a cos 5θ
6 r = a (1 + sin θ)
7 r = a (1 – sin θ)
8 r = a + b sin θ ; a > b
9 r = (3 – 2 cos θ)
10 r = 3 – 2 sin θ
11 r = 1 + 2 sin θ
12 r = 3 + 4 cos θ
13 r = 2 – 3 sin θ
14 $r = a(\sqrt{3} + 2\cos\theta)$
15 $r = \left(\frac{\sqrt{3}}{2} - \cos\frac{q}{2}\right)$
16 $r = \frac{a}{\cos q}$
17 $r^2 = \frac{a^2}{\cos 2q}$
18 r = 2a sin θ
19 r sin θ = a
20 $r^2 = \frac{a^2}{\sin 2q}$
21 $r = \frac{2a}{1 - \cos q}$
22 r = a (sec θ + cos θ)

Answers :

Ex. 1

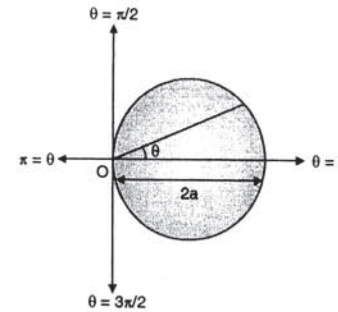


Fig. 1

Ex. 2

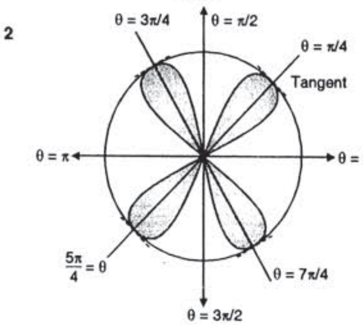


Fig. 2

Ex. 3

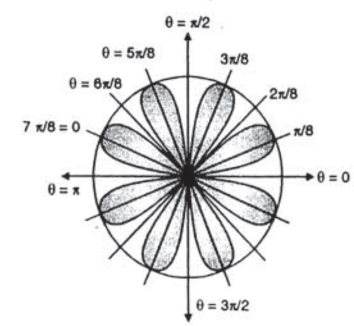


Fig. 3

Ex. 4

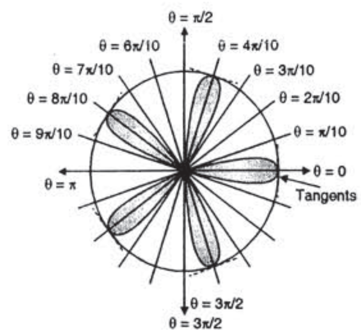


Fig. 4

Ex. 5

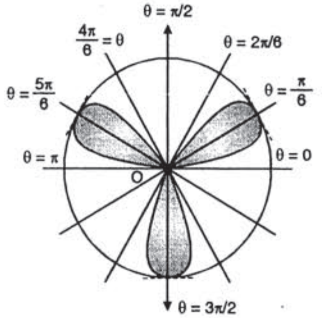


Fig. 5

Ex. 6

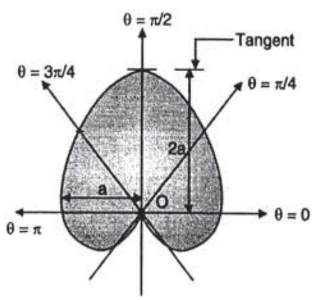


Fig. 6

Ex. 7

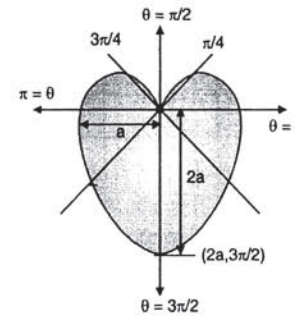


Fig. 7

Ex. 8

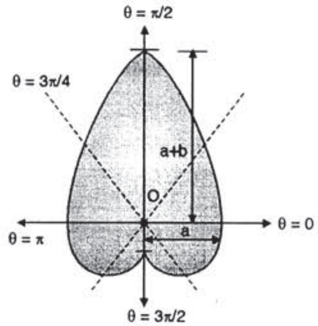


Fig. 8

Ex. 9

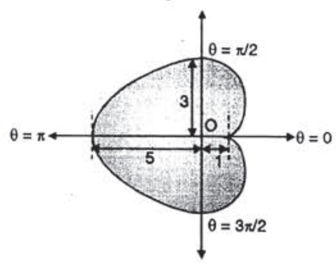


Fig. 9

Ex. 10

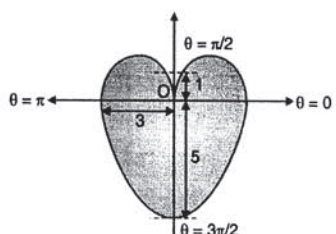


Fig. 10

Ex. 11

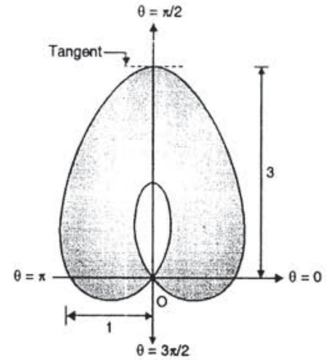


Fig. 11

Ex. 12

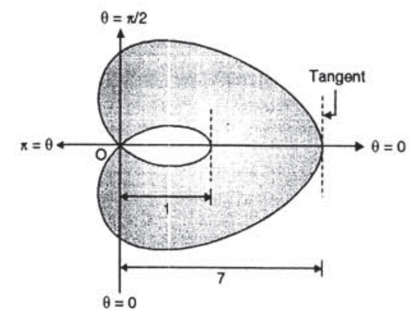


Fig. 12

Ex. 13

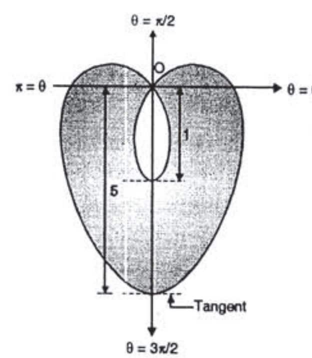


Fig. 13

Ex. 14

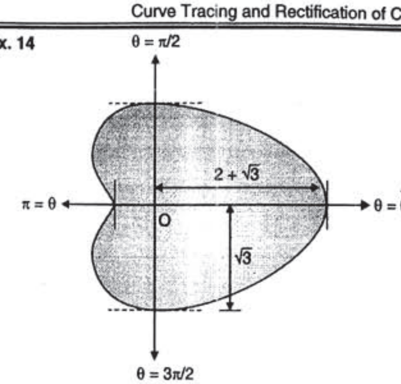


Fig. 14

Ex. 15

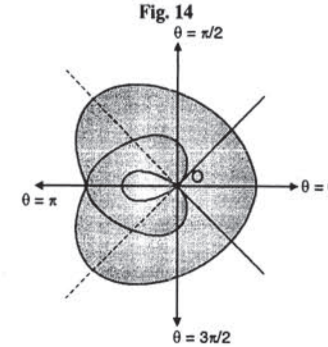


Fig. 15

Ex. 16

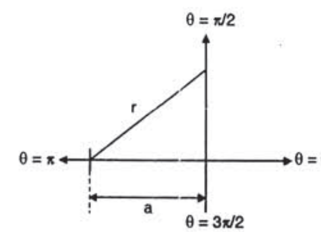


Fig. 16

Ex. 17

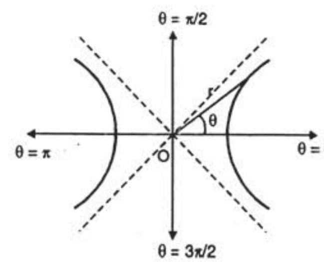


Fig. 17

Ex. 18

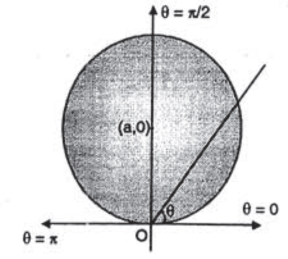


Fig. 18

Ex. 19

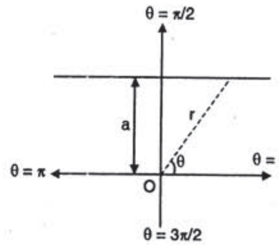


Fig. 19

Ex. 20

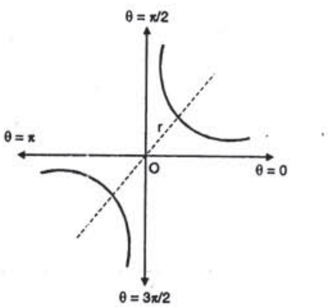


Fig. 20

Ex. 21

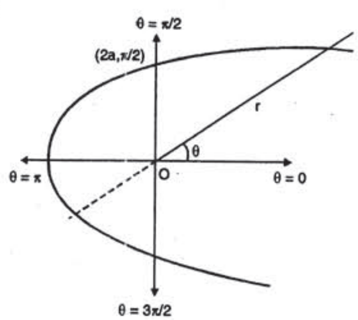


Fig. 21

Ex. 22

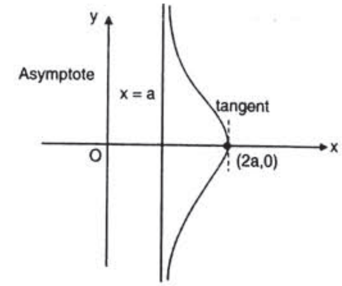


Fig. 22

7.6(A) Tracing of Curves in Parametric Form

($x = f(t)$, $y = g(t)$, where t is a parameter).

Sometimes it is quite difficult to trace the curves in Cartesian or in polar form.

So, convert that curves in parametric form and trace it by using following rules.

Or some parametric curves can be transform to Cartesian or polar form and trace the curve.

7.6(B) To Trace Parametric Curves by Using following Rules

Let, $x = f(t)$ and $y = g(t)$ be the parametric curves.

(I) Symmetry

(i) **About X-axis :** If x is an even function of t and y is an odd function of t, then the curve is symmetric about X-axis.

i. e. If $f(-t) = f(t)$ and $g(-t) = -g(t)$

then the curve is symmetric about X-axis

(ii) **About Y-axis :** If x is an odd function of t and y is an even function of t, then the curve is symmetric about Y-axis.

i.e. If $f(-t) = -f(t)$ and $g(-t) = g(t)$

then the curve is symmetric about Y-axis

(iii) **About opposite quadrant :** If both x and y are an odd function of t then the curve is symmetric about **origin** or **opposite quadrants.**

i.e. if $f(-t) = -f(t)$ and $g(-t) = -g(t)$, then the curve is symmetric about origin or in opposite quadrant

(iv) **About Y-axis :** By putting $t = \pi - t$. If x becomes $-x$ and $y = y$ then also, the given curve is symmetric about Y-axis.

(II) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/df}$

(III) Table : Form a table for t, x, y, $\dfrac{dy}{dx}$

t	0	π/4	π/2	π	3π/2	2π
x						
y						
$\frac{dy}{dx}$						

(IV) Origin : If for some value of t, both x and y are zero at the same time, then the curve passes through origin.

(V) Nature of tangents

1. If $\left(\dfrac{dy}{dx}\right)_t = 0$; then the curve have tangent parallel to X-axis at that value of t.

2. If $\left(\dfrac{dy}{dx}\right)_t = \infty$; then the curve have tangent parallel to Y-axis at that value of t.

(VI) Intersection

Find the X and Y intersects (use table) : Find the value of t for which $x \neq 0$ but $y = 0$, we get X-axis intersection.

Find the value of t, for which $x = 0$ but $y \neq 0$, we get Y-axis intersection.

(VII) Limitations

(i) Find the maximum and minimum values of x and y for any suitable value of t.

(ii) Determine the region where x and y are real.

If x and y are imaginary, then curve does not exist in that region.

> **Note :** If x and y both are periodic functions of t having same period, then the curve can be traced for one period only.

(IX) Find asymptote if-any

Example 7.6.1

Trace the curve : $\left(\dfrac{x}{a}\right)^{2/3} + \left(\dfrac{y}{b}\right)^{2/3} = 1.$

Or $x = a\cos^3 t,\ y = b\sin^3 t$

Or $x^{2/3} + y^{2/3} = a^{2/3}$

Here in above equations just put
$b = a\ (x = a\cos^3 t\ ;\ y = a\sin^3 t)$

Solution :

Step I : This curve is called Astroid or start shaped curve.

(Here, solution given for I[st] type, student should write first parametric equations and then solve)

The parametric equations of the astroid are $x = a\cos^3 t$; $y = b\sin^3 t$

Step II : Symmetry

The given curve is symmetric about X-axis as well as Y-axis (Since, x is an even function and y is an odd function).

Also by putting, $t = \pi - t$, x changes from x to $-x$ and y remains unchanged.

Step III : $\dfrac{dy}{dx}$: Since, $x = a\cos^3 t$, $y = b\sin^3 t$

$$\frac{dx}{dt} = 3a\cos^2 t\,.\,(-\sin t) \quad\text{and}\quad \frac{dy}{dt} = 3b\sin^2 t\cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3b\cos t\sin^2 t}{-3a\sin t\cos^2 t} = -\frac{b\sin t}{a\cos t} = -\frac{b}{a}\tan t$$

Step IV : Table : From a table for t, x, t, $\dfrac{dy}{dx}$

t	0	π/2	π	3π/2	2π
$x = a\cos^3 t$	a	0	– a	0	a
$y = b\sin^3 t$	0	b	0	– b	0
$\frac{dy}{dx} = -\frac{b}{a}\tan t$	0	∞	0	∞	0

Step V : Origin : Both x and y are not zero at the same time for any value t, so this curve does not pass through origin.

Step VI : Intersections

This curve intersect X-axis at (a, 0), (– a, 0) and (a, 0) for $t = 0, \pi$ and 2π respectively.

Also this intersects Y-axis at (0, b), (0, – b) for $t = \pi/2$, $3\pi/2$ respectively.

Step VII : Nature of tangents

$\dfrac{dy}{dx} = 0$ for $t = 0, \pi, 2\pi$

Means, at $t = 0, \pi, 2\pi$, this curve have tangents parallel to X-axis and

$$\rightarrow \quad \frac{dy}{dx} = \infty \quad \text{for } t = \pi/2, 3\pi/2$$

Means, at $t = \pi/2, 3\pi/2$ this curve have tangents parallel to Y-axis.

Step VIII : Region

Limitations : The maximum value of x is a and y is b.

In the I[st] quadrant, when t varies from 0 to $\pi/2$ then x goes on decreases from a to 0 and y goes on increasing from 0 to b.

At $t = 0, \pi/2, \pi, 3\pi/2, \pi$ this curve have different tangents but consider to each other,

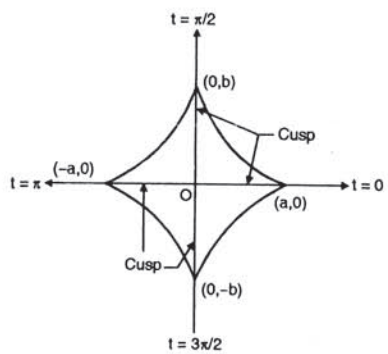


Fig. P.7.6.1

so at these point this curve have cusp.

The rough sketch of the curve is as follows. Astroid (or start shaped curve). (Fig. P.7.6.1)

Example 7.6.2

Trace the curve : $x = a[\cos t + \log(\tan t/2)]$; $y = a \sin t$.

Solution :

Step I : The given curve can be written as

$$x = a\left[\cos t + \frac{1}{2}\log \tan^2(t/2)\right]; \quad y = a \sin t$$

to avoid the logarithm of negative numbers.

Step II : Symmetry : This curve is symmetrical about X-axis. (Since, x is an even function, and y is an odd function). Also symmetric about Y-axis. (Since, for $t = \pi - t$, y is remain same but x changes from x is $-x$)

Step III : $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$\because \; x = a\left[\cos t + \frac{1}{2}\log \tan^2(t/2)\right]; \; y = a \sin t$$

$$\therefore \frac{dx}{dt} = a\left[-\sin t + \frac{1}{2}\cdot\frac{1}{\tan^2 t/2}\, 2\tan(t/2)\cdot \sec^2 t/2 \cdot \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin t/2 \cdot \cos t/2}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right] = a\left[\frac{-\sin^2 t + 1}{\sin t}\right]$$

$$\frac{dx}{dt} = a\left[\frac{\cos^2 t}{\sin t}\right]$$

$$\text{and} \quad \frac{dy}{dt} = a\cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{a\left(\frac{\cos^2 t}{\sin t}\right)} = \tan t$$

Step IV : Table : Form table for t, x, y, $\frac{dy}{dx}$:

t	0	$\pi/2$	π	$3\pi/2$	2π
x	$-\infty$	0	∞	0	$-\infty$
y	0	a	0	$-a$	0
$\frac{dy}{dx}$	0	∞	0	∞	0

Step V : Origin : From table it is clear that, x and y are not zero value at the same time for any value of t.

So this curve does not pass through origin.

Step VI : Intercept

(i) With X-axis : At (0, a), (0, – a) for $t = \pi/2, 3\pi/2$

(ii) With Y-axis : At $(-\infty, 0)$, $(\infty, 0)$, $(-\infty, 0)$ for $i = 0, \pi, 2\pi$ respectively.

Step VII : Nature of tangents : From table $\frac{dy}{dx} = 0$ at $t = 0, \pi, 2\pi$,

So, the curve have tangents parallel to X-axis at $t = 0, \pi, 2\pi$

And $\frac{dy}{dx} = \infty$ at $t = \pi/3, 3\pi/2$. So, the curve have tangents parallel to Y-axis at $t = \pi/2, 3\pi/2$.

Step VIII : Limitations/Region of absence

Since, from table as $t \to 0$ to $\pi/2$, the value of x goes on decreasing. From ∞ to 0 and then again from $t = \pi/2$ to π, x increasing from 0 to ∞.

Step IX : Asymptote

For y = 0; x is infinite, so X-axis itself an asymptote.

The rough sketch of the curve is Fig. P. 7.6.2.

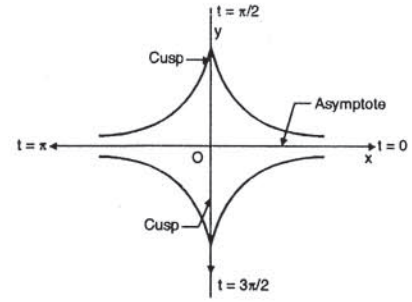


Fig. P.7.6.2

Example 7.6.3

Trace the curve : $x = t^2$; $y = t - \frac{t^3}{3}$.

Solution :

Step I : The given curve is : $x = t^2$; $y = t - \frac{t^3}{3}$

Symmetry : This curve is symmetric about X-axis. (Since, x is an even function and Y is an odd function).

Step II : $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$: Since, $x = t^2$; $y = t - \frac{t^3}{3}$

$$\frac{dx}{dt} = 2t \; ; \; \frac{dy}{dt} = 1 - t^2 \; ; \; \therefore \frac{dy}{dx} = \frac{1-t^2}{2t}$$

Step III : Table : Form a table for t, x, y, $\frac{dy}{dx}$

t	0	1	2	3	4	5
$x = t^2$	0	1	4	9	16	25
$y = t - t^3/3$	0	2/3	– 2/3	0	– 17.33	– 36
$\frac{dy}{dx} = \frac{1-t^2}{2t}$	∞	0	$\frac{-3}{4}$	$\frac{-8}{6} = \frac{-4}{3} = -1.33$	– 1.87	–

As $x = t^2$, for $x < 0$; t^2 = negative

i.e. t becomes imaginary. So, no curve exist for $t < 0$

Step IV : Origin

As both x and y are zero at the same time at t = 0, so this curve passes through origin at t = 0.

Step V : Intersection

This curve intersects X-axis at (9, 0) only.

So, there is a loop between (0, 0) to (9, 0)

Step VI : Nature of tangents

(i) $\frac{dy}{dx} = 0$, at t = 1, this curve have tangents parallel to X-axis.

(ii) $\frac{dy}{dx} = \infty$ at t = 0, this curve have tangent : Parallel to Y-axis at t = 0 (Y-axis).

Step VII : Limitations/region :

This curve exists for $x \geq 0$ and does not exists for $0 < x < -\infty$.

The rough sketch of the curve is Fig. P.7.6.3.

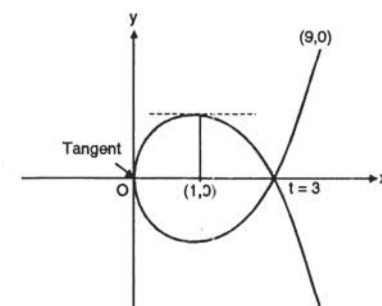


Fig. P.7.6.3

Example 7.6.4

Trace the curve : $x = a(t + \sin t)$; $y = a(1 + \cos t)$.

Solution :

The given curve is :

$X = a(t + \sin t)$; $y = a(1 + \cos t)$

Symmetry : This curve is symmetric about Y-axis.

(Since, x is an odd function and Y is an even function).

Step II : $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a(-\sin t)}{a(1+\cos t)} = \frac{-\sin t}{1+\cos t}$

Step III : Table : Form a table for t, x, y, $\frac{dy}{dx}$

T	0	$\pi/2$	π	$3\frac{\pi}{2}$	2π	$-\pi/2$	$-\pi$	$-3\pi/2$	-2π
X	0	$a\left(\frac{\pi}{2}+1\right)$	$a\pi$	$a\left(\frac{3\pi}{2}-1\right)$	$2\pi a$	$a\left(\frac{-\pi}{2}-1\right)$	$-a\pi$	$a\left(\frac{-3\pi}{2}-1\right)$	$-2a\pi$
Y	2a	a	0	a	2a	a	0	a	2a
$\frac{dy}{dx}$	0	–	∞	–	0	–	∞	–	0

Step IV : Pole : Since, both x and y are never zero, (at the same time) so curve does not passes through origin.

Step V : Nature of tangents :

$\frac{dy}{dx} = 0$ for $t = 0, 2\pi, -2\pi$

So, at $t = 0, 2\pi, -2\pi$, curve have tangents parallel to X-axis.

$\frac{dy}{dx} = \infty$ at $t = \pi, -\pi$

So, curve have tangents parallel to Y-axis at $t = \pi, -\pi$.

Step VI : Intercept : (See Table)

With X-axis : $(a\pi, 0)$ $(-a\pi, 0)$; **With Y-axis :** $(0, 2a)$

Step VII : Limitations / Region of absence : Since, $y = a(1+\cos t)$ and $-1 \le \cos t \le 1$

$\therefore$ y always positive or zero curve does not exist for $y < 0$ and $y > 2a$

The rough sketch of the curve is as follows Fig. P. 7.6.4.

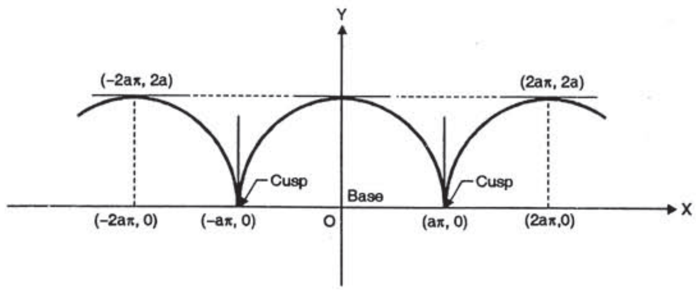


Fig. P. 7.6.4

Example 7.6.5

Trace the curve : $x = a(t + \sin t)$; $y = a(1 - \cos t)$.

Solution :

Step I : The given curve is

$x = a(t + \sin t)$; $y = a(1 - \cos t)$

Symmetry : Symmetry about Y-axis.

($\because$ x is an odd function and Y is an even function)

Step II : $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{(1+\cos t)}$

Step III : Table

t	0	π	2π	$-\pi$	-2π
x	0	$a\pi$	$2\pi a$	$-a\pi$	$-2a\pi$
y	0	2a	0	2a	0
$\frac{dy}{dx}$	0	∞	0	∞	0

Step IV : Origin : Both x and y are zero (at t = 0) at the same time. So, curve passes through origin.

Step V : Nature of tangents :

$\frac{dy}{dx} = 0$ at $t = 0, 2\pi, -2\pi$

Curve have tangents, parallel to X-axis and $\frac{dy}{dx} = \infty$ at $t = \pi, -\pi$

So, curve have tangents parallel to Y-axis.

Step VI : Limitation / region of the curve : Curve lies between y = 0 to y = 2a only. See Fig. P. 7.6.5.

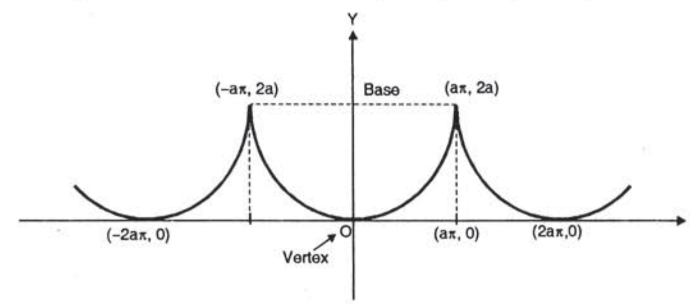


Fig. P.7.6.5

Example 7.6.6

Trace the curve : $x = a(t - \sin t)$; $y = a(1 - \cos t)$.

Solution :

Step I : The given curve is : $x = a(t - \sin t)$; $y = a(1 - \cos t)$

Symmetry : Symmetry about Y-axis.

Step II : $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1-\cos t)} = \frac{\sin t}{(1-\cos t)}$

Step III : Table

t	0	π	2π	$-\pi$	-2π
x	0	$a\pi$	$2\pi a$	$-a\pi$	$-2a\pi$
y	0	2a	0	2a	0
$\frac{dy}{dx}$	∞	0	∞	0	∞

Step IV : Origin : Both x and y are zero at t = 0 at the same time, for t = 0.

So, curve passes through origin at t = 0.

Step V : Nature of tangents

$\frac{dy}{dx} = 0$ at $t = \pi, -\pi$

So, curve have tangents, parallel to X-axis

and $\frac{dy}{dx} = \infty$ at $t = 0, 2\pi, -2\pi$

So, the curve have tangents parallel to Y-axis.

Step VI : Curve lies between y = 0 to y = 2a only. See Fig. P. 7.6.6.

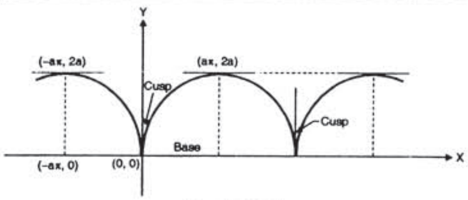


Fig. P. 7.6.6

Exercise 7.3

(a) Trace the following curves :

Example 1 $x = 1 + \sin\theta;\ y = 2\cos 2\theta$

Ans. :

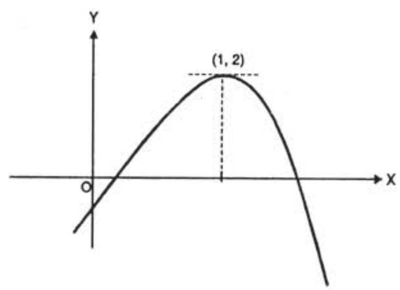


Fig. Ex. 1

Example 2 $x = a \sin 2\theta\,(1 + \cos 2\theta)$

$y = a \cos 2\theta\,(1 - \cos 2\theta)$

Ans. :

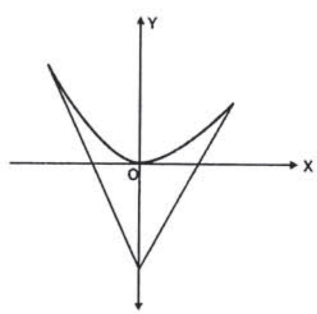


Fig. Ex. 2

Example 3 $x = at\ ;\ y = \frac{a}{t^2}$

Ans. :

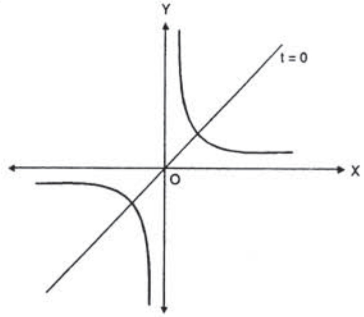


Fig. Ex. 3

Example 4 $x = a \cos t;\ y = b \sin t$

Ans. :

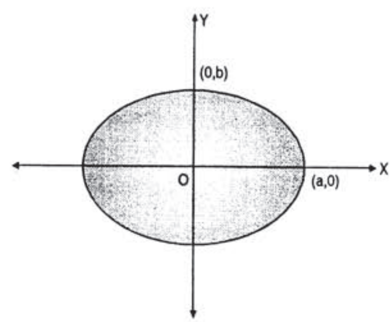


Fig. Ex. 4

7.7 Tracing of Cartesian Curve (Implicit Form)

When one variable cannot be expressed in terms of other then that function is said to be implicit function. For tracing of curve in implicit form, all the rules of tracing of curves in explicit form are applicable. Here only, we have to find the oblique asymptote, which is neither parallel to X-axis nor Y-axis.

Sometimes, we convert Cartesian curves to polar form for better tracing.

Example 7.7.1

Trace the curve : $x^3 + y^3 = 3\,axy$.

Solution :

Step I : The given curve is : $x^3 + y^3 = 3ax\,y$

Symmetry : The given curve is symmetric about $y = x$ line.

(Since, by replacing x by y and y by x no change in the given equation).

Step II : Points of intersections :

(i) **With X-axis :** Put $y = 0$; we get, $x^3 = 0 \Rightarrow x = 0$

This curve intersects X-axis at (0, 0)

(ii) **With Y-axis :** Put $x = 0$; we get $y^3 = 0 \Rightarrow y = 0$.

So, this curve intersects Y-axis at (0, 0).

Hence, the curve passes through origin.

Step III : Equation of tangents :

(i) **With origin :** Equating the lowest degree term/terms to zero.

$\therefore\ 3xy = 0 \Rightarrow x = 0 \quad \text{or} \quad y = 0$

Hence, both X-axis and Y-axis are tangents at origin

(ii) **At** $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$\because\ x^3 + y^3 = 3axy$

Differentiate w.r.t. x

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3a\left[x \cdot \frac{dy}{dx} + y\right]$$

$$\Rightarrow\ x^2 + y^2\frac{dy}{dx} = ax\frac{dy}{dx} + ay$$

$$\Rightarrow\ (y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$$\frac{dy}{dx} = \left(\frac{ay - x^2}{y^2 - ax}\right)$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{-3a^2}{3a^2} = -1$$

The curve have tangent at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ whose slope is -1.

Step IV : Equation of asymptote :

Since, the curve is not symmetric to any axis.

$\therefore$ To find **oblique asymptote.** Let $y = mx + c$ ...(1)

Be the equation of an asymptote.

$\because\ x^3 + y^3 - 3axy = 0$

Highest degree (third) terms : $x^3 + y^3 \equiv \phi_3(x, y)$.

Lowest degree (second) terms : $-3axy \equiv \phi_2(x, y)$

Put $x = 1$ and $y = m$

$\therefore\ \phi_3(m) = 1 + m^3$ and $\phi_2(m) = -3am$

Consider, $\phi_3(m) = 0 \Rightarrow 1 + m^3 = 0$

$\Rightarrow (m + 1)(m^2 - m + 1) = 0$

$\Rightarrow\ m = -1$ and other two roots are complex

Now, find c, $\therefore\ c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\left[\frac{-3am}{3m^2}\right] = \frac{a}{m}$

$$c = \frac{a}{-1} \Rightarrow \mathbf{c = -a}$$

Equation (1) becomes,

$y = -x + a$ or $\mathbf{x + y = -a}$ be the equation of oblique asymptote.

Step V : Region of absence :

If $x > 0$, $y > 0$ then LHS and RHS both positive, so curve exist in I[st] quadrant.

If $x < 0$, $y < 0$, RHS positive but LHS becomes negative. **It is meaningless.** So curve **does not** exist in third quadrant.

The rough sketch of the curve is shown in Fig. P. 7.7.1.

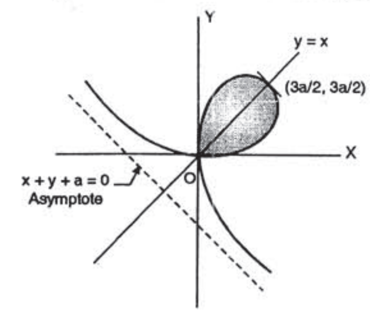


Fig. P. 7.7.1

Example 7.7.2

Trace the curve of : $x^4 + y^4 = 4a\,xy^2$.

Solution :

Step I : The given curve is : $x^4 + y^4 = 4a\,xy^2$

Symmetry : The given curve is symmetric in opposite quadrant

(Since, by putting x = – x and y = – y, the given equation unchanged)

Step II : Points of intersection

(i) With X-axis : Put $y = 0$; $x^4 = 0 \Rightarrow x = 0$

(ii) With Y-axis : Put $x = 0, \Rightarrow y^4 = 0 \Rightarrow y = 0$

Hence, this curve intersects X and Y axis at origin only.

(iii) Origin : put $x = 0, y = 0$, we get $0 = 0$

This curve passes through origin.

Step III : Equation of tangents at origin : Equating lowest degree term to zero.

$\therefore \; 4axy = 0 \Rightarrow x = 0 \quad \text{or} \quad y = 0$

i.e. X-axis and Y-axis, both axes are tangents at origin.

Step IV : Asymptote : Since, this curve is not parallel to any axis.

$\therefore$ So find **oblique asymptote**. Since $x^4 + y^4 = 4axy^2$

Highest (4^{th} degree) degree term $\phi_4 \; x^4 + y^4$;

Lowest (3^{rd} degree) degree term $\phi_3 : 4axy^2$

Put $x = 1 ; y = m$

$$\therefore \; \phi_4(m) = 1 + m^4 \quad , \; \phi_3(m) = 4am^2$$

For m; solve $\phi_4(m) = 0 \Rightarrow 1 + m^4 = 0$

$\Rightarrow$ **m = – 1 and** other roots are of complex roots

For c : $c = -\left[\dfrac{\phi_3(m)}{\phi_4'(m)}\right] = -\left[\dfrac{4am^2}{4m^3}\right] = \dfrac{a}{m} = -a$

$(\because m = -1)$

$$\mathbf{c = -a}$$

$$y = mx + c = -x - a \; ;$$

$\mathbf{x + y = -a}$ be the equation of asymptote

Step V : Region :

For our convenience, we convert the given curve from Cartesian form to polar form,

Since, $x = r\cos\theta, \; y = r\sin\theta$

$$\therefore \; r^4\cos^4\theta + r^4\sin^4\theta = 4ar^3\cos\theta\sin^2\theta$$

$$\Rightarrow r = \frac{4a\cos\theta\sin^2\theta}{\cos^4\theta + \sin^4\theta}$$

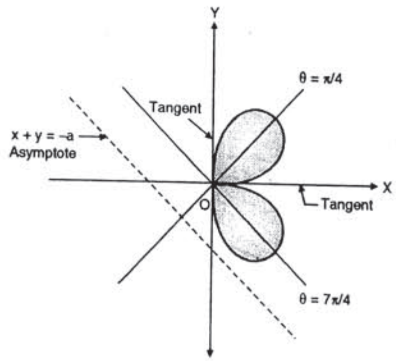


Fig. P.7.7.2

θ	0	π/4	π/2	3π/4	π	5π/4	3π/2	7π/4	2π
r	0	$\frac{4a}{\sqrt{2}}$	0	Negative	Negative	Negative	0	$\frac{4a}{\sqrt{2}}$	0

Table value shows that curves lies in first and fourth quadrant.

Exercise 7.4

(a) Trace the following curve

1 $x^4 + y^4 = 2a^2xy$

2 $x^4 + y^4 = a^2(x^2 - y^2)$

3 $x^5 + y^5 = 5a^2x^2y^2$

4 $y^3 = a^2x - x^3$

5 $x^3 + y^3 = 3ax^2$

Ans. :

Example 1

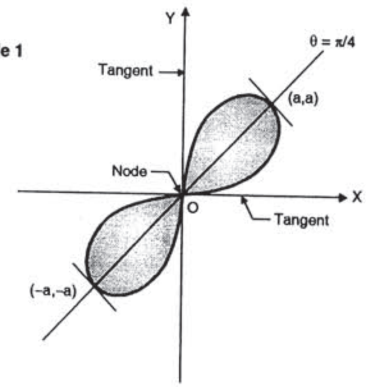


Fig. 1

Example 2

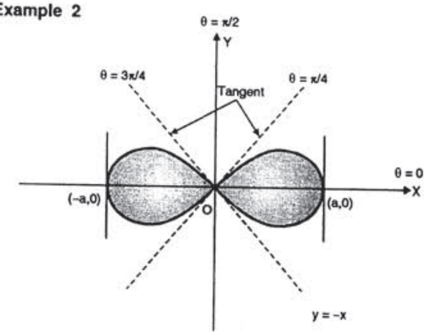


Fig. 2

Example 3

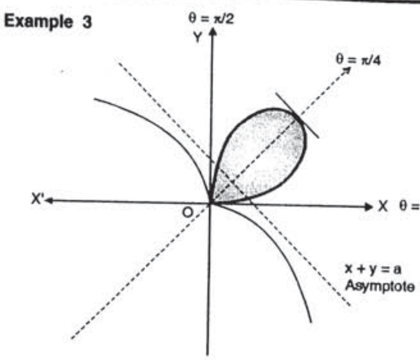


Fig. 3

Example 4

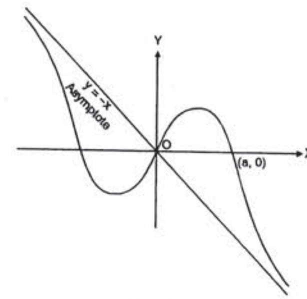


Fig. 4

Example 5

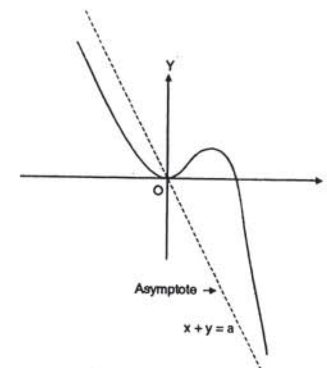


Fig. 5

7.7.1 Spirals

Example 7.7.3

Trace the curve : $r\theta = a, a > 0$.

Solution :

Step I : This curve is called hyperbolic spiral :

$r\theta = a, \quad a > 0$

Step II : Symmetry : Symmetry about Y-axis

(since no change in equation by putting $r = -r$ and $\theta = -\theta$)

Step III $\quad \tan\phi = r\frac{d\theta}{dr} = r\frac{1}{\left(\frac{dr}{d\theta}\right)} = \frac{r}{\left(\frac{-a}{\theta^2}\right)} = \frac{-r\theta^2}{a}$

$= -\frac{a\theta}{\alpha} = -\theta$

$\therefore \tan\theta = 0$ if $\theta = 0$

Step IV : As θ increasing; r goes on decreasing from ∞ to 0 and $\theta \to 0, r \to \infty$

Rough sketch is as follows Fig. P. 7.7.3.

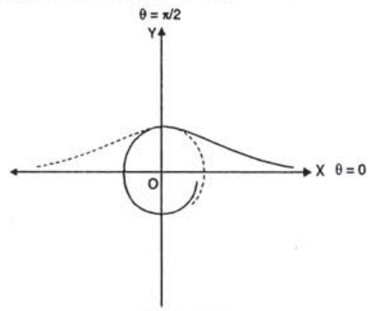


Fig. P. 7.7.3

Example 7.7.4

Trace the curve : $r = a\theta$; $a > 0$.

Solution :

Step I : This curve is called spiral of Archimedes :

$r = a\theta ; \quad a > 0$

Step II : Symmetry : Symmetrical about Y-axis.

($r = -r$ and $\theta = -\theta$ equation remain unchanged)

Step III : Pole : As $\theta = 0, r = 0$

Hence, curve passes through pole.

Step IV : Region : As $\theta \to \infty; r \to \infty$

4. $\quad \tan\phi = r\frac{d\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{r}{(a)} = \frac{a\theta}{a} = \theta$

As $\theta = 0$; $\tan\phi = 0$

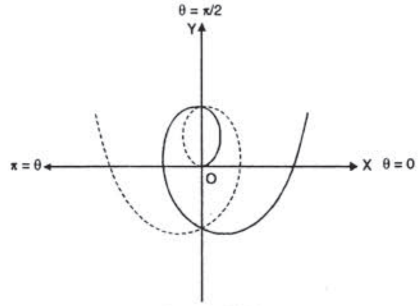


Fig. P. 7.7.4

Hence curve have tangent parallel to radius vector at $\theta = 0$. (Fig. P. 7.7.4)

Example 7.7.5

Trace the curve : $r^2\theta = a^2$.

Solution :

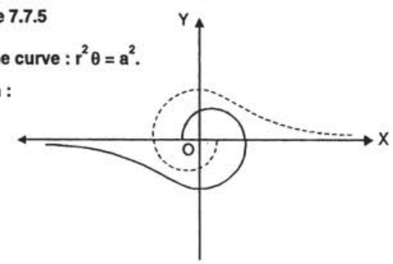


Fig. P. 7.7.5

Example 7.7.6

race the curve $r^2\theta = a^2$.

Solution :

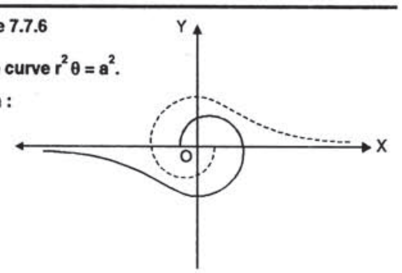


Fig. P. 7.7.6

Chapter Ends

□□□

Multiple Choice Questions (MCQ)

UNIT 4

Reduction Formulae and Curve Tracing

- Short Questions and Answers
- Fill in the Blanks
- Multiple Choice Questions

UNIT IV

Reduction Formulae and Curve Tracing

Multiple Choice Questions for Online Exam

Reduction Formulae

Q. 1 For any positive integer $n > 0$, to evaluate the integral $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$, I_n can be reduced to ____.

(a) $\frac{n}{n-1} I_{n-1}$ (b) $\frac{n-2}{n-1} I_{n-1}$

(c) $\frac{n-1}{n} I_{n-2}$ (d) $\frac{n-2}{n-1} I_{n-2}$ **Ans. : (c)**

Q. 2 For any positive integer $n > 0$, to evaluate the integral $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$, I_n can be reduced to

(a) $\frac{n}{n-1} I_{n-1}$ (b) $\frac{n-2}{n-1} I_{n-1}$

(c) $\frac{n-1}{n} I_{n-2}$ (d) $\frac{n-2}{n-1} I_{n-2}$ **Ans. : (c)**

Q. 3 For any even positive integer $n > 0$, the value of the integral $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$, is calculated by using

(a) $\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \cdot 1$

(b) $\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

(c) $\frac{n-2}{n} \cdot \frac{n-4}{n-2} \cdot \frac{n-6}{n-4} \cdots \frac{4}{5} \frac{2}{3} \frac{\pi}{2}$

(d) $\frac{n-2}{n-1} \cdot \frac{n-4}{n-3} \cdot \frac{n-6}{n-5} \cdots \frac{3}{4} \frac{1}{2} 1$ **Ans. : (b)**

Q. 4 For any odd positive integer $n > 0$, the value of the integral $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$, is calculated by using

(a) $\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \cdot 1$

(b) $\frac{n-2}{n-1} \cdot \frac{n-4}{n-3} \cdot \frac{n-6}{n-5} \cdots \frac{3}{4} \frac{1}{2} 1$

(c) $\frac{n-2}{n} \cdot \frac{n-4}{n-2} \cdot \frac{n-6}{n-4} \cdots \frac{4}{5} \frac{2}{3} \frac{\pi}{2}$

(d) $\frac{n-2}{n-1} \cdot \frac{n-4}{n-3} \cdot \frac{n-6}{n-5} \cdots \frac{3}{4} \frac{1}{2} 1$ **Ans. : (a)**

Q. 5 For any odd positive integer $n > 0$ the value of the integral $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$, is calculated by using

(a) $\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \cdot 1$

(b) $\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

(c) $\frac{n-2}{n} \cdot \frac{n-4}{n-2} \cdot \frac{n-6}{n-4} \cdots \frac{4}{5} \frac{2}{3} \frac{\pi}{2}$

(d) $\frac{n-2}{n-1} \cdot \frac{n-4}{n-3} \cdot \frac{n-6}{n-5} \cdots \frac{3}{4} \frac{1}{2} 1$ **Ans. : (a)**

Q. 6 For any even positive integer $n > 0$, the value of the integral $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$, is calculated by using

(a) $\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \cdot 1$

(b) $\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

(c) $\frac{n-2}{n} \cdot \frac{n-4}{n-2} \cdot \frac{n-6}{n-4} \cdots \frac{4}{5} \frac{2}{3} \frac{\pi}{2}$

(d) $\frac{n-2}{-n-1} \cdot \frac{n-4}{n-3} \cdot \frac{n-6}{n-5} \cdots \frac{3}{4} \frac{1}{2} 1$ **Ans. : (b)**

Q. 7 If m and n are any positive integers and m, n > 0, the value of the integral

$$I_n = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx, \text{ is calculated by using}$$

(a) $\frac{[(m-1)(m-3)(m-5)\ldots\times 2 \text{ or } 1]\,[(n-1)(n-3)(n-5)\ldots\times 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1} \times k$

Where $k = \pi/2$ if m and n both are even = 1 for all other values of m and n

(b) $\frac{[(m)(m-1)(m-3)\ldots\times 2 \text{ or } 1]\,[(n)(n-1)(n-3)\ldots\times 2 \text{ or } 1]}{(m+n)(m+n-1)(m+n-2)\ldots 2 \text{ or } 1} \times k$

Where $k = \pi/2$ if m and n both are even = 1 for all other values of m and n

(c) $\frac{[(m-2)(m-4)(m-6)\ldots\times 2 \text{ or } 1]\,[(n-2)(n-4)(n-6)\ldots\times 2 \text{ or } 1]}{(m+n-1)(m+n-1)(m+n-3)\ldots 2 \text{ or } 1} \times k$

Where $k = \pi/2$ if m and n both are even

= 1 for all other values of m and n

(d) $\frac{[(m+n)(m+n-2)(m+n+4)\ldots 2 \text{ or }]}{[(m-1)(m-3)(m-5)\ldots 2 \text{ or}][(n-1)(n-3)(n-5)\ldots 2 \text{ or } 1]} \times k$

Where $k = \pi/2$ if m and n both are even

= 1 for all other values of m and n

$$k = \begin{cases} \frac{\pi}{2}, & \text{if both m and n are even} \\ 1, & \text{if otherwise} \end{cases}$$

Ans. : (a)

Q. 8 The value of integral $I_n = \int_0^{\frac{\pi}{2}} \sin^5 x \, dx$ is equal to

(a) $\frac{2}{3}$ (b) $\frac{8}{15}$ (c) $\frac{4\pi}{5}$ (d) $\frac{4}{5}$ **Ans. : (b)**

Q. 9 The value of integral $I_n = \int_0^{\frac{\pi}{2}} \sin^4 x \, dx$ is equal to

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{3}{8}$ (d) $\frac{3\pi}{16}$ **Ans. : (d)**

Q. 10 The value of integral $I_n = \int_0^{\frac{\pi}{2}} \cos^3 x \, dx$ is equal to

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{3}{8}$ (d) $\frac{3\pi}{16}$ **Ans. : (a)**

Q. 11 The value of integral $I_n = \int_0^{\frac{\pi}{2}} \cos^8 x \, dx$ is equal to

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{35\pi}{256}$ (d) $\frac{35}{128}$ **Ans. : (c)**

Q. 12 The value of integral $I_n = \int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^4 \theta \, d\theta$ is equal to

(a) $\frac{5}{42}$ (b) $\frac{3}{4}$ (c) $\frac{5\pi}{6}$ (d) $\frac{2}{35}$ **Ans. : (d)**

Q. 13 The value of integral $I_n = \int_0^{\frac{\pi}{6}} \sin^5 3\theta \, d\theta$ is equal to

(a) $\frac{8}{45}$ (b) $\frac{4}{15}$ (c) $\frac{7\pi}{16}$ (d) $\frac{2}{15}$ **Ans. : (a)**

Q. 14 The value of integral $I_n = \int_0^{\frac{\pi}{4}} \sin^7 2\theta \, d\theta$ is equal to

(a) $\frac{35}{8}$ (b) $\frac{8}{35}$ (c) $\frac{16}{35}$ (d) $\frac{8}{15}$ **Ans. : (b)**

Q. 15 The value of integral $I_n = \int_0^{\frac{\pi}{6}} \cos^6 3\theta \, d\theta$ is equal to

(a) $\frac{5\pi}{96}$ (b) $\frac{7}{48}$ (c) $\frac{5\pi}{32}$ (d) $\frac{\pi}{6}$ **Ans. : (a)**

Q. 16 The value of integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx$ is equal to

(a) $\frac{8}{45}$ (b) 0 (c) $\frac{32}{35}$ (d) $\frac{8\pi}{35}$ **Ans. : (b)**

Q. 17 The value of integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x \cos^2 x \, dx$ is equal to

(a) $\frac{1}{16}$ (b) 0 (c) $\frac{\pi}{16}$ (d) $\frac{\pi}{24}$ **Ans. : (c)**

Q. 18 The value of integral $\int_0^{2\pi} \sin^6 x \cos^4 x \, dx$ is equal to

(a) $\frac{3}{8}$ (b) $\frac{15}{1280}$ (c) $\frac{\pi}{16}$ (d) $\frac{3\pi}{128}$ **Ans. : (d)**

Q. 19 The value of integral $\int_0^{2\pi} \sin^4 x \cos^3 x \, dx$ is equal to

(a) $\frac{3}{8}$ (b) $\frac{8}{35}$ (c) 0 (d) $\frac{4}{35}$ **Ans. : (c)**

Q. 20 The value of integral $\int_{-\pi}^{\pi} \sin^2 x \cos^3 x \, dx$ is equal to

(a) 0 (b) $\frac{8}{15}$ (c) $\frac{6}{11}$ (d) $\frac{3}{8}$ **Ans. : (a)**

Q. 21 The value of integral $\int_0^{\pi} \cos^3 x \, dx$ is equal to

(a) 0 (b) $\frac{2}{5}$ (c) $\frac{\pi}{5}$ (d) $\frac{2}{3}$ **Ans. : (a)**

Q. 22 The value of integral $\int_0^{\pi} \cos^6 x \, dx$ is equal to

(a) $\frac{5\pi}{8}$ (b) 0 (c) $\frac{5\pi}{16}$ (d) $\frac{5\pi}{32}$ **Ans. : (c)**

Q. 23 The value of integral $\int_0^{\pi} \sin^4 x \, dx$ is equal to

(a) 0 (b) $\frac{4}{5}$ (c) $\frac{3\pi}{8}$ (d) $\frac{3}{8}$ **Ans. : (c)**

Q. 24 The value of integral $\int_0^{\pi} \sin^3 x \, dx$ is equal to

(a) $\frac{2}{3}$ (b) $\frac{4}{3}$ (c) 0 (d) $\frac{8}{3}$ **Ans. : (b)**

Q. 25 The value of integral $\int_0^{2\pi} \sin^8 x \, dx$ is equal to

(a) $\frac{35\pi}{64}$ (b) $\frac{35\pi}{128}$ (c) 0 (d) $\frac{35\pi}{64}$ **Ans. : (d)**

Q. 26 If $U_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$, and $U_n = \frac{1}{n-1} - U_{n-2}$. Then U_4 is equal to

(a) $-\frac{2}{3} + \frac{\pi}{4}$ (b) $-\frac{13}{15} - \frac{\pi}{4}$

(c) $\frac{2}{3} + \frac{\pi}{8}$ (d) $\frac{13}{15} + \frac{\pi}{4}$ **Ans. : (a)**

Q. 27 If $I_n = \int_0^{\frac{\pi}{2}} x \cos^n x \, dx$, and $I_n = -\frac{1}{n^2} + \frac{n-1}{n} I_{n-2}$

Then I_3 is equal to

(a) $-\frac{2}{3} + \frac{\pi}{3}$ (b) $-\frac{7}{9} + \frac{\pi}{3}$

(c) $\frac{11}{15} + \frac{\pi}{2}$ (d) $\frac{13}{9} + \frac{\pi}{2}$ **Ans. : (b)**

Q. 28 If $I_n = \int_0^{\frac{\pi}{4}} \sin^{2n} x \, dx$, and $I_n = -\frac{1}{n\,2^{n+1}} + \frac{2n-1}{2n} I_{n-1}$

Then I_2 is equal to

(a) $-\frac{2}{3} + \frac{\pi}{3}$ (b) $-\frac{3}{8} + \frac{5\pi}{8}$

(c) $\frac{1}{4} + \frac{5\pi}{16}$ (d) $-\frac{1}{4} + \frac{3\pi}{32}$ **Ans. : (d)**

Q. 29 If $I_n = \int_0^{\frac{\pi}{2}} \theta \sin^n \theta \, d\theta$ and $I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n^2}$. Then value of I_5.

(a) $-\frac{149}{225} + \frac{\pi}{4}$ (b) $\frac{64}{11} + \frac{\pi}{4}$

(c) $\frac{149}{225}$ (d) $\frac{64}{11}$ **Ans. : (c)**

Q. 30 If $I_n = \int_0^{\frac{\pi}{4}} \sec^n \theta \, d\theta$, and $I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$

Then $\int_0^{\frac{\pi}{4}} \sec^6 \theta \, d\theta$ is equal to

(a) $\frac{7}{30}$ (b) $\frac{28}{15}$ (c) $\frac{5}{16}$ (d) $\frac{14}{15}$ **Ans. : (b)**

Q. 31 If $U_n = \int x^n e^x$ then which of the following is true

(a) $U_n = x^n e^x - n\,U_{n-1}$

(b) $U_n = x^n e^x - n\,U_{n-2}$

(c) $U_n = x^n e^x - (n-1)\,U_{n-1}$

(d) $U_n = x^n e^x - (n-1)\,U_{n-2}$ **Ans. : (a)**

Curve Tracing

Q. 32 If the portion of the curve on both sides of point P lies above the tangent at P, then the curve is called

(a) Concave upward (convex downward)

(b) Concave downward (convex upwards)

(c) Inflexional

(d) None of the above. **Ans. : (a)**

Q. 33 If the portion of the curve on both sides of point P lies below the tangent at P, then the curve is called

(a) Concave upward (convex downward)

(b) Concave downward (convex upwards)

(c) Inflexional

(d) None of the above. **Ans. : (b)**

Q. 34 A point through which the number of branches of curve passes, that point is called

(a) Point of inflexion (b) Double point

(c) Multiple point (d) Conjugate point

Ans. : (c)

Q. 35 A double point through which curve passes, are real and the tangents at the point of intersection are real and different then that point is called

(a) Conjugate point (b) Cusp

(c) Conjugate point (d) Node **Ans. : (d)**

Q. 36 A double point through which curve passes are, real and the tangents at the point of intersection are real and coincide then that point is called

(a) Node (b) Cusp

(c) Conjugate point (d) Conjugate point

Ans. : (b)

Q. 37 If all powers of y are even in the given Cartesian equation then the curve is symmetric about

(a) X-axis. (b) Y-axis.

(c) Y = X line (d) Origin **Ans. : (a)**

Q. 38 If all powers of x are even in the given Cartesian equation then the curve is symmetric about

(a) X-axis. (b) Y-axis.

(c) Y=X line (d) Origin **Ans. : (b)**

Q. 39 If all powers of both x and y are even in the given Cartesian equation then the curve is symmetric about

(a) X-axis. (b) Y-axis.

(c) both X and Y-axis (d) Origin **Ans. : (c)**

Q. 40 By interchanging x and y i.e. put x = y and y = x, if the given Cartesian equation is remain unchanged then the given curve is symmetric about

(a) X-axis. (b) Y-axis.

(c) both X and Y-axis (d) Y = X line **Ans. : (d)**

Q. 41 By interchanging, x by – y and y by – x, if the given Cartesian equation of the curve remains unchanged then the curve is symmetric about

(a) y = – x line. (b) Y-axis.

(c) both X and Y-axis (d) y = x **Ans. : (a)**

Q. 42 By interchanging x by – x and y by – y, if the given Cartesian equation of the curve remains unchanged then the curve is symmetric about

(a) X-axis. (b) Y-axis.

(c) both X and Y-axis (d) Origin **Ans. : (d)**

Q. 43 The equation of X-axis is

(a) x = 0 (b) y = 0. (c) y = x (d) xy = c

Ans. : (b)

Q. 44 The equation of y-axis is

(a) x = 0 (b) y = 0. (c) y = x (d) xy = c **Ans. : (a)**

Q. 45 In Cartesian form, the equation of tangent at origin is obtained by equating the

(a) highest degree term/s in equation to zero

(b) lowest degree term/s in equation to zero

(c) co-efficient of lowest degree term/s in equation to zero.

(d) co-efficient of highest degree term/s in equation to zero. **Ans. : (b)**

Q. 46 In Cartesian form , if $\left(\frac{dy}{dx}\right)_{(a,\,b)} = 0$, then at (a, b), tangent to the curve is

(a) parallel to y = x line (b) parallel to y-axis

(c) parallel to x-axis. (d) parallel to x = – y

Ans. : (c)

Q. 47 In Cartesian form , if $\left(\frac{dy}{dx}\right)_{(a,\,b)} = \pm\infty$, then at (a, b), tangent to the curve is

(a) parallel to y = x line (b) parallel to y-axis

(c) parallel to x-axis. (d) parallel to x = – y

Ans. : (b)

Q. 48 In Cartesian form , the given curve attains its maximum or minimum values at (a, b), if

(a) $\left(\frac{dy}{dx}\right)_{(a,\,b)} > 0$ (b) $\left(\frac{dy}{dx}\right)_{(a,\,b)} = 0$

(c) $\left(\frac{dy}{dx}\right)_{(a,\,b)} < 0$ (d) $\left(\frac{dy}{dx}\right)_{(a,\,b)} = \pm\infty$,

Ans. : (b)

Q. 49 In Cartesian form , a curve of degree 'n' have

(a) n or less than n number of asymptotes

(b) Exactly n number of asymptote

(c) less than n number of asymptote

(d) greater than n number of asymptote **Ans. : (a)**

Q. 50 In Cartesian form ,the asymptote parallel to X-axis if exist is obtained by equating the

(a) co-efficient of lowest degree term/s of x to zero

(b) co-efficient of highest degree term/s of y to zero

(c) co-efficient of highest degree term/s of x to zero

(d) co-efficient of lowest degree term/s of y to zero

Ans. : (c)

Q. 51 In Cartesian form ,the asymptote parallel to Y-axis if exist is obtained by equating the

(a) co-efficient of lowest degree term/s of x to zero

(b) co-efficient of lowest degree term/s of y to zero

(c) co-efficient of highest degree term/s of x to zero

(d) co-efficient of highest degree term/s of y to zero

Ans. : (d)

Q. 52 For Cartesian curves, as $x \to \infty$ for $y \to a$ then, y = a is is an asymptote parallel to

(a) X-axis. (b) Y-axis

(c) y = x line (d) y = - x line **Ans. : (a)**

Q. 53 For Cartesian curves, as $y \to \infty$ for $x \to b$ then, x = b is is an asymptote parallel to

(a) X-axis. (b) Y-axis

(c) y = x line (d) y = - x line **Ans. : (b)**

Q. 54 Oblique asymptote is obtained when the curve is

(a) symmetrical about X-axis.

(b) symmetrical about Y-axis

(c) symmetrical about both X-axis and Y-axis

(d) not symmetrical about both X-axis and Y-axis

Ans. : (d)

Q. 55 In Cartesian form Oblique asymptote is obtained by putting y = mx + c in the given equation and then equate the

(a) coefficients of the two highest powers of x to zero

(b) co-efficient of highest degree term/s of y to zero

(c) co-efficient of highest degree term/s of x to zero

(d) co-efficient of lowest degree term/s of y to zero

Ans. : (a)

Q. 56 The circle

(a) have asymptote parallel to X-axis.

(b) have asymptote parallel to Y-axis

(c) have asymptote parallel to y = x line

(d) asymptote does not exist **Ans. : (d)**

Q. 57 In Cartesian form, if the coefficient of the highest degree term of x is constant or if it's all linear factors are imaginary then

(a) no asymptote parallel to X-axis.

(b) no asymptote parallel to Y-axis

(c) have asymptote parallel to y = x line

(d) asymptote does not exist **Ans. : (a)**

Q. 58 In Cartesian form, if the coefficient of the highest degree term of y is constant or if it's all linear factors are imaginary then

(a) no asymptote parallel to X-axis.

(b) no asymptote parallel to Y-axis

(c) have asymptote parallel to y = x line

(d) asymptote does not exist **Ans. : (b)**

Q. 59 The Cartesian parametric curve x = f (t) and y = g (t) is symmetric about X-axis if

(a) x is an even function of t and y is an odd function of t

(b) x is an odd function of t and y is an even function of t

(c) both x and y are an odd function of t

(d) both x and y are an even function of t **Ans. : (a)**

Q. 60 The Cartesian parametric curve x = f (t) and y = g (t) is symmetric about Y-axis if

(a) x is an even function of t and y is an odd function of t

(b) x is an odd function of t and y is an even function of t

(c) both x and y are an odd function of t

(d) both x and y are an even function of t **Ans. : (b)**

Q. 61 The Cartesian parametric curve x = f (t) and y = g (t) is symmetric about opposite quadrant if

(a) x is an even function of t and y is an odd function of t

(b) x is an odd function of t and y is an even function of t
(c) both x and y are an odd function of t
(d) both x and y are an even function of t **Ans. : (c)**

Q. 62 By putting $t = \pi - t$, the Cartesian parametric curve $x = f(t)$ and $y = g(t)$, is symmetric about Y-axis if
(a) x is an even function of t and y is an odd function of t
(b) x is an odd function of t and y is an even function of t
(c) both x and y are an odd function of t
(d) both x and y are an even function of t **Ans. : (b)**

Q. 63 In Cartesian parametric curve $x = f(t)$ and $y = g(t)$, if for some value of t, both x and y are zero at the same time, then always the curve
(a) does not passes through origin
(b) x is an even function of t and y is an odd function of t
(c) passes through origin.
(d) is symmetric about opposite quadrant **Ans. : (c)**

Q. 64 In Cartesian parametric curve $x = f(t)$ and $y = g(t)$, if $\left(\frac{dy}{dx}\right)_{t=a} = 0$; then at t = a, the curve have tangent.
(a) perpendicular to X-axis at t = a.
(b) perpendicular to Y-x line at t = a.
(c) parallel to Y-axis at t = a.
(d) parallel to X-axis at t = a. **Ans. : (d)**

Q. 65 In Cartesian parametric curve $x = f(t)$ and $y = g(t)$, if $\left(\frac{dy}{dx}\right)_{t=b} = \infty$; then at t = b, the curve have tangent.
(a) perpendicular to X-x line t = b.
(b) perpendicular to Y-axis t = b.
(c) parallel to Y-axis t = b.
(d) parallel to X-axis t = b. **Ans. : (c)**

Q. 66 If the polar equation of curve $r = f(\theta)$ remains unchanged by replacing θ by $-\theta$ then the curve is symmetric about
(a) initial line $\theta = 0$ (b) $\theta = \pi/4$
(c) $\theta = \pi/2$ (d) pole **Ans. : (a)**

Q. 67 If the polar equation of curve $r = f(\theta)$ remains unchanged by replacing r by – r (or powers of r are even) then the curve is symmetric about
(a) initial line $\theta = 0$ (b) pole
(c) $\theta = \pi/2$ (d) $\theta = \pi/4$ **Ans. : (b)**

Q. 68 If the polar equation of curve $r = f(\theta)$ remains unchanged by replacing r by – r and θ by $-\theta$ at the same time then the curve is symmetric about
(a) initial line $\theta = 0$ (b) pole
(c) $\theta = \pi/2$ (d) $\theta = \pi/4$ **Ans. : (c)**

Q. 69 If the polar equation of curve $r = f(\theta)$ remains unchanged by replacing θ by $\pi - \theta$ then the curve is symmetric about
(a) initial line $\theta = 0$ (b) pole
(c) $\theta = \pi/2$ (d) opposite quadrant
Ans. : (c)

Q. 70 If the polar equation of curve $r = f(\theta)$ remains unchanged by replacing r by – r and θ by $\pi - \theta$ then the curve is symmetric about
(a) initial line $\theta = 0$ (b) $\theta = \pi/4$
(c) $\theta = \pi/2$ (d) opposite quadrant **Ans. : (d)**

Q. 71 If the polar equation of curve $r = f(\theta)$ remains unchanged by replacing $\theta = \pi/2 - \theta$, then the curve is symmetric about
(a) initial line $\theta = 0$ (b) $\theta = \pi/4$
(c) $\theta = \pi/2$ (d) opposite quadrant
Ans. : (b)

Q. 72 In the polar equation of curve $r = f(\theta)$, pole will lie on the curve if for some value of θ
(a) $r = 0$ (b) $r > 0$ (c) $r < 0$ (d) none of the above
Ans. : (a)

Q. 73 In the polar equation of curve $r = f(\theta)$, equation of tangents at pole, if exist can be obtained by putting
(a) $\theta = \pi/4$ (b) $r = 0$
(c) $\theta = \pi/2$ (d) $r > 0$ **Ans. : (b)**

Q. 74 For the polar equation of curve $r = f(\theta)$, the angle ϕ between radius vector and tangent line is obtained by the formula
(a) $\tan\phi = \frac{r\,dr}{d\theta}$ (b) $\cot\phi = \frac{r\,d\theta}{dr}$
(c) $\tan\phi = \frac{r\,d\theta}{dr}$ (d) $\cos\phi = \frac{r\,d\theta}{dr}$ **Ans. : (c)**

Q. 75 In the polar equation of curve $r = f(\theta)$, if $\tan\phi = \frac{r\,d\theta}{dr} = 0$ for $\theta = \theta_1$ then at $\theta = \theta_1$ the curve have a tangent
(a) parallel to radius vector r.
(b) perpendicular to radius vector r..
(c) parallel to $\theta = \pi/2$.

(d) perpendicular to $\theta = \pi/2$. **Ans. : (a)**

Q. 76 In the polar equation of curve $r = f(\theta)$, if $\tan\phi = \frac{r\,d\theta}{dr} = \infty$ for $\theta = \theta_1$ then at $\theta = \theta_1$ the curve have a tangent
(a) parallel to radius vector r.
(b) perpendicular to radius vector r.
(c) parallel to $\theta = \pi/2$.
(d) perpendicular to $\theta = \pi/2$. **Ans. : (b)**

Q. 77 In the polar equation of curve $r = f(\theta)$, if $\frac{dr}{d\theta} > 0$, then
(a) r increases as θ increases
(b) r decreases as θ increasing
(c) the tangent is parallel to radius vector r.
(d) the tangent is perpendicular to radius vector r.
Ans. : (a)

Q. 78 In the polar equation of curve $r = f(\theta)$, if $\frac{dr}{d\theta} < 0$, then
(a) r increases as θ increases
(b) r decreases as θ increasing
(c) the tangent is parallel to radius vector r.
(d) the tangent is perpendicular to radius vector r.
Ans. : (b)

Q. 79 For the rose curve $r = a \sin n\theta$ or $r = a \cos n\theta$ if n is an odd then the curve consists of
(a) (n - 1) similar loops (b) (n + 1) similar loops
(c) 2n similar loops . (d) n similar loops
Ans. : (d)

Q. 80 For the rose curve $r = a \sin n\theta$ or $r = a \cos n\theta$ if n is an even then the curve consists of
(a) (n - 1) similar loops (b) (n + 1) similar loops
(c) 2n similar loops (d) n similar loops
Ans. : (c)

Q. 81 For n = 1 ,the rose curve $r = a \sin n\theta$ or $r = a \cos n\theta$ becomes
(a) Ellipse (b) Parabola (c) circle. (d) Hyperbola
Ans. : (c)

Q. 82 For the rose curve $r = a \sin n\theta$ first loop is drawn along
(a) $\theta = 0$ (b) $\theta = \pi/4$ (c) $\theta = \pi/2n$ (d) $\theta = \pi$ **Ans. : (c)**

Q. 83 For the rose curve $r = a \cos n\theta$ first loop is drawn along
(a) $\theta = 0$ (b) $\theta = \pi/4$ (c) $\theta = \pi/2n$ (d) $\theta = \pi$ **Ans. : (a)**

Q. 84 The curve represented by the equation $xy^2 = 4a^2(a - x)$ is symmetric about
(a) X-axis. (b) Y-axis.
(c) both X and Y-axis (d) Y = X line **Ans. : (a)**

Q. 85 The curve represented by the equation $xy^2 = 4a^2(a - x)$, at the point (a , 0) , curve have a tangent
(a) parallel to X-axis.
(b) parallel to both X and Y-axis.
(c) parallel to Y-axis.
(d) parallel to Y = X line. **Ans. : (b)**

Q. 86 The curve represented by the equation $xy^2 = 4a^2(a - x)$ have an asymptote
(a) parallel to X-axis. (b) parallel to Y = X line.
(c) Y-axis. (d) Does not exist.
Ans. : (c)

Q. 87 The asymptote of the curve represented by the equation $xy^2 = 4a^2(a - x)$, is
(a) Does not exist. (b) x = 0.
(c) y = x . (d) y = 0 **Ans. : (b)**

Q. 88 The curve represented by the equation $xy^2 = 4a^2(a - x)$ does exists in the range
(a) $0 \le x \le a$. (b) $x > a$. (c) $x < 0$. (d) $y = 0$
Ans. : (a)

Q. 89 The curve represented by the equation $xy^2 = 4a^2(a - x)$ is
(a) symmetric about X-axis and passes through origin
(b) symmetric about X-axis and does not passes through origin
(c) symmetric about Y-axis and passes through origin
(d) symmetric about Y-axis and does not passes through origin **Ans. : (b)**

Q. 90 The curve represented by the equation $y^2(2a - x) = x^3$ is symmetric about
(a) X-axis. (b) Y-axis.
(c) both X and Y-axis (d) Y = X line **Ans. : (a)**

Q. 91 The curve represented by the equation $y^2(2a-x) = x^3$, at the point (0 , 0) , curve have a tangent
(a) parallel to X-axis.

(b) parallel to both X and Y-axis.
(c) parallel to Y-axis.
(d) parallel to Y = X line. **Ans. : (a)**

Q. 92 The curve represented by the equation $y^2(2a-x)=x^3$ have an asymptote
(a) parallel to X-axis (b) parallel to Y = X line.
(c) Y-axis. (d) Does not exist.
Ans. : (c)

Q. 93 The asymptote of the curve represented by the equation $y^2(2a-x)=x^3$, is
(a) Does not exist. (b) $x = 2a$
(c) $y = x$. (d) $y = 0$ **Ans. : (b)**

Q. 94 The curve represented by the equation $y^2(2a-x)=x^3$ does exists in the range
(a) $0 \le x \le 2a$ (b) $x > 2a$.
(c) $x < -2a$. (d) $y = 0$ **Ans. : (a)**

Q. 95 The curve represented by the equation $y^2(2a-x)=x^3$ is
(a) symmetric about X-axis and passes through origin
(b) symmetric about X-axis and does not passes through origin
(c) symmetric about Y-axis and passes through origin
(d) symmetric about Y-axis and does not passes through origin **Ans. : (a)**

Q. 96 At origin (the point of intersection) the curve represented by the equation $y^2(2a-x)=x^3$ have
(a) node (b) cusp
(c) asymptote (d) conjugate point **Ans. : (b)**

Q. 97 The tangent of the curve $xy^2 = a(x^2 - a^2)$ with $\frac{dy}{dx} = \left[\frac{ax^2 + a^3}{x^3 \cdot 2y}\right]$ at (a, 0) is
(a) parallel to X-axis. (b) parallel to both X and Y-axis.
(c) parallel to Y-axis. (d) parallel to Y = X line.
Ans. : (c)

Q. 98 The curve represented by the equation $a^2y^2 = x^2(a^2 - x^2)$ is symmetric about
(a) X-axis. (b) Y-axis.
(c) both X and Y-axis (d) Y = X line **Ans. : (c)**

Q. 99 The equation of tangent represented by curve $a^2y^2 = x^2(a^2 - x^2)$, at the point (0, 0), is
(a) $y = 0$. (b) $x = 0$. (c) $x = a$. (d) $y = \pm x$
Ans. : (d)

Q. 100 The asymptote parallel to X-axis of the curve $xy^2 = 4a^2(a - x)$, is
(a) Does not exist. (b) $x = 2a$
(c) $y = x$ (d) $y = 0$ **Ans. : (a)**

Q. 101 The curve represented by the equation $a^2y^2 = x^2(a^2 - x^2)$ does exists in the range
(a) $0 \le x \le 2a$ (b) $x > a$.
(c) $-a \le x \le a$. (d) $x > a$ and $x < -a$ **Ans. : (c)**

Q. 102 The curve represented by the equation $a^2y^2 = x^2(a^2 - x^2)$ is
(a) symmetric about X-axis and passes through origin
(b) symmetric about X-axis and does not passes through origin
(c) symmetric about both X and Y-axis and passes through origin
(d) symmetric about Y-axis and does not passes through origin **Ans. : (c)**

Q. 103 The tangent of the curve $a^2y^2 = x^2(a^2 - x^2)$ with $\frac{dy}{dx} = \frac{-x^3 + x(a^2 - x^2)}{a^2 y}$ at (a, 0) is
(a) parallel to X-axis.
(b) parallel to both X and Y-axis.
(c) parallel to Y-axis
(d) parallel to Y = X line. **Ans. : (c)**

Q. 104 The curve represented by the equation $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is symmetric about
(a) X-axis. (b) Y-axis.
(c) both X and Y-axis (d) Y = X line **Ans. : (d)**

Q. 105 The curve represented by the equation $\sqrt{x} + \sqrt{y} = \sqrt{a}$, at the point (0, 0), curve have
(a) a tangent parallel to X-axis
(b) a tangent parallel to both X and Y-axis.
(c) a tangent parallel to Y-axis
(d) no tangent at origin. **Ans. : (d)**

Q. 106 The curve represented by the equation $\sqrt{x} + \sqrt{y} = \sqrt{a}$ have an asymptote
(a) Does not exist. (b) parallel to Y = X line.
(c) Y-axis. (d) parallel to X-axis.
Ans. : (a)

Q. 107 The curve represented by the equation $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is
(a) symmetric about X-axis and passes through origin

(b) symmetric about the line Y = X and does not passes through origin
(c) symmetric about Y-axis and passes through origin
(d) symmetric about Y-axis and does not passes through origin **Ans. : (b)**

Q. 108 The tangent of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ with $\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$ at (a, 0) is
(a) parallel to X-axis.
(b) parallel to both X and Y-axis.
(c) parallel to Y-axis.
(d) parallel to Y = X line. **Ans. : (a)**

Q. 109 The tangent of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ with $\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$ at (0, a) is
(a) parallel to X-axis.
(b) parallel to both X and Y-axis.
(c) parallel to Y-axis.
(d) parallel to Y = X line. **Ans. : (c)**

Q. 110 The curve represented by the equation $y^2 = (x-1)(x-2)(x-3)$ is symmetric about
(a) X-axis. (b) Y-axis.
(c) both X and Y-axis (d) Y = X line **Ans. : (a)**

Q. 111 The curve represented by the equation $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ is symmetric about
(a) X-axis. (b) Y-axis.
(c) both X and Y-axis (d) Y = X line **Ans. : (c)**

Q. 112 The curve represented by the equation $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ is
(a) symmetric about both Y-axis and passes through origin
(b) symmetric about X and Y-axis and does not passes through origin
(c) symmetric about Y-axis and passes through origin
(d) symmetric about Y-axis and does not passes through origin **Ans. : (b)**

Q. 113 The tangent of the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ with $\frac{dy}{dx} = -\frac{b}{a}\tan t$ at $t = 0, \pi, 2\pi$ is
(a) parallel to X-axis (b) parallel to both X and Y-axis.
(c) parallel to Y-axis (d) parallel to Y = X line.
Ans. : (a)

Q. 114 The tangent of the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ with $\frac{dy}{dx} = -\frac{b}{a}\tan t$ at $t = \pi/2, 3\pi/2$ is
(a) parallel to X-axis (b) parallel to both X and Y-axis.
(c) parallel to Y-axis (d) parallel to Y = X line.
Ans. : (c)

Q. 115 In the I[st] quadrant, the curve represented by the equation $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$, when t varies from 0 to $\pi/2$ then
(a) x goes on increasing from 0 to a only.
(b) x goes on decreases from a to 0 and y is constant.
(c) x is constant and y goes on increasing from 0 to b.
(d) x goes on decreases from a to 0 and y goes on increasing from 0 to b. **Ans. : (d)**

Q. 116 At point $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$ (the point of intersection) the tangents of the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ are different and coincides to each others, then at this point the curve have
(a) node (b) cusp
(c) asymptote (d) conjugate point **Ans. : (b)**

Q. 117 The curve represented by the equation $x = a[\cos t + \log(\tan^2 t/2)]$; $y = a \sin t$ is symmetric about
(a) X-axis. (b) Y-axis.
(c) both X and Y-axis (d) Y = X line **Ans. : (c)**

Q. 118 The curve represented by the equation $x = a[\cos t + \log(\tan t/2)]$; $y = a \sin t$ is
(a) symmetric about X-axis and passes through origin
(b) symmetric about both-axis and does not passes through origin
(c) symmetric about Y-axis and passes through origin
(d) symmetric about Y-axis and does not passes through origin **Ans. : (b)**

Q. 119 The tangent of the curve $x = a[\cos t + \log(\tan t/2)]$; $y = a \sin t$ with $\frac{dy}{dx} = \tan t$ at $t = 0, \pi, 2\pi$ is
(a) parallel to X-axis. (b) parallel to both X and Y-axis.
(c) parallel to Y-axis. (d) parallel to Y = X line.
Ans. : (a)

Q. 120 The tangent of the curve $x = a[\cos t + \log(\tan t/2)]$; $y = a \sin t$ with $\frac{dy}{dx} = \tan t$ at $t = \pi/3, 3\pi/2$ is

(a) parallel to X-axis. (b) parallel to both X and Y-axis.
(c) parallel to Y-axis. (d) parallel to Y = X line.

Ans. : (c)

Q. 121 The curve represented by the equation $x = t^2$; $y = t - \frac{t^3}{3}$ is symmetric about

(a) X-axis. (b) Y-axis.
(c) both X and Y-axis (d) Y = X line **Ans. : (a)**

Q. 122 The curve represented by the equation $x = t^2$; $y = t - \frac{t^3}{3}$ is

(a) symmetric about X-axis and passes through origin
(b) symmetric about X-axis and does not passes through origin
(c) symmetric about Y-axis and passes through origin
(d) symmetric about Y-axis and does not passes through origin **Ans. : (a)**

Q. 123 The tangent of the curve $x = t^2$; $y = t - \frac{t^3}{3}$ with $\frac{dy}{dx} = \frac{1-t^2}{2t}$ at $t = 0$ is

(a) parallel to X-axis.
(b) parallel to both X and Y-axis.
(c) parallel to Y-axis.
(d) parallel to Y = X line. **Ans. : (c)**

Q. 124 The tangent of the curve $x = t^2$; $y = t - \frac{t^3}{3}$ with $\frac{dy}{dx} = 0$ at $t = 1$ is

(a) parallel to X-axis (b) parallel to both X and Y-axis.
(c) parallel to Y-axis (d) parallel to Y = X line.

Ans. : (a)

Q. 125 The curve represented by the equation $x = a(t + \sin t)$; $y = a(1 + \cos t)$ is symmetric about

(a) X-axis. (b) Y-axis.
(c) both X and Y-axis (d) Y = X line **Ans. : (b)**

Q. 126 The curve represented by the equation $x = t^2$; $y = t - \frac{t^3}{3}$ is

(a) passes through origin
(b) does not passes through origin
(c) symmetric about Y-axis and passes through origin
(d) symmetric about Y-axis and does not passes through origin **Ans. : (a)**

Q. 127 The curve represented by the equation $x = a(t + \sin t)$; $y = a(1 - \cos t)$ is symmetric about

(a) X-axis. (b) Y-axis.
(c) both X and Y-axis (d) Y = X line **Ans. : (b)**

Q. 128 The curve represented by the equation $x^3 + y^3 = 3axy$ is symmetric about

(a) X-axis. (b) Y-axis.
(c) both X and Y-axis (d) Y = X line **Ans. : (d)**

Q. 129 The tangents of the curve $x^3 + y^3 = 3axy$ at origin

(a) parallel to X-axis only .
(b) parallel to both X and Y-axis.
(c) parallel to Y-axis only.
(d) parallel to Y = X line. **Ans. : (b)**

Q. 130 The curve represented by the equation $x^3 + y^3 = 3axy$ have an asymptote

(a) parallel to X-axis. (b) parallel to Y = X line.
(c) Y-axis. (d) oblique asymptote.

Ans. : (d)

Q. 131 The equation of asymptote of the curve $x^3 + y^3 = 3axy$ having $m = 1$ and $c = -a$ is

(a) $y = x + a$. (b) $y = -a$.
(c) $y = x - a$. (d) $x + a = 0$ **Ans. : (d)**

Q. 132 The curve represented by the equation $x^5 + y^5 - 5a^2x^2y = 0$ is symmetric about

(a) Opposite quadrant. (b) Y-axis.
(c) both X and Y-axis (d) Y = X line **Ans. : (a)**

Q. 133 The curve represented by the equation $x^5 + y^5 - 5a^2x^2y = 0$ is

(a) symmetric about X-axis and passes through origin
(b) symmetric in opposite quadrant and does not passes through origin
(c) symmetric about in opposite quadrant. and passes through origin
(d) symmetric about Y-axis and does not passes through origin **Ans. : (c)**

Q. 134 At origin (the point of intersection) the tangents of the curve $x^5 + y^5 - 5a^2x^2y = 0$ are real and different from each others ,then at this point the curve have

(a) node (b) cusp
(c) asymptote (d) conjugate point **Ans. : (a)**

Q. 135 The equation of asymptote of the curve $x^5 + y^5 - 5a^2x^2y = 0$ having $m = -1$ and $c = 0$ is

(a) $y = -1$ (b) $y = -x$.
(c) $y = x$ (d) $x = 0$ **Ans. : (b)**

Q. 136 The curve represented by the equation $y = c\cosh\frac{x}{c}$ is

(a) symmetric about X-axis and passes through origin
(b) symmetric in opposite quadrant and does not passes through origin
(c) symmetric about in opposite quadrant. and passes through origin
(d) symmetric about Y-axis and does not passes through origin **Ans. : (d)**

Q. 137 The curve represented by the equation $r = a\cos 2\theta$ is

(a) symmetric about the initial line ($\theta = 0$) and the line $\theta = \pi/2$
(b) symmetric about the initial line ($\theta = 0$) only
(c) symmetric about the line $\theta = \pi/2$ only
(d) symmetric in opposite quadrant **Ans. : (a)**

Q. 138 The curve represented by the equation $r = a\cos 2\theta$ have a tangent at pole, at

(a) $\theta = 0$ and the line $\theta = \pi/2$ (b) $\theta = \pi/4$
(c) the line $\theta = \pi/2$ only (d) $\theta = \pi/6$

Ans. : (b)

Q. 139 If $\tan\phi = \frac{-1}{2}\cot\theta$ for the curve $r = a\cos 2\theta$ then the tangents are perpendicular to radius vector at

(a) $\pi/3$ (b) $\theta = \pi/4$ (c) $\theta = 0, \pi/2$ (d) $\pi/6$

Ans. : (c)

Q. 140 The curve represented by the equation $r = a\cos 2\theta$ have

(a) 2 loops (b) one loop
(c) 5 loops (d) 4 loops **Ans. : (d)**

Q. 141 The curve represented by the equation $r = 2\sin 3\theta$ is

(a) symmetric about the initial line ($\theta = 0$)
(b) symmetric about the line $\theta = \pi/2$
(c) symmetric about both the initial line and the line $\theta = \pi/2$
(d) symmetric in opposite quadrant **Ans. : (b)**

Q. 142 The curve represented by the equation $r = 2\sin 3\theta$ have a tangent at origin at

(a) $\theta = \pi/2$ (b) $\theta = \pi/4$
(c) the line $\theta = \pi/2$ only (d) $0, \pi/3, 2\pi/3, \pi$

Ans. : (b)

Q. 143 If $\tan\phi = \frac{1}{3}\tan 3\theta$ of the curve $r = 2\sin 3\theta$, then the tangents are perpendicular to radius vector at

(a) $\pi/3$ (b) $\theta = \pi/4$ (c) $\theta = \pi/6$, (d) $\theta = 0$

Ans. : (c)

Q. 144 If $\tan\phi = \frac{1}{3}\tan 3\theta$ of the curve $r = 2\sin 3\theta$, then the tangents are parallel to radius vector at

(a) $\pi/2$ (b) $\theta = 0$ (c) $\theta = \pi/6$, (d) $\theta = \pi/4$

Ans. : (b)

Q. 145 The curve represented by the equation $r = 2\sin 3\theta$ have

(a) 3 loops (b) one loop (c) 5 loops (d) 4 loops

Ans. : (d)

Q. 146 The curve represented by the equation $r = a\cos 3\theta$ is

(a) symmetric about the initial line ($\theta = 0$) and passes through pole
(b) symmetric about the line $\theta = \pi/2$ and passes through pole
(c) symmetric about the initial line ($\theta = 0$) and does not passes through pole
(d) symmetric about the line $\theta = \pi/2$ and does not passes through pole **Ans. : (a)**

Q. 147 The curve represented by the equation $r^2 = a^2\cos 2\theta$ is

(a) symmetric about the initial line ($\theta = 0$) and the line $\theta = \pi/2$, does not passes through pole
(b) symmetric about the initial line ($\theta = 0$) and the line $\theta = \pi/2$, passes through pole
(c) symmetric in opposite quadrant and passes through pole

(d) symmetric in opposite quadrant **Ans. : (b)**

Q. 148 The curve represented by the equation

$r = a + b\cos\theta \;;\; a > b$ is

(a) symmetric about the initial line ($\theta = 0$) and passes through pole

(b) symmetric about the line $\theta = \pi/2$ and passes through pole

(c) symmetric in opposite quadrant and passes through pole

(d) symmetric about the initial line ($\theta = 0$) and does not passes through pole **Ans. : (d)**

Q. 149 The curve represented by the equation

$r = a + b\cos\theta \;;\; a < b$ is

(a) symmetric about the initial line ($\theta = 0$) and passes through pole

(b) symmetric about the line $\theta = \pi/2$ and passes through pole

(c) symmetric in opposite quadrant and passes through pole

(d) symmetric about the initial line ($\theta = 0$) and does not passes through pole **Ans. : (a)**

Q. 150 The curve represented by the equation

$r = \frac{a}{2}(1 + \cos\theta)$ is

(a) symmetric about the initial line ($\theta = 0$) and passes through pole

(b) symmetric about the line $\theta = \pi/2$ and passes through pole

(c) symmetric in opposite quadrant and passes through pole

(d) symmetric about the initial line ($\theta = 0$) and does not passes through pole **Ans. : (a)**

Q. 151 The curve represented by the equation

$r = 3-2\sin\theta$ is

(a) symmetric about the initial line ($\theta = 0$) and passes through pole

(b) symmetric about the line $\theta = \pi/2$ and does not passes through pole

(c) symmetric about the line $\theta = \pi/2$, passes through pole

(d) symmetric about the initial line ($\theta = 0$) and does not passes through pole **Ans. : (b)**

Q. 152 The curve represented by the equation

$r = \frac{2a}{1 + \cos\theta}$ is

(a) symmetric about the initial line ($\theta = 0$) and passes through pole

(b) symmetric about the line $\theta = \pi/2$ and does not passes through pole

(c) symmetric about the line $\theta = \pi/2$, passes through pole

(d) symmetric about the initial line ($\theta = 0$) and does not passes through pole **Ans. : (d)**

Q. 153 The curve represented by the equation

$r = \frac{2a}{1 + \sin\theta}$ is

(a) symmetric about the initial line ($\theta = 0$) and passes through pole

(b) symmetric about the line $\theta = \pi/2$ and does not passes through pole

(c) symmetric about the line $\theta = \pi/2$, passes through pole

(d) symmetric about the initial line ($\theta = 0$) and does not passes through pole **Ans. : (b)**

Q. 154 The curve represented by the equation $r\,\theta = a$, $a > 0$ is

(a) symmetric about the initial line ($\theta = 0$) and passes through pole

(b) symmetric about the line $\theta = \pi/4$ and does not passes through pole

(c) symmetric about the line $\theta = \pi/2$ and passes through pole

(d) symmetric in opposite quadrant **Ans. : (c)**

□□□

UNIT 5

Multiple Integrals

>> Syllabus :

Double integration in Cartesian and polar co-ordinates; Evaluation of double integrals by changing the order of integration and changing to polar form; Triple integral.

Applications of multiple integrals to find area as double integral , volume as triple integral and surface area.

- **Chapter 8 : Multiple Integrals and their Application**
- **Chapter 9 : Applications of Multiple Integrals**

CHAPTER 8

UNIT V

Multiple Integrals and their Application

Syllabus

Double integration in Cartesian and polar co-ordinates; Evaluation of double integrals by changing the order of integration and changing to polar form; Triple integral.

8.1 Introduction

Integration is very important tool due to its large number of applications in different branches. The students are very much familiar with the integration of single variable. When the function contains several variables, then we need multiple integration. Multiple integration can be used to find area of the curve, volume, mass of lamina, moment of inertia, centre of gravity etc. In this topic, we will see the double and triple integration and evaluation of it by different methods.

8.2 Double Integration

Since, we know, $\int_a^b f(x)\, dx$ has been defined as the limit of sum as,

$$\int_a^b f(x)\, dx = \lim_{\substack{n \to \infty \\ \delta x_r \to 0}} \sum_{r=1}^{n} f(x_r)\, \delta x_r$$

This concept has been extended to define double and triple integration.

Let, f(x, y) be any continuous and single valued function defined in the region R, where x and y are independent variables.

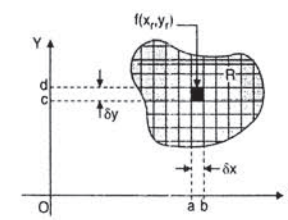


Fig. 8.2.1

Divide the region R into n sub regions (into rectangles) by drawing lines parallel to coordinate axes.

Let, $R_1, R_2, R_3, \ldots\ldots, R_n$ be the subregions and $A_1, A_2, \ldots, A_n$ are the areas respectively.

The limit of sum of

$$f_1(x_1, y_1) A_1 + f_2(x_2, y_2) A_2 + \ldots + f_r(x_r, y_r) A_r$$

$$= \sum_{r=1}^{n} f(x_r, y_r) A_r$$

as number of sub region (n) $\to \infty$, area ($\delta x_r\, \delta y_x =$) $A_r \to 0$ is called double integral of f(x, y) over the region R. (Region of integration)

$$\therefore \quad \iint f(x, y)\, dx\, dy = \lim_{\substack{n \to \infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\, \delta A_r$$