1. COMPLEX NUMBERS

1.1 Introductory Remarks

The solution in radicals (without trigonometric functions) of a general cubic equation contains the square root of the negative numbers when all the three roots are real numbers; a situation that cannot be rectified by factoring aided by the rational root test if the cubic is irreducible. This conundrum led the Italian Mathematician Gerolamo Cardano (1501-1576) to conceive of the complex numbers in around 1545, though his understanding was rudimentary.

The work on the problem of general polynomials ultimately led to the fundamental theorem of algebra, which shows that with complex numbers, a solution exists to every polynomial equation of degree one or higher. The complex numbers thus form an algebraically closed field, where any polynomial equation has a root.

Many mathematicians contributed to the full development of complex numbers. The rules for addition, subtraction, multiplication and division of complex numbers were developed by the Italian mathematician Rafael Bombelli (1526-1572). A more abstract formalism for the complex numbers was further developed by the Irish mathematician William Rowan Hamilton (1805-1865), who extended this abstraction to the theory of quarternions.

1.2 Definition of Complex Number

A complex number is a number that can be expressed in the form $x + iy$, where x and y are real numbers and i is the imaginary unit satisfying the equation $i^2 = -1$. In this expression, x is the real part and y is the imaginary part of the complex number. A complex number is generally denoted by z. The symbol z, which can stand for a set of complex numbers, is called a complex variable. Two complex numbers $z = x + iy$ and $\bar{z} = x - iy$ are said to be conjugate of each other.

1.3 Equality of Complex Numbers

If two complex numbers are equal, then their real and imaginary parts will respectively be equal.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers such that $z_1 = z_2$. Then

$$x_1 + iy_1 = x_2 + iy_2$$

$$\Rightarrow (x_1 - x_2) = i(y_2 - y_1)$$

$$\Rightarrow (x_1 - x_2)^2 = -(y_2 - y_1)^2 \quad (\because i^2 = -1)$$

$$\Rightarrow (x_1 - x_2)^2 + (y_2 - y_1)^2 = 0$$

$$\Rightarrow x_1 - x_2 = 0\,, y_2 - y_1 = 0$$

$$\Rightarrow x_1 = x_2\,, y_1 = y_2$$

This proves the required result.

1.4 Geometrical Representation of a Complex Number

A complex number can be viewed as a point or position vector in a two dimensional Cartesian coordinate system called the complex plane or Argand diagram, named after the French mathematician J.R. Argand (1768 - 1822) who published the idea of geometrical representation of complex numbers in the year 1806.

The complex numbers are conventionally plotted using the real part as the horizontal component, and imaginary part as vertical (see Fig. 1.1).

Any complex number $z = x + iy$ can be represented as the point $P(x, y)$ in the xy-plane.

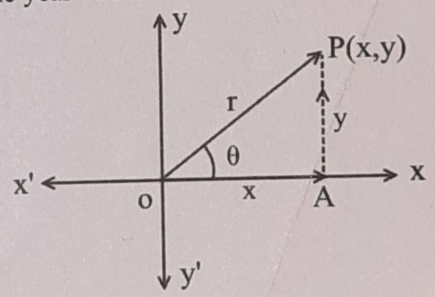

Fig. 1.1

1.5 Polar Form of a Complex Number

The polar equivalent of the complex number $z = x + iy$ is given by

$$z = x + iy = r\cos\theta + i(r\sin\theta) = r(\cos\theta + i\sin\theta)$$

where $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$.

Here the quantity r is known as the modulus of the complex number z and is represented as

$$r = |x + iy| = \sqrt{x^2 + y^2}.$$

The angle θ, which defines the position of the vector $\overrightarrow{OP}$, is called the amplitude or argument of the complex number $z = x + iy$ and is given by amp. of $(= x + iy) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$.

If OP be turned in anti-clockwise direction through multiples of 2π, the point $P(x, y)$ for all such rotations will have the same position as before. Hence for all such positions of OP, the point P always represents the same complex number $= x + iy$.

The complex number $z = x + iy$ has, therefore, various polar forms, when the amplitude is increased by multiples of 2π. As a result, the general polar form of $z = x + iy$ is given by

$$z = x + iy = r[\cos(2m\pi + \theta) + i\sin(2m\pi + \theta)]; \quad r = 0, 1, 2, 3, \ldots\ldots$$

Here $2m\pi + \theta$ is known as the general amplitude of $= x + iy$, and θ which lies between $-\pi$ and π is known as the principal value of the amplitude.

1.6 Polar Form of $z = x + iy$ for Different Signs of x, y

(a) $\boldsymbol{z = x + iy}$ $(x > 0, y > 0)$: The point $P(x, y)$, in this case, will lie in the first quadrant.

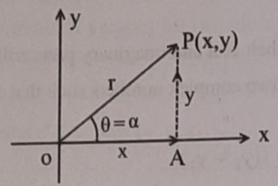

Fig. 1.2

Here $r = \sqrt{(x^2 + y^2)}$ and $= \alpha$. Hence $z = x + iy = \sqrt{(x^2 + y^2)}\{\cos\alpha + i\sin\alpha\}$, where $\tan\alpha = \frac{y}{x}$.

Example: $1 + i\sqrt{3} = 2\left\{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right\}$

(b) $\boldsymbol{z = -x + iy}$ $(x > 0, y > 0)$: The point $P(-x, y)$ lies in the second quadrant and $\angle AOP = \alpha$ where $\tan\alpha = \frac{y}{x}$.

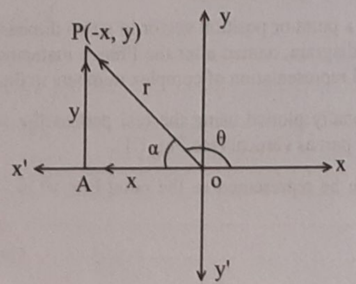

Fig. 1.3

The amplitude of the complex number will be $\theta = \pi - \alpha$ and the modulus will be $= \sqrt{(x^2 + y^2)}$.

Hence $-x + iy = \sqrt{(x^2 + y^2)}\{\cos(\pi - \alpha) + i\sin(\pi - \alpha)\}$

Example: $-1 + i\sqrt{3} = 2\left\{\cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right)\right\}$

$$= 2\left\{\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right\}$$

(c) $\boldsymbol{z = -x - iy}$ $(x > 0, y > 0)$: The point $P(-x, -y)$ lies in the third quadrant and $\angle AOP = \alpha$ where $\tan\alpha = \frac{y}{x}$ and the amplitude of the complex number being $\theta = \pi + \alpha$ and $r = \sqrt{(x^2 + y^2)}$.

$$\therefore z = -x - iy = \sqrt{(x^2 + y^2)}\{\cos(\pi + \alpha) + i\sin(\pi + \alpha)\}$$

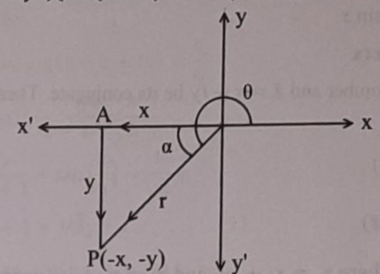

Fig. 1.4

Example: $z = -1 - i\sqrt{3} = 2\left\{\cos\left(\pi + \frac{\pi}{3}\right) + i\sin\left(\pi + \frac{\pi}{3}\right)\right\}$

$$= 2\left\{\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right\}$$

(d) $\boldsymbol{z = x - iy}$ $(x > 0, y > 0)$: The point $P(x, -y)$ lies in the fourth quadrant and $\angle AOP = \alpha$ where $\tan\alpha = \frac{y}{x}$ and the amplitude of the complex number being $\theta = 2\pi - \alpha$ or $-\alpha$ and $r = \sqrt{(x^2 + y^2)}$.

Hence $z = x - iy = \sqrt{(x^2 + y^2)}\{\cos(2\pi - \alpha) + i\sin(2\pi - \alpha)\}$

$$= \sqrt{(x^2 + y^2)}\{\cos\alpha - i\sin\alpha\}$$

Example: $z = 1 - i\sqrt{3} = 2\left\{\cos\left(2\pi - \frac{\pi}{3}\right) + i\sin\left(2\pi - \frac{\pi}{3}\right)\right\}$

$$= 2\left\{\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right\}$$

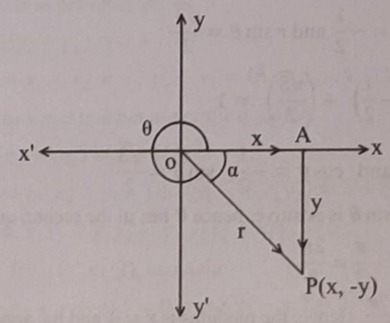

Fig. 1.5

1.7 Exponential Form of Complex Numbers

When x is real, we have $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\cdots\infty$

$$\Rightarrow e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots\cdots\cdots\infty$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots\right)$$

$$= \cos x + i\sin x \text{ (Euler's Formula)}$$

Thus, $\quad z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

Also, $\quad e^{-ix} = \cos x - i\sin x.$

1.8 Properties of Complex Numbers

Let $z = x + iy$ be a complex number and $\bar{z} = x - iy$ be its conjugate. Then, we have the following properties of the complex numbers:

(a) $Re(z) = x = \frac{1}{2}(z + \bar{z})$

(b) $Im(z) = y = \frac{1}{2i}(z - \bar{z})$

(c) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers

(d) $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$

(e) $\overline{(z_1/z_2)} = \frac{\bar{z}_1}{\bar{z}_2}$

(f) $z\bar{z} = |z|^2 = |\bar{z}|^2$

(g) $|z_1 z_2| = |z_1||z_2|$ and $arg(z_1 z_2) = arg(z_1) + arg(z_2)$

(h) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$

1.9 Solved Examples

1. Find the modulus and argument of $= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Solution: Here $z = x + iy = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = r(\cos\theta + i\sin\theta)$

$$\Rightarrow r\cos\theta = -\frac{1}{2} \text{ and } r\sin\theta = \frac{\sqrt{3}}{2}$$

$tan\theta = \frac{\sqrt{3}}{2}$

$$\Rightarrow r^2 = \left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$$

$\pi - \frac{\pi}{3}$

$$\therefore r = 1 \text{ and } \cos\theta = -\frac{1}{2},\ \sin\theta = \frac{\sqrt{3}}{2}$$

$\frac{2\pi}{3}$

Since $\cos\theta$ is negative and $\sin\theta$ is positive, hence θ lies in the second quadrant.

$$\therefore \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

So, the principal value of $= \frac{2\pi}{3}$. Hence, the modulus is $r = 1$ and the argument is $\theta = \frac{2\pi}{3}$.

2. If $\alpha + i\beta = \frac{1}{a - ib}$, prove that $(\alpha^2 + \beta^2)(a^2 + b^2) = 1$.

Solution: Given that $\alpha + i\beta = \frac{1}{a-ib} = \frac{a+ib}{a^2+b^2} = \frac{a}{a^2+b^2} + i\left(\frac{b}{a^2+b^2}\right)$

$$\Rightarrow \alpha = \frac{a}{a^2 + b^2},\ \beta = \frac{b}{a^2 + b^2}$$

$$\Rightarrow \alpha^2 + \beta^2 = \left(\frac{a}{a^2 + b^2}\right)^2 + \left(\frac{b}{a^2 + b^2}\right)^2 = \frac{a^2 + b^2}{(a^2 + b^2)^2} = \frac{1}{a^2 + b^2}$$

$$\Rightarrow (\alpha^2 + \beta^2)(a^2 + b^2) = 1$$

3. If $\arg(z + 1) = \frac{\pi}{6}$ and $\arg(z - 1) = \frac{2\pi}{3}$, find z.

Solution: Let $z = x + iy$, then

$$\arg(z + 1) = \frac{\pi}{6} \Rightarrow \arg(x + 1 + iy) = \frac{\pi}{6}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x + 1}\right) = \frac{\pi}{6}$$

$$\Rightarrow \frac{y}{x + 1} = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow x + 1 = \sqrt{3}y \ldots\ldots\ldots\ldots (1)$$

Similarly, $\arg(z - 1) = \frac{2\pi}{3} \Rightarrow \arg(x - 1 + iy) = \frac{2\pi}{3}$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x - 1}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \frac{y}{x - 1} = \tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$$

$$\Rightarrow -x\sqrt{3} + \sqrt{3} = y$$

$$\Rightarrow x\sqrt{3} + y = \sqrt{3} \ldots\ldots\ldots\ldots (2)$$

From (1) and (2), we have $x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$.

4. If the sum and product of two complex numbers are real, show that those two numbers must be either real or conjugate.

Solution: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be the two complex numbers.

Then, $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) = u(say)$

Then, u will be a real number as per the question.

So, we have $(x_1 + x_2) + i(y_1 + y_2) = u + i(0)$

$$\Rightarrow x_1 + x_2 = u;\ y_1 + y_2 = 0 \ldots\ldots\ldots\ldots (1)$$

Let $z_1 z_2 = v$, then v will be a real number as per the question.

So, we have $\quad (x_1 + iy_1)(x_2 + iy_2) = v + i(0)$

$$\Rightarrow (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) = v + i(0)$$

$$\Rightarrow x_1x_2 - y_1y_2 = v;\ y_1x_2 + x_1y_2 = 0 \ldots\ldots\ldots\ldots (2)$$

Let us substitute $y_2 = -y_1$ from (1) in (2), to obtain

$$y_1x_2 + x_1(-y_1) = 0 \Rightarrow y_1(x_2 - x_1) = 0$$

$$\Rightarrow y_1 = 0, x_1 = x_2$$

Case II. When n is a negative integer

Let $n = -m$, where m is a positive integer. Then,

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos\theta - i\sin m\theta)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos(-m)\theta + i\sin(-m)\theta$$

$$= \cos n\theta + i\sin n\theta$$

Hence, the theorem holds true when n is a negative integer.

Case III. When n is a fraction

Let $n = \frac{p}{q}$, where $\frac{p}{q}$ may be a positive or negative fraction.

From the cases I and II, we have

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos\theta + i\sin\theta$$

$$\Rightarrow (\cos\theta + i\sin\theta)^{\frac{1}{q}} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{\frac{p}{q}} = \left\{(\cos\theta + i\sin\theta)^{\frac{1}{q}}\right\}^p$$

$$= \left\{\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right\}^p = \cos p\left(\frac{\theta}{q}\right) + i\sin p\left(\frac{\theta}{q}\right)$$

$$= \cos\left(\frac{p}{q}\theta\right) + i\sin\left(\frac{p}{q}\theta\right)$$

$$= \cos n\theta + i\sin n\theta$$

Thus, the theorem holds true in case n is a fraction.

Thus, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, when n is a real number.

1.11 Applications of De-Moivre's Theorem

1. If $z = \cos\theta + i\sin\theta$, then $\frac{1}{z} = \cos\theta - i\sin\theta$.

Proof: $\frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos(-1)\theta + i\sin(-1)\theta$

$$= \cos(-\theta) + i\sin(-\theta)$$

$$= \cos\theta - i\sin\theta$$

2. $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$.

Proof: $(\cos\theta - i\sin\theta)^n = \{\cos(-\theta) + i\sin(-\theta)\}^n = \cos n(-\theta) + i\sin n(-\theta)$

$$= \cos(-n\theta) + i\sin(-n\theta)$$

$$= \cos n\theta - i\sin n\theta$$

3. If $z_1 = \cos\theta + i\sin\theta$ and $z_2 = \cos\phi + i\sin\phi$, then $\frac{z_1}{z_2} = \cos(\theta - \phi) + i\sin(\theta - \phi)$.

Proof: $\frac{z_1}{z_2} = \frac{\cos\theta + i\sin\theta}{\cos\phi + i\sin\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)^{-1}$

$$= (\cos\theta + i\sin\theta)(\cos\phi - i\sin\phi)$$

$$= (\cos\theta\cos\phi + \sin\theta\sin\phi) + i(\sin\theta\cos\phi - \cos\theta\sin\phi)$$

$$= \cos(\theta - \phi) + i\sin(\theta - \phi)$$

4. $(\sin\theta + i\cos\theta)^n = \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$

Proof: $(\sin\theta + i\cos\theta)^n = \left\{\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right\}^n = \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$.

1.12 Solved Examples

1. Prove that $\left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n = \cos\left(\frac{n\pi}{2} - n\alpha\right) + i\sin\left(\frac{n\pi}{2} - n\alpha\right)$.

Solution: $LHS = \left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n = \left\{\frac{1+\cos\left(\frac{\pi}{2}-\alpha\right)+i\sin\left(\frac{\pi}{2}-\alpha\right)}{1+\cos\left(\frac{\pi}{2}-\alpha\right)-i\sin\left(\frac{\pi}{2}-\alpha\right)}\right\}^n$

$$= \left\{\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) + 2i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) - 2i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right\}^n$$

$\cos 2\theta = 2\cos^2\theta/2 - 1 = 1 - 2\sin^2\theta/2$

$$= \left\{\frac{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) - i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right\}^n$$

$$= \left\{\frac{e^{i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}}{e^{-i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}}\right\}^n = \left\{e^{2i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right\}^n$$

$$= e^{2ni\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)} = e^{in\left(\frac{\pi}{2}-\alpha\right)}$$

$$= \cos n\left(\frac{\pi}{2} - \alpha\right) + i\sin n\left(\frac{\pi}{2} - \alpha\right)$$

$$= \cos\left(\frac{n\pi}{2} - n\alpha\right) + i\sin\left(\frac{n\pi}{2} - n\alpha\right)$$

$$= RHS$$

2. If $x + \frac{1}{x} = 2\cos\theta$, $y + \frac{1}{y} = 2\cos\phi$ *and* $z + \frac{1}{z} = 2\cos\psi$, then prove that

(a) $xyz + \frac{1}{xyz} = 2\cos(\theta + \phi + \psi)$

(b) $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\phi)$

(c) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$.

Solution: (a) Here $x + \frac{1}{x} = 2\cos\theta \Rightarrow x^2 - 2x\cos\theta + 1 = 0$

$$\Rightarrow x = \cos\theta \pm i\sin\theta$$

Let $x = \cos\theta + i\sin\theta$

Similarly, $y = \cos\phi + i\sin\phi$ and, $z = \cos\psi + i\sin\psi$

$$\therefore xyz = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)(\cos\psi + i\sin\psi)$$
$$= \cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi)$$
$$\Rightarrow \frac{1}{xyz} = \frac{1}{\cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi)}$$
$$= \{\cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi)\}^{-1}$$
$$= \cos(\theta+\phi+\psi) - i\sin(\theta+\phi+\psi)$$

Hence $xyz + \frac{1}{xyz} = 2\cos(\theta+\phi+\psi)$

(b) $\frac{x^m}{y^n} + \frac{y^n}{x^m} = \frac{(\cos\theta + i\sin\theta)^m}{(\cos\phi + i\sin\phi)^n} + \frac{(\cos\phi + i\sin\phi)^n}{(\cos\theta + i\sin\theta)^m}$

$$= \frac{\cos m\theta + i\sin m\theta}{\cos n\phi + i\sin n\phi} + \frac{\cos n\phi + i\sin n\phi}{\cos m\theta + i\sin m\theta}$$
$$= (\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)^{-1} + (\cos n\phi + i\sin n\phi)(\cos m\theta + i\sin m\theta)^{-1}$$
$$= (\cos m\theta + i\sin m\theta)(\cos n\phi - i\sin n\phi) + (\cos n\phi + i\sin n\phi)(\cos m\theta - i\sin m\theta)$$
$$= (\cos m\theta\cos n\phi + \sin m\theta\sin n\phi) + i(\sin m\theta\cos n\phi - \cos m\theta\sin n\phi)$$
$$+ (\cos n\phi\cos m\theta + \sin n\phi\sin m\theta) + i(-\cos n\phi\sin m\theta + \sin n\phi\cos m\theta)$$
$$= \cos(m\theta - n\phi) + i\sin(m\theta - n\phi) + \cos(m\theta - n\phi) + i\sin(n\phi - m\theta)$$
$$= 2\cos(m\theta - n\phi) + i\sin(m\theta - n\phi) - i\sin(m\theta - n\phi)$$
$$= 2\cos(m\theta - n\phi)$$

(d) $x^m y^n + \frac{1}{x^m y^n} = (\cos\theta + i\sin\theta)^m(\cos\phi + i\sin\phi)^n + \frac{1}{(\cos\theta + i\sin\theta)^m(\cos\phi + i\sin\phi)^n}$

$$= (\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi) + \frac{1}{(\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)}$$
$$= (\cos m\theta\cos n\phi - \sin m\theta\sin n\phi) + i(\sin m\theta\cos n\phi + \cos m\theta\sin n\phi)$$
$$+ \frac{1}{(\cos m\theta\cos n\phi - \sin m\theta\sin n\phi) + i(\sin m\theta\cos n\phi + \cos m\theta\sin n\phi)}$$
$$= \cos(m\theta + n\phi) + i\sin(m\theta + n\phi) + \frac{1}{\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)}$$
$$= \cos(m\theta + n\phi) + i\sin(m\theta + n\phi) + [\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)]^{-1}$$
$$= \cos(m\theta + n\phi) + i\sin(m\theta + n\phi) + \cos(m\theta + n\phi) - i\sin(m\theta + n\phi)$$
$$= 2\cos(m\theta + n\phi)$$

3. If $\sin\alpha + \sin\beta + \sin\gamma = 0$ and $\cos\alpha + \cos\beta + \cos\gamma = 0$, then prove that

(a) $\cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha) = 0$ *and* $\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) = 0$

(b) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$ *and* $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

(c) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$

(d) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$ *and* $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

Solution: Let $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$ *and* $z = \cos\gamma + i\sin\gamma$

$$\Rightarrow x + y + z = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$$
$$\Rightarrow x + y + z = 0 + i(0) = 0 \ldots\ldots\ldots\ldots (1)$$

Now, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = x^{-1} + y^{-1} + z^{-1} = (\cos\alpha + i\sin\alpha)^{-1} + (\cos\beta + i\sin\beta)^{-1} + (\cos\gamma + i\sin\gamma)^{-1}$

$$= (\cos\alpha - i\sin\alpha) + (\cos\beta - i\sin\beta) + (\cos\gamma - i\sin\gamma)$$
$$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$$
$$\therefore \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0 - i(0) = 0 \ldots\ldots\ldots\ldots (2)$$

(a) From (2), we have $yz + zx + xy = 0 \ldots\ldots\ldots\ldots\ldots (3)$

$$\Rightarrow (\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma) + (\cos\gamma + i\sin\gamma)(\cos\alpha + i\sin\alpha)$$
$$+ (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = 0$$
$$\Rightarrow \{\cos(\beta+\gamma) + i\sin(\beta+\gamma)\} + \{\cos(\gamma+\alpha) + i\sin(\gamma+\alpha)\} + \{\cos(\alpha+\beta) + i\sin(\alpha+\beta)\} = 0$$
$$\Rightarrow \{\cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha)\} + i\{\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha)\} = 0$$
$$\Rightarrow \cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha) = 0;\ \sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) = 0$$

(b) $x + y + z = 0 \Rightarrow (x + y + z)^2 = 0$

$$\Rightarrow x^2 + y^2 + z^2 + 2(xy + yz + zx) = 0$$
$$\Rightarrow x^2 + y^2 + z^2 + 2(0) = 0$$
$$\Rightarrow x^2 + y^2 + z^2 = 0$$
$$\therefore (\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$$
$$\Rightarrow (\cos 2\alpha + i\sin 2\alpha) + (\cos 2\beta + i\sin 2\beta) + (\cos 2\gamma + i\sin 2\gamma) = 0$$
$$\Rightarrow (\cos 2\alpha + \cos 2\beta + \cos 2\gamma) + i(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0$$
$$\Rightarrow \cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0 \text{ and } \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$$

(c) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{1}{2}[2\cos^2\alpha + 2\cos^2\beta + 2\cos^2\gamma]$

$$= \frac{1}{2}[\{\cos 2\alpha + 1\} + \{\cos 2\beta + 1\} + \{\cos 2\gamma + 1\}]$$
$$= \frac{1}{2}[\{\cos 2\alpha + \cos 2\beta + \cos 2\gamma\} + 3]$$
$$= \frac{1}{2}[\{0\} + 3] = \frac{3}{2}$$

(d) $x + y + z = 0 \Rightarrow x + y = -z$

$$\Rightarrow (x + y)^3 = (-z)^3 = -z^3$$
$$\Rightarrow x^3 + y^3 + 3xy(x + y) = -z^3$$
$$\Rightarrow x^3 + y^3 + 3xy(-z) = -z^3$$
$$\Rightarrow x^3 + y^3 + z^3 = 3xyz$$
$$\therefore (\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$$
$$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \times (\cos\gamma + i\sin\gamma)$$
$$\Rightarrow (\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta) + (\cos 3\gamma + i\sin 3\gamma)$$
$$= 3\{\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)\}$$
$$\Rightarrow (\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$$
$$= 3\cos(\alpha+\beta+\gamma) + i\{3\sin(\alpha+\beta+\gamma)\}$$
$$\Rightarrow \cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma) \text{ and } \sin 3\alpha + \sin 3\beta + \sin 3\gamma$$
$$= 3\sin(\alpha+\beta+\gamma).$$

4. If α and β are the roots of the equation: $z^2 \sin^2\theta - z\sin 2\theta + 1 = 0$, then prove that $\alpha^n + \beta^n = 2\cos n\theta \, cosec^n\,\theta$, where '$n$' is a positive integer.

Solution: Given that $z^2 \sin^2\theta - z\sin 2\theta + 1 = 0$,

we have $z = \frac{\sin 2\theta \pm \sqrt{(\sin 2\theta)^2 - 4\sin^2\theta}}{2\sin^2\theta}$

$$= \frac{2\sin\theta\cos\theta \pm \sqrt{4\sin^2\theta\cos^2\theta - 4\sin^2\theta}}{2\sin^2\theta}$$

$$= \frac{\cos\theta \pm \sqrt{\cos^2\theta - 1}}{\sin\theta} = \frac{\cos\theta \pm i\sin\theta}{\sin\theta}$$

$$= e^{\pm i\theta}\, cosec\,\theta$$

Let $\alpha = e^{i\theta}\, cosec\,\theta$ and $\beta = e^{-i\theta}\, cosec\,\theta$

$$\Rightarrow \alpha^n + \beta^n = \left\{e^{i\theta}\, cosec\,\theta\right\}^n + \left\{e^{-i\theta}\, cosec\,\theta\right\}^n$$

$$= e^{ni\theta}\, cosec^n\,\theta + e^{-ni\theta}\, cosec^n\,\theta$$

$$= 2\, cosec^n\,\theta \left\{\frac{e^{ni\theta} + e^{-ni\theta}}{2}\right\}$$

$$= 2\, cosec^n\,\theta\{\cos n\theta\}$$

$$= 2\cos n\theta\, cosec^n\,\theta$$

5. If $\alpha = 1 + i, \beta = 1 - i$ and $\cot\theta = 1 + x$, prove that $(x+\alpha)^n + (x+\beta)^n = 2\cos n\theta\, cosec^n\,\theta$.

Solution: Here $\cot\theta = 1 + x$ gives

$$x = \cot\theta - 1 \ldots\ldots\ldots\ldots (1)$$

$$\Rightarrow x + \alpha = \cot\theta - 1 + \alpha = \cot\theta - 1 + (1 + i) = \cot\theta + i$$

$$\Rightarrow x + \alpha = \frac{\cos\theta + i\sin\theta}{\sin\theta} = e^{i\theta}\, cosec\,\theta$$

$$\Rightarrow (x+\alpha)^n = e^{in\theta}\, cosec^n\theta \ldots\ldots\ldots\ldots (2)$$

From (2), we have

$$x + \beta = \cot\theta - 1 + \beta = \cot\theta - 1 + (1 - i) = \cot\theta - i$$

$$\Rightarrow x + \beta = \frac{\cos\theta - i\sin\theta}{\sin\theta} = e^{-i\theta}\, cosec\,\theta$$

$$\Rightarrow (x+\beta)^n = e^{-in\theta}\, cosec^n\theta$$

Hence, we have

$$(x+\alpha)^n + (x+\beta)^n = \left(e^{in\theta} + e^{-in\theta}\right) cosec^n\theta$$

$$= 2\left(\frac{e^{in\theta} + e^{-in\theta}}{2}\right) cosec^n\theta$$

$$= 2\cos n\theta\, cosec^n\theta$$

6. If α, β are the roots of the equation $x^2 - 2\sqrt{3}x + 4 = 0$, prove that $\alpha^3 + \beta^3 = 0$.

Solution: Given that $x^2 - 2\sqrt{3}x + 4 = 0$, we have

$$x = \frac{2\sqrt{3} \pm \sqrt{12 - 16}}{2} = \frac{2\sqrt{3} \pm 2i}{2} = \sqrt{3} \pm i$$

Let $\alpha = \sqrt{3} + i = r(\cos\theta + i\sin\theta)$

$$\Rightarrow r\cos\theta = \sqrt{3} \textit{ and } r\sin\theta = 1$$

$$\Rightarrow r^2 = 4 \Rightarrow r = 2$$

Also, $\tan\theta = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{6}$

$$\therefore \alpha = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let $\beta = \sqrt{3} - i = s(\cos\phi + i\sin\phi)$

$$\Rightarrow s\cos\phi = \sqrt{3} \textit{ and } s\sin\phi = -1$$

$$\Rightarrow s^2 = 4 \textit{ and } \tan\phi = -\frac{1}{\sqrt{3}} = \tan\left(-\frac{\pi}{6}\right)$$

$$\Rightarrow s = 2, \phi = -\frac{\pi}{6}$$

Hence $\beta = 2\left\{\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right\} = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) \ldots\ldots\ldots (2)$

Hence $\alpha^3 + \beta^3 = 2^3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3 + 2^3\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^3$

$$= 2^3\left\{\cos 3\left(\frac{\pi}{6}\right) + i\sin 3\left(\frac{\pi}{6}\right)\right\} + 2^3\left\{\cos 3\left(\frac{\pi}{6}\right) - i\sin 3\left(\frac{\pi}{6}\right)\right\}$$

$$= 2^3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) + 2^3\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$$

$$= 2^3(i) + 2^3(-i) = 0.$$

7. If $(a_1 + ib_1)(a_2 + ib_2)(a_3 + ib_3)\ldots(a_n + ib_n) = A + iB$, then prove that $(a_1^2 + b_1^2)(a_2^2 + b_2^2)(a_3^2 + b_3^2)\ldots\ldots(a_n^2 + b_n^2) = A^2 + B^2$ and $\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \tan^{-1}\left(\frac{b_3}{a_3}\right) + \cdots\ldots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{B}{A}\right)$.

Solution: Let $a_k + ib_k = r_k(\cos\theta_k + i\sin\theta_k)$,

Then $a_k^2 + b_k^2 = r_k^2$ and $\theta_k = \tan^{-1}\left(\frac{b_k}{a_k}\right)$ *for* $k = 1, 2, 3, 4, \ldots\ldots\ldots, n$

Hence $(a_1 + ib_1)(a_2 + ib_2)(a_3 + ib_3)\ldots\ldots(a_n + ib_n)$

$$= r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)\cdot r_3(\cos\theta_3 + i\sin\theta_3)\ldots\ldots r_n(\cos\theta_n + i\sin\theta_n)$$

$$= r_1 r_2 r_3 \ldots\ldots r_n\{\cos(\theta_1 + \theta_2 + \theta_3 + \cdots + \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \cdots\ldots\theta_n)\}$$

$$= A + iB(given)$$

$$= R(\cos\phi + i\sin\phi)$$

$$\Rightarrow R = r_1 r_2 r_3 \ldots\ldots r_n, \theta_1 + \theta_2 + \theta_3 + \cdots\ldots\theta_n = \phi \textit{ and } \phi = \tan^{-1}\left(\frac{B}{A}\right)$$

$$\Rightarrow r_1^2 r_2^2 r_3^2 \ldots\ldots r_n^2 = R^2 \textit{ and } \tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \tan^{-1}\left(\frac{b_3}{a_3}\right) + \cdots\ldots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \phi = \tan^{-1}\left(\frac{B}{A}\right)$$

$$\Rightarrow (a_1^2 + b_1^2)(a_2^2 + b_2^2)(a_3^2 + b_3^2)\ldots\ldots(a_n^2 + b_n^2)$$

$$= A^2 + B^2 \textit{ and } \tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \tan^{-1}\left(\frac{b_3}{a_3}\right) + \cdots\ldots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{B}{A}\right)$$

8. Evaluate $\left[\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right]^n$.

Solution: $\left[\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right]^n = \left[\frac{1+\cos\left(\frac{\pi}{2}-\alpha\right)+i\sin\left(\frac{\pi}{2}-\alpha\right)}{1+\cos\left(\frac{\pi}{2}-\alpha\right)-i\sin\left(\frac{\pi}{2}-\alpha\right)}\right]^n$

$$= \left[\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+2i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)-2i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right]^n$$

$$= \left[\frac{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)-i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right]^n$$

$$= \left[\frac{\left\{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right\}^2}{\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+\sin^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right]^n$$

$$= \left[\cos 2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+i\sin 2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right]^n = \left[\cos\left(\frac{\pi}{2}-\alpha\right)+i\sin\left(\frac{\pi}{2}-\alpha\right)\right]^n$$

$$= \cos n\left(\frac{\pi}{2}-\alpha\right)+i\sin n\left(\frac{\pi}{2}-\alpha\right)$$

$$= \cos\left(\frac{n\pi}{2}-n\alpha\right)+i\sin\left(\frac{n\pi}{2}-n\alpha\right).$$

9. If $\cos\alpha+2\cos\beta+3\cos\gamma = \sin\alpha+2\sin\beta+3\sin\gamma = 0$, **prove that** $\cos 3\alpha+8\cos 3\beta+27\cos 3\gamma = 18\cos(\alpha+\beta+\gamma)$ **and** $\sin 3\alpha+8\sin 3\beta+27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma)$.

Solution: Let $x = \cos\alpha + i\sin\alpha$

$$y = 2(\cos\beta + i\sin\beta)$$

$$z = 3(\cos\gamma + i\sin\gamma)$$

$$\Rightarrow x+y+z = (\cos\alpha+2\cos\beta+3\cos\gamma)+i(\sin\alpha+2\sin\beta+3\sin\gamma)$$

$$= 0 + i(0) = 0$$

Now $x+y+z=0 \Rightarrow x^3+y^3+z^3 = 3xyz$

$$\Rightarrow (\cos\alpha+i\sin\alpha)^3+2^3(\cos\beta+i\sin\beta)^3+3^3(\cos\gamma+i\sin\gamma)^3$$

$$= 3(\cos\alpha+i\sin\alpha)\cdot 2(\cos\beta+i\sin\beta)\cdot 3(\cos\gamma+i\sin\gamma)$$

$$= 18\{\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)\}$$

$$\Rightarrow (\cos 3\alpha+i\sin 3\alpha)+8(\cos 3\beta+i\sin 3\beta)+27(\cos 3\gamma+i\sin 3\gamma)$$

$$= 18\cos(\alpha+\beta+\gamma)+18\,i\sin(\alpha+\beta+\gamma)$$

$$\Rightarrow (\cos 3\alpha+8\cos 3\beta+27\cos 3\gamma)+i(\sin 3\alpha+8\sin 3\beta+27\sin 3\gamma)$$

$$= 18\cos(\alpha+\beta+\gamma)+18\,i\sin(\alpha+\beta+\gamma)$$

$\Rightarrow \cos 3\alpha+8\cos 3\beta+27\cos 3\gamma = 18\cos(\alpha+\beta+\gamma)$ *and* $\sin 3\alpha+8\sin 3\beta+27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma)$.

10. Prove that the $(4n)^{th}$ **power of** $\frac{1+7i}{(2-i)^2}$ **is equal to** $(-4)^n$ **where** $'n'$ **is a positive integer.**

Solution: Here $\frac{1+7i}{(2-i)^2} = \frac{1+7i}{4+i^2-4i}$

$$= \frac{1+7i}{4-1-4i} = \frac{1+7i}{3-4i}$$

$$= \frac{(1+7i)(3+4i)}{(3-4i)(3+4i)}$$

$$= \frac{-25+25i}{25}$$

$$= -1+i$$

Hence $\left[\frac{1+7i}{(2-i)^2}\right]^{4n} = (-1+i)^{4n}$ .... (1)

Let $-1+i = r(\cos\theta + i\sin\theta)$, then

$$r = \sqrt{2} \ and\ \theta = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1) = \frac{3\pi}{4}$$

[$\because$ θ lies in the second quadrant]

$$\therefore -1+i = \sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

$$\Rightarrow (-1+i)^{4n} = \left\{\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\right\}^{4n}$$

$$= \left(\sqrt{2}\right)^{4n}\left\{\cos 4n\left(\frac{3\pi}{4}\right)+i\sin 4n\left(\frac{3\pi}{4}\right)\right\}$$

$$= (4)^n\{\cos 3n\pi + i\sin 3n\pi\}$$

$$= (4)^n\{\cos 3\pi + i\sin 3\pi\}^n$$

$$= (4)^n\{-1+i(0)\}^n$$

$$= (4)^n(-1)^n = (-4)^n.$$

$\cos 3\pi = -1$
$\sin 3\pi = 0$

From (1), we have therefore

$$\left[\frac{1+7i}{(2-i)^2}\right]^{4n} = (-4)^n.$$

EXERCISE - 1.2

1. Simplify

   (a). $\dfrac{(\cos 2\theta + i\sin 2\theta)^7 (\cos 3\theta + i\sin 3\theta)^{-5}}{(\cos 4\theta + i\sin 4\theta)^{12} (\cos 5\theta - i\sin 5\theta)^{-6}}$ Ans.: $\cos 107\theta - i\sin 107\theta$

   (b). $\dfrac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5}$.   Ans.: $\sin 9\theta + i\cos 9\theta$

2. If $x + \frac{1}{x} = 2\cos\theta$, $y + \frac{1}{y} = 2\cos\phi$ *and* $z + \frac{1}{z} = 2\cos\psi$, then prove that

   (a). $xyz + \dfrac{1}{xyz} = 2\cos(\theta + \phi + \psi)$

   (b). $\dfrac{x^m}{y^n} + \dfrac{y^n}{x^m} = 2\cos(m\theta - n\phi)$.

3. If $x_r = \cos\left(\frac{\pi}{2^r}\right) + i\sin\left(\frac{\pi}{2^r}\right)$, prove that

   (a). $x_1 x_2 x_3 \ldots\ldots \infty = -1$

   (b). $x_0 x_1 x_2 \ldots\ldots\ldots \infty = 1$.

4. If $a\cos\alpha + b\cos\beta + c\cos\gamma = a\sin\alpha + b\sin\beta + c\sin\gamma = 0$, prove that

   (a). $a^3\cos 3\alpha + b^3\cos 3\beta + c^3\cos 3\gamma = 3abc\cos(\alpha + \beta + \gamma)$

   (b). $a^3\sin 3\alpha + b^3\sin 3\beta + c^3\sin 3\gamma = 3abc\sin(\alpha + \beta + \gamma)$.

5. Show that $[(\cos\theta + \cos\phi) + i(\sin\theta + \sin\phi)]^n + [(\cos\theta + \cos\phi) - i(\sin\theta + \sin\phi)]^n = 2^{n+1}$ $\cos^n\left(\dfrac{\theta - \phi}{2}\right)\cos\left(\dfrac{n\theta + n\phi}{2}\right)$.

6. Show that $\left(\dfrac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right)^n = \cos n\theta + i\sin n\theta$.

7. If $x_r = \cos\frac{\pi}{3^r} + i\sin\frac{\pi}{3^r}$, prove that

   (a). $x_1 x_2 x_3 \ldots\ldots \infty = i$

   (b). $x_0 x_1 x_2 \ldots\ldots\ldots \infty = -i$.

8. Prove that $\left(1 + i\sqrt{3}\right)^n + \left(1 - i\sqrt{3}\right)^n = 2^{n+1}\cos\frac{n\pi}{3}$ where 'n' is a positive integer.

9. If α *and* β are the roots of the equation $x^2 - \left(2\sqrt{3}\right)x + 4 = 0$, prove that $\alpha^3 + \beta^3 = 0$.

10. If $\alpha = 1 + i$, $\beta = 1 - i$ *and* $\cot\theta = x + 1$, prove that $(x + \alpha)^n - (x + \beta)^n = (\alpha - \beta)\sin n\theta\, cosec^n\,\theta$.

11. If $a = \cos 2\alpha + i\sin 2\alpha$, $b = \cos 2\beta + i\sin 2\beta$ *and* $c = \cos 2\gamma + i\sin 2\gamma$, prove that

    $\sqrt{\dfrac{ab}{c}} + \sqrt{\dfrac{c}{ab}} = 2\cos(\alpha + \beta - \gamma)$.

12. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ *and* $c = \cos\gamma + i\sin\gamma$, prove that

    $\dfrac{(a+b)(b+c)(c+a)}{abc} = 8\cos\left(\dfrac{\alpha - \beta}{2}\right)\cos\left(\dfrac{\beta - \gamma}{2}\right)\cos\left(\dfrac{\gamma - \alpha}{2}\right)$.

1.13 Expansion of $\sin n\theta$, $\cos n\theta$ in the powers of $\sin\theta$, $\cos\theta$

By De-Moivre's theorem, we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \ldots\ldots\ldots\ldots\ldots (1)$$

By Binomial theorem, we have

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= {}^nc_0\cos^n\theta\,(i\sin\theta)^0 + {}^nc_1\cos^{n-1}\theta\,(i\sin\theta)^1 \\ &\quad + {}^nc_2\cos^{n-2}\theta\,(i\sin\theta)^2 + {}^nc_3\cos^{n-3}\theta\,(i\sin\theta)^3 + \cdots .. + {}^nc_n\cos^0\theta\,(i\sin\theta)^n \\ &= \cos^n\theta + i\{{}^nc_1\cos^{n-1}\theta\sin\theta\} - {}^nc_2\cos^{n-2}\theta\sin^2\theta - i\{{}^nc_3\cos^{n-3}\theta\sin^3\theta\} \\ &\quad + {}^nc_4\cos^{n-4}\theta\sin^4\theta + \cdots\ldots\ldots(-1)^{\frac{n}{2}}\sin^n\theta \ldots\ldots (2)\end{aligned}$$

Equating the real and imaginary parts, we have

$$\cos n\theta = \cos^n\theta - {}^nc_2\cos^{n-2}\theta\sin^2\theta + {}^nc_4\cos^{n-4}\theta\sin^4\theta - \cdots\ldots$$

$$\sin n\theta = {}^nc_1\cos^{n-1}\theta\sin\theta - {}^nc_3\cos^{n-3}\theta\sin^3\theta + \cdots\ldots$$

1.14 Solved Examples

1. **Show that**

   (a) $\boldsymbol{\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta}$

   (b) $\boldsymbol{\cos 5\theta = 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta}$.

Solution: By De-Moivre's theorem, we have

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta \ldots\ldots\ldots\ldots (1)$$

By Binomial theorem, we have

$$\begin{aligned}(\cos\theta + i\sin\theta)^5 &= \cos^5\theta + {}^5c_1\cos^4\theta\,(i\sin\theta) + {}^5c_2\cos^3\theta\,(i\sin\theta)^2 + {}^5c_3\cos^2\theta\,(i\sin\theta)^3 \\ &\quad + {}^5c_4\cos\theta\,(i\sin\theta)^4 + {}^5c_5(i\sin\theta)^5 \\ &= \cos^5\theta + i\{5\cos^4\theta\,\sin\theta\} - 10\cos^3\theta\sin^2\theta - i\{10\cos^2\theta\sin^3\theta\} \\ &\quad + 5\cos\theta\sin^4\theta + i\sin^5\theta \ldots\ldots (2)\end{aligned}$$

Equating the real and imaginary parts from (1) and (2), we have

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \ldots\ldots\ldots (3)$$

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \ldots\ldots\ldots (4)$$

From (3), we have

$$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\,(1 - \cos^2\theta) + 5\cos\theta\,(1 - \cos^2\theta)^2 \\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta\,(1 + \cos^4\theta - 2\cos^2\theta) \\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta + 5\cos^5\theta - 10\cos^3\theta \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta \\ &= 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta\end{aligned}$$

From (4), we have

$$\begin{aligned}\sin 5\theta &= 5(1 - \sin^2\theta)^2\sin\theta - 10(1 - \sin^2\theta)\sin^3\theta + \sin^5\theta \\ &= 5(1 + \sin^4\theta - 2\sin^2\theta)\sin\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\ &= 5\sin\theta + 5\sin^5\theta - 10\sin^3\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\ &= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta\end{aligned}$$

2. Show that $\tan 7\theta = \dfrac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$.

Solution: By De-Moivre's theorem, we have

$$(\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta \;\ldots\ldots\ldots\ldots (1)$$

By Binomial theorem, we have

$$\begin{aligned}(\cos\theta + i\sin\theta)^7 &= \cos^7\theta + {}^7c_1\cos^6\theta\,(i\sin\theta) + {}^7c_2\cos^5\theta\,(i\sin\theta)^2 + {}^7c_3\cos^4\theta\,(i\sin\theta)^3 \\ &\quad + {}^7c_4\cos^3\theta\,(i\sin\theta)^4 + {}^7c_5\cos^2\theta\,(i\sin\theta)^5 + {}^7c_6\cos\theta\,(i\sin\theta)^6 + {}^7c_7(i\sin\theta)^7 \\ &= \cos^7\theta + i\{7\cos^6\theta\sin\theta\} - 21\cos^5\theta\sin^2\theta - i\{35\cos^4\theta\sin^3\theta\} \\ &\quad + 35\cos^3\theta\sin^4\theta + i\{21\cos^2\theta\sin^5\theta\} - 7\cos\theta\sin^6\theta - i\sin^7\theta \;\ldots\ldots\ldots (2)\end{aligned}$$

On equating the real and imaginary parts from (1) and (2), we have

$$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta \;\ldots\ldots\ldots (3)$$

$$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta \;\ldots\ldots\ldots (4)$$

From (3) and (4), we have

$$\begin{aligned}\tan 7\theta &= \frac{7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta}{\cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta} \\ &= \frac{\cos^7\theta\,[7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta]}{\cos^7\theta\,[1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta]} \\ &= \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}.\end{aligned}$$

3. Using De-Moivre's theorem, prove that $\dfrac{\sin 6\theta}{\sin 2\theta} = 16\cos^4\theta - 16\cos^2\theta + 3$.

Solution: By Binomial theorem, we have

$$\begin{aligned}(\cos\theta + i\sin\theta)^6 &= \cos^6\theta + {}^6c_1\cos^5\theta\,(i\sin\theta) + {}^6c_2\cos^4\theta\,(i\sin\theta)^2 + {}^6c_3\cos^3\theta\,(i\sin\theta)^3 \\ &\quad + {}^6c_4\cos^2\theta\,(i\sin\theta)^2 + {}^6c_5\cos\theta\,(i\sin\theta)^5 + {}^6c_6(i\sin\theta)^6 \\ &= \cos^6\theta + i\{6\cos^5\theta\sin\theta\} - 15\cos^4\theta\sin^2\theta - i\{20\cos^3\theta\sin^3\theta\} \\ &\quad - 15\cos^2\theta\sin^2\theta + i\{6\cos\theta\sin^5\theta\} - \sin^6\theta\end{aligned}$$

$$\begin{aligned}\Rightarrow \cos 6\theta + i\sin 6\theta &= \cos^6\theta - 15\cos^4\theta\sin^2\theta - 15\cos^2\theta\sin^2\theta - \sin^6\theta \\ &\quad + i\{6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta\}\end{aligned}$$

$$\Rightarrow \sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$$

$$\begin{aligned}\therefore \frac{\sin 6\theta}{\sin 2\theta} &= \frac{6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta}{2\sin\theta\cos\theta} \\ &= 3\cos^4\theta - 10\cos^2\theta\sin^2\theta + 3\sin^4\theta \\ &= 3\cos^4\theta - 10\cos^2\theta\,(1 - \cos^2\theta) + 3(1 - \cos^2\theta)^2 \\ &= 3\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 3(1 + \cos^4\theta - 2\cos^2\theta) \\ &= 3\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 3 + 3\cos^4\theta - 6\cos^2\theta \\ &= 16\cos^4\theta - 16\cos^2\theta + 3.\end{aligned}$$

4. Using De-Moivre's theorem, express $\dfrac{\sin 7\theta}{\sin\theta}$ **in powers of** $\sin\theta$.

Solution: We know that $\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$

$$\begin{aligned}\therefore \frac{\sin 7\theta}{\sin\theta} &= 7\cos^6\theta - 35\cos^4\theta\sin^2\theta + 21\cos^2\theta\sin^4\theta - \sin^6\theta \\ &= 7(1 - \sin^2\theta)^3 - 35(1 - \sin^2\theta)^2\sin^2\theta + 21(1 - \sin^2\theta)\sin^4\theta - \sin^6\theta\end{aligned}$$

$$= 7\{1 - \sin^6\theta - 3\sin^2\theta + 3\sin^4\theta\} - 35(1 + \sin^4\theta - 2\sin^2\theta)\sin^2\theta + 21\sin^4\theta - 21\sin^6\theta - \sin^6\theta$$

$$= 7 - 7\sin^6\theta - 21\sin^2\theta + 21\sin^4\theta - (35 + 35\sin^4\theta - 70\sin^2\theta)\sin^2\theta + 21\sin^4\theta - 22\sin^6\theta$$

$$= 7 - 7\sin^6\theta - 21\sin^2\theta + 21\sin^4\theta - 35\sin^2\theta - 35\sin^6\theta + 70\sin^4\theta + 21\sin^4\theta - 22\sin^6\theta$$

$$= 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

1.15 Expansion of $\sin^n\theta$ and $\cos^n\theta$ in terms of Sines and Cosines of Multiples of θ

If $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$$\therefore x + \frac{1}{x} = 2\cos\theta \text{ and } x - \frac{1}{x} = 2i\sin\theta$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^n = 2^n\cos^n\theta \text{ and } \left(x - \frac{1}{x}\right)^n = (2i\sin\theta)^n = 2^n i^n \sin^n\theta$$

1. Show that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

Solution: Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$$\Rightarrow 2i\sin\theta = x - \frac{1}{x}$$

$$\begin{aligned}\Rightarrow (2i\sin\theta)^5 &= \left(x - \frac{1}{x}\right)^5 = x^5 + {}^5c_1x^4\left(-\frac{1}{x}\right) + {}^5c_2x^3\left(-\frac{1}{x}\right)^2 + {}^5c_3x^2\left(-\frac{1}{x}\right)^3 + {}^5c_4x\left(-\frac{1}{x}\right)^4 + {}^5c_5x^0\left(-\frac{1}{x}\right)^5 \\ &= x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}\end{aligned}$$

$$\begin{aligned}\Rightarrow 32i\sin^5\theta &= \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right) \\ &= 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)\end{aligned}$$

$$\Rightarrow 16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$$

$$\Rightarrow \sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

2. Expand $\sin^7\theta$ **in a series of sines of multiples of** θ.

Solution: Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$$\Rightarrow 2i\sin\theta = x - \frac{1}{x}$$

$$\begin{aligned}\Rightarrow (2i\sin\theta)^7 &= \left(x - \frac{1}{x}\right)^7 = x^7 + {}^7c_1x^6\left(-\frac{1}{x}\right) + {}^7c_2x^5\left(-\frac{1}{x}\right)^2 + {}^7c_3x^4\left(-\frac{1}{x}\right)^3 + {}^7c_4x^3\left(-\frac{1}{x}\right)^4 \\ &\quad + {}^7c_5x^2\left(-\frac{1}{x}\right)^5 + {}^7c_6x\left(-\frac{1}{x}\right)^6 + {}^7c_7x^0\left(-\frac{1}{x}\right)^7 \\ &= x^7 - 7x^5 + 21x^3 - 35x + 35\left(\frac{1}{x}\right) - 21\left(\frac{1}{x^3}\right) + 7\left(\frac{1}{x^5}\right) - \frac{1}{x^7}\end{aligned}$$

$$\Rightarrow -2i(64\sin^7\theta) = \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right)$$

$$= 2i\sin 7\theta - 7(2i\sin 5\theta) + 21(2i\sin 3\theta) - 35(2i\sin\theta)$$

$$\Rightarrow 64\sin^7\theta = -\sin 7\theta + 7\sin 5\theta - 21\sin 3\theta + 35\sin\theta$$

$$\Rightarrow \sin^7\theta = \frac{1}{64}[35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta].$$

3. If $\sin^4\theta\cos^3\theta = a\cos\theta + b\cos 3\theta + c\cos 5\theta + d\cos 7\theta$, find the values of $a, b, c,$ and d.

Solution: Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$$\therefore 2\cos\theta = x + \frac{1}{x} \text{ and } 2i\sin\theta = x - \frac{1}{x}$$

$$\Rightarrow (2\cos\theta)^3 = \left(x + \frac{1}{x}\right)^3 \text{ and } (2i\sin\theta)^4 = \left(x - \frac{1}{x}\right)^4$$

$$\Rightarrow 2^3\cos^3\theta = \left(x + \frac{1}{x}\right)^3 \text{ and } 2^4\sin^4\theta = \left(x - \frac{1}{x}\right)^4$$

$$\Rightarrow (2^3\cos^3\theta)(2^4\sin^4\theta) = \left(x + \frac{1}{x}\right)^3\left(x - \frac{1}{x}\right)^4$$

$$\Rightarrow 2^7\cos^3\theta\sin^4\theta = \left(x + \frac{1}{x}\right)^3\left(x - \frac{1}{x}\right)^4$$

$$= \left(x + \frac{1}{x}\right)^3\left(x - \frac{1}{x}\right)^3\left(x - \frac{1}{x}\right)$$

$$= \left(x^2 - \frac{1}{x^2}\right)^3\left(x - \frac{1}{x}\right)$$

$$= \left(x^6 - \frac{1}{x^6} - 3x^2 + \frac{3}{x^2}\right)\left(x - \frac{1}{x}\right)$$

$$= x^7 - \frac{1}{x^5} - 3x^3 + \frac{3}{x} - x^5 + \frac{1}{x^7} + 3x - \frac{3}{x^3}$$

$$= \left(x^7 + \frac{1}{x^7}\right) - \left(x^5 + \frac{1}{x^5}\right) - 3\left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$$

$$\Rightarrow 2^7\cos^3\theta\sin^4\theta = 2\cos 7\theta - 2\cos 5\theta - 3(2\cos 3\theta) + 3(2\cos\theta)$$

$$\Rightarrow \cos^3\theta\sin^4\theta = \frac{1}{2^6}[\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta]$$

4. Prove that $\cos^6\theta - \sin^6\theta = \frac{1}{16}(\cos 6\theta + 15\sin 2\theta)$.

Solution: Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$$\therefore 2\cos\theta = x + \frac{1}{x}$$

$$\Rightarrow (2\cos\theta)^6 = \left(x + \frac{1}{x}\right)^6$$

$$= x^6 + {}^6c_1x^5\left(\frac{1}{x}\right) + {}^6c_2x^4\left(\frac{1}{x}\right)^2 + {}^6c_3x^3\left(\frac{1}{x}\right)^3 + {}^6c_4x^2\left(\frac{1}{x}\right)^4 + {}^6c_5x\left(\frac{1}{x}\right)^5 + {}^6c_6x^0\left(\frac{1}{x}\right)^6$$

$$= x^6 + 6x^4 + 15x^2 + 20 + 15\left(\frac{1}{x^2}\right) + 6\left(\frac{1}{x^4}\right) + \frac{1}{x^6}$$

$$= \left(x^6 + \frac{1}{x^6}\right) + 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) + 20$$

$$\Rightarrow 2^6\cos^6\theta = 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$$

$$\Rightarrow \cos^6\theta = \frac{1}{2^5}[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10] \text{..........(1)}$$

Also, $2i\sin\theta = x - \frac{1}{x}$

$$\Rightarrow (2i\sin\theta)^6 = \left(x - \frac{1}{x}\right)^6 = x^6 + {}^6c_1x^5\left(-\frac{1}{x}\right) + {}^6c_2x^4\left(-\frac{1}{x}\right)^2$$

$$+ {}^6c_3x^3\left(-\frac{1}{x}\right)^3 + {}^6c_4x^2\left(-\frac{1}{x}\right)^4 + {}^6c_5x\left(-\frac{1}{x}\right)^5 + {}^6c_6\left(-\frac{1}{x}\right)^6$$

$$= x^6 - 6x^4 + 15x^2 - 20 + 15\left(\frac{1}{x^2}\right) - 6\left(\frac{1}{x^4}\right) + \frac{1}{x^6}$$

$$= \left(x^6 + \frac{1}{x^6}\right) - 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) - 20$$

$$\Rightarrow -2^6\sin^6\theta = 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$$

$$\Rightarrow \sin^6\theta = -\frac{1}{2^5}[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10] \text{..........(2)}$$

From (1) and (2), we have

$$\cos^6\theta - \sin^6\theta = \frac{1}{2^5}[\{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\} + \{\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\}]$$

$$= \frac{1}{2^5}[2\cos 6\theta + 30\cos 2\theta] = \frac{1}{16}[\cos 6\theta + 15\cos 2\theta].$$

5. If $\sin^4\theta\cos^3\theta = A_1\cos\theta + A_3\cos 3\theta + A_5\cos 5\theta + A_7\cos 7\theta$, prove that $A_1 + 9A_3 + 25A_5 + 49A_7 = 0$.

Solution: Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$$\Rightarrow 2\cos\theta = x + \frac{1}{x}, 2i\sin\theta = x - \frac{1}{x}$$

$$\Rightarrow (2\cos\theta)^3 = \left(x + \frac{1}{x}\right)^3 = x^3 + {}^3c_1x^2\left(\frac{1}{x}\right) + {}^3c_2x\left(\frac{1}{x}\right)^2 + {}^3c_3\left(\frac{1}{x}\right)^3$$

$$\Rightarrow 8\cos^3\theta = x^3 + 3x + 3\left(\frac{1}{x}\right) + \frac{1}{x^3} \text{..........(1)}$$

Similarly,

$$(2i\sin\theta)^4 = \left(x - \frac{1}{x}\right)^4 = x^4 + {}^4c_1x^3\left(-\frac{1}{x}\right) + {}^4c_2x^2\left(-\frac{1}{x}\right)^2 + {}^4c_3x\left(-\frac{1}{x}\right)^3 + {}^4c_4\left(-\frac{1}{x}\right)^4$$

$$\Rightarrow 16\sin^4\theta = x^4 - 4x^2 + 6 - 4\left(\frac{1}{x^2}\right) + \frac{1}{x^4} \text{..........(2)}$$

From (1) and (2), we have

$$(8\cos^3\theta)(16\sin^4\theta) = \left(x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}\right)\left(x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}\right)$$

$$= \left(x^7 - 4x^5 + 6x^3 - 4x + \frac{1}{x}\right) + \left(3x^5 - 12x^3 + 18x^2 - \frac{12}{x} + \frac{3}{x^3}\right)$$

$$+ \left(3x^3 - 12x + \frac{18}{x} - \frac{12}{x^3} + \frac{3}{x^5}\right) + \left(x - \frac{4}{x} + \frac{6}{x^3} - \frac{4}{x^5} + \frac{1}{x^7}\right)$$

$$\Rightarrow 128\cos^3\theta\sin^4\theta = \left(x^7+\frac{1}{x^7}\right)-\left(x^5+\frac{1}{x^5}\right)-3\left(x^3+\frac{1}{x^3}\right)+3\left(x+\frac{1}{x}\right)$$

$$= 2\cos 7\theta - 2\cos 5\theta - 3(2\cos 3\theta) + 3(2\cos\theta)$$

$$\Rightarrow 64\cos^3\theta\sin^4\theta = \cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta$$

$$\Rightarrow \cos^3\theta\sin^4\theta = \frac{1}{64}\cos 7\theta - \frac{1}{64}\cos 5\theta - \frac{3}{64}\cos 3\theta + \frac{3}{64}\cos\theta$$

$$= A_1\cos\theta + A_3\cos 3\theta + A_5\cos 5\theta + A_7\cos 7\theta \ (given)$$

$$\Rightarrow A_1 = \frac{3}{64}, A_3 = -\frac{3}{64}, A_5 = -\frac{1}{64}, A_7 = \frac{1}{64}$$

Hence

$$A_1 + 9A_3 + 25A_5 + 49A_7 = \frac{3}{64} + 9\left(-\frac{3}{64}\right) + 25\left(-\frac{1}{64}\right) + 49\left(\frac{1}{64}\right)$$

$$= \frac{3}{64} - \frac{27}{64} - \frac{25}{64} + \frac{49}{64}$$

$$= 0$$

EXERCISE 1.3

1. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$, show that $xy + yz + zx = 1$.

2. Expand $\cos^8\theta$ in a series of cosines of multiples of .

   Ans.: $\cos^8\theta = \frac{1}{128}(\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35)$

3. Expand $\sin^7\theta\cos^3\theta$ in a series of sines of multiples of .

   Ans.: $\sin^7\theta\cos^3\theta = -\frac{1}{2^9}(\sin 10\theta - 4\sin 8\theta + 3\sin 6\theta + 8\sin 4\theta - 14\sin 2\theta)$

4. Prove that $\frac{1+\cos 7\theta}{1+\cos\theta} = (x^3 - x^2 - 2x + 1)^2$, where $x = 2\cos\theta$.

5. Prove that $2(1+\cos 8\theta) = (x^4 - 4x^2 + 2)^2$, where $x = 2\cos\theta$.

6. Prove that $\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$.

7. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, show that $x + y + z = xyz$.

8. Prove that

   (a). $\cos^7\theta = \frac{1}{16}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$

   (b). $\sin^8\theta = \frac{1}{2^7}(\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35)$

   (c). $32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

   (d). $\sin^5\theta\cos^2\theta = \frac{1}{64}(\sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta)$.

9. Prove that $\cos^8\theta + \sin^8\theta = \frac{1}{64}(\cos 8\theta + 28\cos 4\theta + 35)$.

10. If $\sin 6\theta = a\cos^5\theta\sin\theta + b\cos^3\theta\sin^3\theta + c\cos\theta\sin^5\theta$, then find the values of a, b and c.

    Ans.: $a = 6, b = -20, c = 6$

11. Prove that $\sin^6\theta + \cos^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$.

1.16 Roots of a Complex Number

Theorem: If p and q are integers relatively prime to each other, then $(\cos\theta + i\sin\theta)^{p/q}$ has exactly q distinct values that can be arranged in geometrical progression.

Proof: By De-Moivre's theorem, one of the values of

$$(\cos\theta + i\sin\theta)^{p/q} \text{ is } \cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}.$$

Since $\cos\theta$ and $\sin\theta$ are periodic functions, we have

$$\cos\theta = \cos(2n\pi + \theta) \text{ and } \sin\theta = \sin(2n\pi + \theta) \text{where } n \text{ is any integer.}$$

$$\therefore (\cos\theta + i\sin\theta)^{p/q} = \{\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)\}^{p/q}$$

$$\Rightarrow (\cos\theta + i\sin\theta)^{p/q} = \cos\frac{p}{q}(2n\pi + \theta) + i\sin\frac{p}{q}(2n\pi + \theta) \ldots\ldots\ldots (1)$$

Let us take $n = 0, 1, 2, 3, \ldots\ldots\ldots, q-1$ in (1) to obtain the q values of $(\cos\theta + i\sin\theta)^{p/q}$ as

$$\alpha_0 = \cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q} \quad when\ n = 0$$

$$\alpha_1 = \cos\frac{p(2\pi + \theta)}{q} + i\sin\frac{p(2\pi + \theta)}{q} \quad when\ n = 1$$

$$\alpha_2 = \cos\frac{p(4\pi + \theta)}{q} + i\sin\frac{p(4\pi + \theta)}{q} \quad when\ n = 2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\alpha_{n-1} = \cos\frac{p(2\overline{q-1}\pi + \theta)}{q} + i\sin\frac{p(2\overline{q-1}\pi + \theta)}{q} \quad when\ n = q-1$$

Note: 1. The highest value that can be given to n is $q-1$. If we give integral values greater than $(q-1)$, we will not get any new values but the same values as given by $n = 0, 1, 2, \ldots\ldots, q-1$.

For Example, let us take $n = q$ in (1) to obtain

$$(\cos\theta + i\sin\theta)^{p/q} = \cos\frac{p}{q}(2q\pi + \theta) + i\sin\frac{p}{q}(2q\pi + \theta)$$

$$= \cos\left(2p\pi + \frac{p\theta}{q}\right) + i\sin\left(2p\pi + \frac{p\theta}{q}\right)$$

$$= \cos\left(\frac{p\theta}{q}\right) + i\sin\left(\frac{p\theta}{q}\right)$$

which is the same value as obtained by putting $n = 0$.

Also, $\alpha_1 = \cos\frac{p}{q}(2\pi + \theta) + i\sin\frac{p}{q}(2\pi + \theta) = \left(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}\right)\left(\cos\frac{2\pi p}{q} + i\sin\frac{2\pi p}{q}\right) = \alpha_0\beta$

where $\beta = \cos\frac{2\pi p}{q} + i\sin\frac{2\pi p}{q}$

$$\alpha_2 = \cos\frac{p}{q}(4\pi + \theta) + i\sin\frac{p}{q}(4\pi + \theta) = \left(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}\right)\left(\cos\frac{2\pi p}{q} + i\sin\frac{2\pi p}{q}\right)^2 = \alpha_0\beta^2$$

Thus, the q values of $(\cos\theta + i\sin\theta)^{p/q}$ are $\alpha_0, \alpha_0\beta, \alpha_0\beta^2, \alpha_0\beta^3, \ldots\ldots\ldots, \alpha_0\beta^{q-1}$ which are obviously in geometrical progression.

Note: 2. It may be noted that $(\cos\theta + i\sin\theta)^{2/16}$ has only 8 distinct values and not 16 because $\frac{2}{16}$ in its lowest term is $\frac{1}{8}$. In order to find the distinct values of $(\cos\theta + i\sin\theta)^{p/q}$, we should therefore see that $\frac{p}{q}$ is in its lowest terms (i.e. p and q are relatively prime to each other). If $\frac{p}{q}$ is not reduced to its lowest terms, the values will be repeated.

1.17 Solved Examples

1. **Find all the values of $(i)^{1/4}$.**

Solution: We have $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$$= \cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)$$

$$= \cos\left(\frac{4n\pi + \pi}{2}\right) + i\sin\left(\frac{4n\pi + \pi}{2}\right)$$

$$\Rightarrow (i)^{1/4} = \left[\cos\left(\frac{4n\pi + \pi}{2}\right) + i\sin\left(\frac{4n\pi + \pi}{2}\right)\right]^{1/4}$$

$$= \left[\cos\frac{1}{4}\left(\frac{4n\pi + \pi}{2}\right) + i\sin\frac{1}{4}\left(\frac{4n\pi + \pi}{2}\right)\right]$$

$$= \left[\cos\left(\frac{n\pi}{2} + \frac{\pi}{8}\right) + i\sin\left(\frac{n\pi}{2} + \frac{\pi}{8}\right)\right] \ldots\ldots\ldots\ldots (1)$$

Putting $n = 0, 1, 2, 3$, we get the required values as follows

$$\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}, \cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}, \cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8} \textit{ and } \cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}.$$

2. **Find all the values of $(1 + i)^{1/3}$ and represent these values on the Argand's Diagram.**

Solution: Let $1 + i = r(\cos\theta + i\sin\theta)$. Then, $r\cos\theta = 1 \textit{ and } r\sin\theta = 1$

On squaring and adding, we have $r^2 = 2$ i.e. $r = \sqrt{2}$.

On dividing, we have $\tan\theta = 1$ i.e. $\theta = \frac{\pi}{4}$.

$$\therefore 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left\{\cos\left(2n\pi + \frac{\pi}{4}\right) + i\sin\left(2n\pi + \frac{\pi}{4}\right)\right\}$$

$$\Rightarrow (1 + i)^{1/3} = \left[\sqrt{2}\left\{\cos\left(2n\pi + \frac{\pi}{4}\right) + i\sin\left(2n\pi + \frac{\pi}{4}\right)\right\}\right]^{1/3}$$

$$= 2^{1/6}\left[\cos\left(\frac{2n\pi}{3} + \frac{\pi}{12}\right) + i\sin\left(\frac{2n\pi}{3} + \frac{\pi}{12}\right)\right]$$

$$\Rightarrow (1 + i)^{1/3} = 2^{1/6}\left[\cos\left(\frac{8n\pi + \pi}{12}\right) + i\sin\left(\frac{8n\pi + \pi}{12}\right)\right]$$

On putting $n = 0, 1, 2$, we obtain

$$\alpha_0 = 2^{1/6}\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right]$$

$$\alpha_1 = 2^{1/6}\left[\cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}\right]$$

$$\alpha_2 = 2^{1/6}\left[\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right].$$

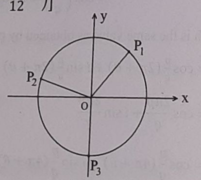

Fig. 1.6

On the Argand diagram (see Fig. 1.6), these three points $P_1, P_2, P_3 (say)$ will lie on a circle of radius $2^{1/6}$, each separated from the other two by an angle $\frac{2\pi}{3}$.

3. **Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that their continued product is unity.**

Solution: Let $\frac{1}{2} + i\frac{\sqrt{3}}{2} = r(\cos\theta + i\sin\theta)$

Then $r\cos\theta = \frac{1}{2}$ and $r\sin\theta = \frac{\sqrt{3}}{2}$, giving $r = 1$ and $\tan\theta = \sqrt{3}$

i.e. $\theta = \frac{\pi}{3}$, so that

$$\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right) = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \cos\left(2n\pi + \frac{\pi}{3}\right) + i\sin\left(2n\pi + \frac{\pi}{3}\right)$$

$$\Rightarrow \left\{\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right\}^{3/4} = \left[\cos\left(2n\pi + \frac{\pi}{3}\right) + i\sin\left(2n\pi + \frac{\pi}{3}\right)\right]^{3/4}$$

$$= \cos\frac{3}{4}\left(2n\pi + \frac{\pi}{3}\right) + i\sin\frac{3}{4}\left(2n\pi + \frac{\pi}{3}\right)$$

$$= \cos\left(\frac{6n\pi}{4} + \frac{\pi}{4}\right) + i\sin\left(\frac{6n\pi}{4} + \frac{\pi}{4}\right)$$

$$\Rightarrow \left\{\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right\}^{3/4} = \cos\left(\frac{3n\pi}{2} + \frac{\pi}{4}\right) + i\sin\left(\frac{3n\pi}{2} + \frac{\pi}{4}\right) \ldots\ldots\ldots\ldots (1)$$

Let us put $n = 0, 1, 2, 3$, to obtain the different values of (1) as

$$\alpha_0 = \left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

$$\alpha_1 = \left[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right]$$

$$\alpha_2 = \left[\cos\frac{13\pi}{4} + i\sin\frac{13\pi}{4}\right]$$

$$\alpha_3 = \left[\cos\frac{19\pi}{4} + i\sin\frac{19\pi}{4}\right].$$

Hence $\alpha_0\alpha_1\alpha_2\alpha_3 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)\left(\cos\frac{13\pi}{4} + i\sin\frac{13\pi}{4}\right)\left(\cos\frac{19\pi}{4} + i\sin\frac{19\pi}{4}\right)$

$$= \cos\left(\frac{\pi}{4} + \frac{7\pi}{4} + \frac{13\pi}{4} + \frac{19\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{7\pi}{4} + \frac{13\pi}{4} + \frac{19\pi}{4}\right)$$

$$= \cos 10\pi + i\sin 10\pi$$

$$= 1 + i(0)$$

$$= 1$$

4. **Prove that the n, n^{th} roots of unity form a series in geometric progression. Also, show that the sum of these n roots is zero and their product is $(-1)^{n-1}$.**

Solution: We can write $1 = \cos 0 + i\sin 0 = \cos 2r\pi + i\sin 2r\pi \quad \textit{where } r = 0, 1, 2, 3, \ldots\ldots, n - 1$

$$\Rightarrow (1)^{1/n} = \{\cos 2r\pi + i\sin 2r\pi\}^{1/n} = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n} \ldots\ldots\ldots (1)$$

Let us put $r = 0, 1, 2, 3, \ldots\ldots\ldots, n - 1$ in (1) to obtain the n roots as

$$\alpha_0 = \cos 0 + i\sin 0 = 1 + i(0) = 1$$

$$\alpha_1 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

$$\alpha_2 = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n} = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^2 = \alpha_1^2$$

$$\alpha_3 = \cos\frac{6\pi}{n} + i\sin\frac{6\pi}{n} = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^3 = \alpha_1^3$$

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$$\alpha_{n-1} = \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n} = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{n-1} = \alpha_1^{n-1}$$

Hence, the roots $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots, \alpha_{n-1}$ are in geometric progression.

The sum of these n roots is given by

$$1 + \alpha_1 + \alpha_1^2 + \alpha_1^3 + \cdots\ldots + \alpha_1^{n-1}$$

$$= \frac{a(1-r^n)}{1-r} = \frac{(1)(1-\alpha_1^n)}{1-\alpha_1} = \frac{1-\alpha_1^n}{1-\alpha_1} = \frac{1-\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n}{1-\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)}$$

$$= \frac{1-(\cos 2\pi + i\sin 2\pi)}{1-\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)}$$

$$= \frac{1-\{1+i(0)\}}{1-\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)} = 0$$

Also, the product of the roots is given by

$$\alpha_0\alpha_1\alpha_2\alpha_3 \ldots\ldots\ldots \alpha_{n-1} = (\alpha_1^0)(\alpha_1^1)(\alpha_1^2)(\alpha_1^3)\ldots\ldots\ldots(\alpha_1^{n-1})$$

$$= \alpha_1^{0+1+2+3+\cdots\ldots+n-1} = \alpha_1^{\frac{n(n-1)}{2}}$$

$$= \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{\frac{n(n-1)}{2}}$$

$$= \left\{\cos\left(\frac{n(n-1)}{2}\cdot\frac{2\pi}{n}\right) + i\sin\left(\frac{n(n-1)}{2}\cdot\frac{2\pi}{n}\right)\right\}$$

$$= \cos\frac{2\pi(n-1)}{2} + i\sin\frac{2\pi(n-1)}{2}$$

$$= \cos(n-1)\pi + i\sin(n-1)\pi = (-1)^{n-1}.$$

5. **Prove that the roots of the equation $x^2 - 2ax\cos\theta + a^2 = 0$ are also the roots of the equation $x^{2n} - 2a^n x^n \cos n\theta + a^{2n} = 0$, where n is a positive integer.**

Solution: It is sufficient to prove that $x^2 - 2ax\cos\theta + a^2$ is a factor of $x^{2n} - 2a^n x^n \cos n\theta + a^{2n} = 0$.

Let $x^n = y$, then $x^{2n} - 2a^n x^n \cos n\theta + a^{2n}$

$$= y^2 - 2a^n y\cos n\theta + a^{2n}$$

$$= (y - a^n\cos n\theta)^2 - a^{2n}\cos^2 n\theta + a^{2n}$$

$$= (y - a^n\cos n\theta)^2 + a^{2n}(1-\cos^2 n\theta)$$

$$= (y - a^n\cos n\theta)^2 - (ia^n\sin n\theta)^2$$

$$= (y - a^n\cos n\theta + ia^n\sin n\theta)(y - a^n\cos n\theta - ia^n\sin n\theta)$$

Now $y - a^n\cos n\theta - ia^n\sin n\theta = x^n - a^n(\cos n\theta + i\sin n\theta)$

$$= x^n - a^n(\cos\theta + i\sin\theta)^n$$

which is of the form $x^n - \lambda^n$ and thus contains $x - \lambda$ i.e. $x - a(\cos\theta + i\sin\theta)$ as a factor.

Similarly, the expression $y - a^n\cos n\theta + ia^n\sin n\theta$ contains $x - a(\cos\theta - i\sin\theta)$ as a factor.

The product of these two factors is given by

$$\{x - a(\cos\theta + i\sin\theta)\}\{x - a(\cos\theta - i\sin\theta)\} = (x - a\cos\theta)^2 + a^2\sin^2\theta$$

$$= x^2 + a^2 - 2ax\cos\theta$$

Thus, the problem is solved.

6. **Find the cube roots of unity.**

Solution: Let us consider the cubic equation $x^3 = 1$. Its roots are obviously the cube roots of unity.

We write this equation as $x^3 - 1 = 0$

$$\Rightarrow (x-1)(x^2+x+1) = 0$$

$$\Rightarrow x = 1, x^2+x+1 = 0$$

But $x^2 + x + 1 = 0 \Rightarrow x = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm\sqrt{3}i}{2}$

$$\Rightarrow x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

Thus, we have three cube roots of unity, viz. 1(real) and $\frac{-1\pm\sqrt{3}i}{2}$ (complex).

Let $\omega = \frac{-1+\sqrt{3}i}{2}$, then $\omega^2 = \left(\frac{-1+\sqrt{3}i}{2}\right)^2 = \frac{1+3i^2-2\sqrt{3}i}{4}$

$$= \frac{-2-2\sqrt{3}i}{4}$$

$$= \frac{-1-\sqrt{3}i}{2}$$

which is the third root that is complex in nature.

Similarly, if we take $\omega = \frac{-1-\sqrt{3}i}{2}$, then $\omega^2 = \frac{-1+\sqrt{3}i}{2}$.

Thus, the cube roots of unity are $1, \omega, \omega^2$.

We may note that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.

7. **If ω is a complex cube-root of unity, prove that $(1-\omega)^6 = -27$.**

Solution: We know that ω is a complex root of the equation

$$x^3 - 1 = 0 \Rightarrow (x-1)(x^2+x+1) = 0$$

So, ω is a root of $x^2 + x + 1 = 0$, so that

$$\omega^2 + \omega + 1 = 0 \ldots\ldots\ldots\ldots\ldots (1)$$

$$and \quad \omega^3 = 1 \ldots\ldots\ldots\ldots\ldots (2)$$

Now, $(1-\omega)^2 = 1 + \omega^2 - 2\omega = -\omega - 2\omega = -3\omega$ [From (1)]

$\Rightarrow (1-\omega)^6 = (-3\omega)^3 = -27\omega^3 = -27$ [From (2)].

8. Show that the roots of $z^5 = 1$ can be written as $1, \alpha, \alpha^2, \alpha^3, \alpha^4$. Hence prove that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$.

Solution: We have $z^5 = 1 = \cos 0 + i \sin 0$

$$= \cos(2n\pi + 0) + i\sin(2n\pi + 0)$$

$$= \cos 2n\pi + i \sin 2n\pi$$

$$\Rightarrow z = (\cos 2n\pi + i \sin 2n\pi)^{1/5} = \cos\frac{2n\pi}{5} + i \sin\frac{2n\pi}{5}$$

Setting $n = 0, 1, 2, 3, 4$ in succession, the required roots are

$$z_0 = \cos 0 + i \sin 0 = 1$$

$$z_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \alpha(say)$$

$$z_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^2 = \alpha^2$$

$$z_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^3 = \alpha^3$$

$$z_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^4 = \alpha^4$$

Hence, the required roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$.

For the next part, we notice that since $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ are the roots of the equation $z^5 - 1 = 0$, hence

$$z^5 - 1 = (z-1)(z-\alpha)(z-\alpha^2)(z-\alpha^3)(z-\alpha^4)$$

$$\Rightarrow (z-\alpha)(z-\alpha^2)(z-\alpha^3)(z-\alpha^4) = \frac{z^5-1}{z-1} = z^4 + z^3 + z^2 + z + 1$$

Setting $z = 1$ in this result, we obtain

$$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = (1)^4 + (1)^3 + (1)^2 + (1) + 1 = 5.$$

9. Prove that $\sqrt[n]{a+bi} + \sqrt[n]{a-bi}$ has n real values and hence find those of $\sqrt[3]{1+i\sqrt{3}} + \sqrt[3]{1-i\sqrt{3}}$.

Solution: Let $a = r\cos\theta$ and $b = r\sin\theta$, then

$$r = \sqrt{a^2+b^2} \text{ and } \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\therefore a + bi = r(\cos\theta + i\sin\theta) = r[\cos(2m\pi+\theta) + i\sin(2m\pi+\theta)]$$

$$\Rightarrow (a+bi)^{1/n} = r^{1/n}\left[\cos\left(\frac{2m\pi+\theta}{n}\right) + i\sin\left(\frac{2m\pi+\theta}{n}\right)\right] \dots\dots\dots (1)$$

where $= 0, 1, 2, 3, \dots\dots, n-1$.

From (1), we have therefore

$$(a-bi)^{1/n} = r^{1/n}\left[\cos\left(\frac{2m\pi+\theta}{n}\right) - i\sin\left(\frac{2m\pi+\theta}{n}\right)\right] \dots\dots\dots (2)$$

From (1) and (2), we have

$$(a+bi)^{1/n} + (a-bi)^{1/n} = 2r^{1/n}\left[\cos\left(\frac{2m\pi+\theta}{n}\right)\right] \dots\dots\dots (3)$$

where $= 0, 1, 2, 3, \dots\dots, n-1$.

Hence, $\sqrt[3]{1+i\sqrt{3}} + \sqrt[3]{1-i\sqrt{3}} = 2(2)^{1/3}\cos\left(\frac{2m\pi+\frac{\pi}{3}}{3}\right) = 2^{4/3}\cos\left(\frac{2m\pi}{3}+\frac{\pi}{9}\right)$

Hence three roots of the expressions are: $2^{4/3}\cos\frac{\pi}{9}, 2^{4/3}\cos\frac{7\pi}{9}, 2^{4/3}\cos\frac{13\pi}{9}$.

10. Use De-Moivre's theorem to solve $x^8 + x^5 + x^3 + 1 = 0$.

Solution: The given equation is $x^8 + x^5 + x^3 + 1 = 0$

$$\Rightarrow (x^5+1)(x^3+1) = 0$$

$$\Rightarrow x^5 + 1 = 0, x^3 + 1 = 0$$

Now $x^5 + 1 = 0 \Rightarrow x^5 = -1 = \cos\pi + i\sin\pi = \cos(2n\pi+\pi) + i\sin(2n\pi+\pi)$

$$\therefore x = (-1)^{1/5} = \{\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)\}^{1/5}$$

$$\text{or, } x = (-1)^{1/5} = \cos\left(\frac{2n\pi+\pi}{5}\right) + i\sin\left(\frac{2n\pi+\pi}{5}\right)$$

Putting $n = 0, 1, 2, 3, 4$, we get the roots of $x^5 + 1 = 0$ as

$$\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}, \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}, \cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5}, \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}, \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$$

i.e. $\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}, \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}, \cos\pi + i\sin\pi, \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}, \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$

i.e. $\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}, \cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}, \cos\pi + i\sin\pi$

Similarly, $x^3 + 1 = 0 \Rightarrow x^3 = -1 = \cos\pi + i\sin\pi = \cos(2n\pi+\pi) + i\sin(2n\pi+\pi)$

$$x = (-1)^{1/3} = \cos\left(\frac{2n\pi+\pi}{3}\right) + i\sin\left(\frac{2n\pi+\pi}{3}\right)$$

where $= 0, 1, 2$.

Hence, the roots of $x^3 + 1 = 0$ are

$$\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}, \cos\pi + i\sin\pi, \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

i.e. $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}, \cos\pi + i\sin\pi, \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$

or that $\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}, \cos\pi + i\sin\pi$.

11. Solve the equation $z^3 = (z+1)^3$ and hence show that $z = -\frac{1}{2} + \frac{i}{2}\cot\left(\frac{\theta}{2}\right)$ where $= \frac{2n\pi}{3}$.

Solution: We have here $z^3 = (z+1)^3$

i.e. $\left(\frac{z}{z+1}\right)^3 = 1 = \cos 0 + i\sin 0 = \cos 2n\pi + i\sin 2n\pi$

$$\Rightarrow \frac{z}{z+1} = (\cos 2n\pi + i\sin 2n\pi)^{1/3} = \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$$

$$= \cos\theta + i\sin\theta, \text{ where } = \frac{2n\pi}{3}.$$

$$\Rightarrow \frac{z}{(z+1)-z} = \frac{\cos\theta + i\sin\theta}{1 - (\cos\theta + i\sin\theta)}$$

$$\Rightarrow z = \frac{\cos\theta + i\sin\theta}{1 - \cos\theta - i\sin\theta} = \frac{\cos\theta + i\sin\theta}{2\sin^2\frac{\theta}{2} - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{\cos\theta + i\sin\theta}{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)}$$

$$= \frac{(\cos\theta + i\sin\theta)\left(\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$= \frac{\cos\theta\sin\frac{\theta}{2} + i\cos\theta\cos\frac{\theta}{2} + i\sin\theta\sin\frac{\theta}{2} - \sin\theta\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

$$= \frac{\left(\cos\theta\sin\frac{\theta}{2} - \sin\theta\cos\frac{\theta}{2}\right) + i\left(\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$= \frac{\sin\left(\frac{\theta}{2} - \theta\right) + i\cos\left(\theta - \frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}} = \frac{-\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

$$= -\frac{1}{2} + \frac{1}{2}i\cot\frac{\theta}{2}.$$

12. Show that all the roots of $(x+1)^6 + (x-1)^6 = 0$ are given by $-i\cot\frac{(2k+1)\pi}{12}$, $k = 0, 1, 2, 3, 4, 5$.

Solution: Given that $(x+1)^6 + (x-1)^6 = 0$,

we have $\frac{(x+1)^6}{(x-1)^6} = -1 = \cos\pi + i\sin\pi = \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$

$$\Rightarrow \left(\frac{x+1}{x-1}\right)^6 = \cos(2k+1)\pi + i\sin(2k+1)\pi$$

$$\Rightarrow \frac{x+1}{x-1} = [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/6}$$

$$= \cos\frac{(2k+1)\pi}{6} + i\sin\frac{(2k+1)\pi}{6}\ ; k = 0, 1, 2, 3, 4, 5$$

Let $\frac{(2k+1)\pi}{6} = \theta$, then $\frac{x+1}{x-1} = \cos\theta + i\sin\theta = \frac{\cos\theta + i\sin\theta}{1}$

$$\Rightarrow \frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{1+\cos\theta + i\sin\theta}{\cos\theta + i\sin\theta - 1} = \frac{2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{-2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$\Rightarrow \frac{2x}{2} = \frac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}\left(-\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right)}$$

$$= \frac{\cot\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\left(-\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)}{\left(-\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right)\left(-\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)}$$

$$= \cot\frac{\theta}{2}\left(-\sin\frac{\theta}{2}\cos\frac{\theta}{2} - i\cos^2\frac{\theta}{2} - i\sin^2\frac{\theta}{2} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$= \cot\frac{\theta}{2}(-i) = -i\cot\frac{\theta}{2} = -i\cot\frac{(2k+1)\pi}{12}.$$

13. Solve the equation $x^6 - i = 0$.

Solution: $x^6 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$$= \cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right), \qquad \text{where } k = 0, 1, 2, 3, 4, 5.$$

$$= \cos\left(\frac{4k\pi + \pi}{2}\right) + i\sin\left(\frac{4k\pi + \pi}{2}\right)$$

$$\Rightarrow x = \left[\cos\left(\frac{4k\pi + \pi}{2}\right) + i\sin\left(\frac{4k\pi + \pi}{2}\right)\right]^{\frac{1}{6}}$$

$$= \cos\left(\frac{4k\pi + \pi}{12}\right) + i\sin\left(\frac{4k\pi + \pi}{12}\right) \ \ldots\ldots\ldots \text{Using De} - \text{Moivre's theorem}$$

Putting $k = 0, 1, 2, 3, 4, 5$, we get all the six roots of the given equation in the following manner:

$$k = 0;\ x_1 = \cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)$$

$$k = 1;\ x_2 = \cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)$$

$$k = 2;\ x_3 = \cos\left(\frac{9\pi}{12}\right) + i\sin\left(\frac{9\pi}{12}\right)$$

$$k = 3;\ x_4 = \cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right) = \cos\left(\pi + \frac{\pi}{12}\right) + i\sin\left(\pi + \frac{\pi}{12}\right) = -\cos\left(\frac{\pi}{12}\right) - i\sin\left(\frac{\pi}{12}\right)$$

$$k = 4;\ x_5 = \cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right) = \cos\left(\pi + \frac{5\pi}{12}\right) + i\sin\left(\pi + \frac{5\pi}{12}\right) = -\cos\left(\frac{5\pi}{12}\right) - i\sin\left(\frac{5\pi}{12}\right)$$

$$k = 5;\ x_6 = \cos\left(\frac{21\pi}{12}\right) + i\sin\left(\frac{21\pi}{12}\right) = \cos\left(\pi + \frac{9\pi}{12}\right) + i\sin\left(\pi + \frac{9\pi}{12}\right) = -\cos\left(\frac{9\pi}{12}\right) - i\sin\left(\frac{9\pi}{12}\right)$$

14. Solve the equation $z^3 = (z+1)^3$, then show that $z = -\frac{1}{2} + \frac{i}{2}\cot\left(\frac{\theta}{2}\right)$, where $\theta = 2n\frac{\pi}{3}$.

Solution: We have $\left(\frac{z}{z+1}\right)^3 = 1 = \cos 0 + i\sin 0 = \cos 2n\pi + i\sin 2n\pi$

$$\Rightarrow \left(\frac{z}{z+1}\right) = (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{3}}$$

$$= \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$$

$$= \cos\theta + i\sin\theta, \qquad \text{where } \theta = 2n\frac{\pi}{3}$$

Subtracting numerator from denominator on both sides, we get

$$\frac{z}{z+1-z} = \frac{\cos\theta + i\sin\theta}{1 - \cos\theta - i\sin\theta}$$

$$\Rightarrow z = \frac{\cos\theta + i\sin\theta}{2\sin^2\left(\frac{\theta}{2}\right) - 2i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow z = \frac{\cos\theta + i\sin\theta}{2\sin\left(\frac{\theta}{2}\right)\left[\sin\left(\frac{\theta}{2}\right) - i\cos\left(\frac{\theta}{2}\right)\right]} \cdot \frac{\left[\sin\left(\frac{\theta}{2}\right) + i\cos\left(\frac{\theta}{2}\right)\right]}{\left[\sin\left(\frac{\theta}{2}\right) + i\cos\left(\frac{\theta}{2}\right)\right]}$$

$$\Rightarrow z = \frac{(\cos\theta + i\sin\theta)\left[\sin\left(\frac{\theta}{2}\right) + i\cos\left(\frac{\theta}{2}\right)\right]}{2\sin\left(\frac{\theta}{2}\right)\left[\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)\right]}$$

$$\Rightarrow z = \frac{\cos\theta \sin\left(\frac{\theta}{2}\right) + i\cos\theta\cos\left(\frac{\theta}{2}\right) + i\sin\theta\sin\left(\frac{\theta}{2}\right) - \sin\theta\cos\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{-\left[\sin\theta\cos\left(\frac{\theta}{2}\right) - \cos\theta\sin\left(\frac{\theta}{2}\right)\right] + i\left[\cos\theta\cos\left(\frac{\theta}{2}\right) + \sin\theta\sin\left(\frac{\theta}{2}\right)\right]}{2\sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{-\sin\left(\frac{\theta}{2}\right) + i\cos\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} = -\frac{1}{2} + \frac{i}{2}\cot\left(\frac{\theta}{2}\right), \textit{ where } \theta = 2n\frac{\pi}{3}.$$

15. Solve the equation $x^7 + x^4 + i(x^3 + 1) = 0$.

Solution: The given equation is

$$x^7 + x^4 + i(x^3 + 1) = 0$$
$$\Rightarrow x^4(x^3 + 1) + i(x^3 + 1) = 0$$
$$\Rightarrow (x^4 + i)(x^3 + 1) = 0.$$

Thus, the roots of given equation are the roots of $(x^4 + i) = 0$ and $(x^3 + 1) = 0$.

Considering $(x^4 + i) = 0 \Rightarrow x^4 = -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

$$x^4 = \cos\left(2k\pi + \frac{\pi}{2}\right) - i\sin\left(2k\pi + \frac{\pi}{2}\right), \quad \textit{where } k = 0, 1, 2, 3,$$

$$x = \left[\cos\left(2k\pi + \frac{\pi}{2}\right) - i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{\frac{1}{4}}$$

$$= \left[\cos\frac{(4k+1)\pi}{8} - i\sin\frac{(4k+1)\pi}{8}\right]$$

Putting $k = 0, 1, 2, 3$, we get all the roots of equation $(x^4 + i) = 0$ in the following manner:

$$k = 0, \quad x_1 = \cos\left(\frac{\pi}{8}\right) - i\sin\left(\frac{\pi}{8}\right)$$

$$k = 1, \quad x_2 = \cos\left(\frac{5\pi}{8}\right) - i\sin\left(\frac{5\pi}{8}\right)$$

$$k = 2, \quad x_3 = \cos\left(\frac{9\pi}{8}\right) - i\sin\left(\frac{9\pi}{8}\right)$$

$$k = 3, \quad x_4 = \cos\left(\frac{13\pi}{8}\right) - i\sin\left(\frac{13\pi}{8}\right)$$

Now, $(x^3 + 1) = 0 \Rightarrow x^3 = -1 = \cos\pi + i\sin\pi$

$$\Rightarrow x^3 = \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$$
$$\Rightarrow x = [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{3}}$$
$$\Rightarrow x = \left[\cos\frac{(2k\pi + \pi)}{3} + i\sin\frac{(2k\pi + \pi)}{3}\right]$$

Putting $k = 0, 1, 2$, we get all the three roots of the equation $(x^3 + 1) = 0$ in the following manner:

$$k = 0, \quad x_5 = \cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$k = 1, \quad x_6 = \cos\pi - i\sin\pi = -1$$

$$k = 2, \quad x_7 = \cos\left(\frac{5\pi}{3}\right) - i\sin\left(\frac{9\pi}{8}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

16. Show that all the roots of $(x + 1)^6 + (x - 1)^6 = 0$ are given by $-i\cot\frac{(2k+1)\pi}{12}$, $k = 0, 1, 2, 3, 4, 5$.

Solution: We have $(x + 1)^6 = -(x - 1)^6$

$$\Rightarrow \left(\frac{x+1}{x-1}\right)^6 = -1 = \cos\pi + i\sin\pi$$

$$\Rightarrow \left(\frac{x+1}{x-1}\right)^6 = \cos(2k+1)\pi + i\sin(2k+1)\pi = e^{(2k+1)\pi i}$$

$$\Rightarrow \frac{x+1}{x-1} = e^{i(2k+1)\pi/6}; \; k = 0, 1, 2, 3, 4, 5$$

Let $\frac{(2k+1)\pi}{6} = \theta$, then

$$\frac{x+1}{x-1} = e^{i\theta} = \cos\theta + i\sin\theta = \frac{\cos\theta + i\sin\theta}{1}$$

$$\Rightarrow \frac{x+1+x-1}{x-1-x-1} = \frac{\cos\theta + i\sin\theta + 1}{1 - \cos\theta - i\sin\theta}$$

$$\Rightarrow \frac{x}{-1} = \frac{\cos\theta + i\sin\theta + 1}{1 - \cos\theta - i\sin\theta} = \frac{2\cos^2\left(\frac{\theta}{2}\right) + 2i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right) - 2i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} = \frac{\cos\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right]}{\sin\left(\frac{\theta}{2}\right)\left[\sin\left(\frac{\theta}{2}\right) - i\cos\left(\frac{\theta}{2}\right)\right]}$$

$$\Rightarrow \frac{x}{-1} = \cot\left(\frac{\theta}{2}\right)\frac{\left[\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right]}{\left[\sin\left(\frac{\theta}{2}\right) - i\cos\left(\frac{\theta}{2}\right)\right]}$$

$$\Rightarrow x = -\cot\left(\frac{\theta}{2}\right)\frac{\left[\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right]\left[\sin\left(\frac{\theta}{2}\right) + i\cos\left(\frac{\theta}{2}\right)\right]}{\left[\sin\left(\frac{\theta}{2}\right) - i\cos\left(\frac{\theta}{2}\right)\right]\left[\sin\left(\frac{\theta}{2}\right) + i\cos\left(\frac{\theta}{2}\right)\right]}$$

$$\Rightarrow x = -\cot\left(\frac{\theta}{2}\right)\frac{\left[\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) + i\sin^2\left(\frac{\theta}{2}\right) + i\cos^2\left(\frac{\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\right]}{\left[\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)\right]}$$

$$\Rightarrow x = -i\cot\left(\frac{\theta}{2}\right) = -i\cot\left[\frac{(2k+1)\pi}{12}\right], \qquad k = 0, 1, 2, 3, 4, 5.$$

17. Solve the equation $x^9 + x^5 - x^4 - 1 = 0$.

Solution: We have $x^5(x^4 + 1) - 1(x^4 + 1) = 0$

$$\Rightarrow (x^5 - 1)(x^4 + 1) = 0$$

Thus, the roots of given equation are the roots of $(x^4 + 1) = 0$ and $(x^5 - 1) = 0$.

Now, considering $(x^4 + 1) = 0 \Rightarrow x^4 = -1 = \cos\pi + i\sin\pi$

$$\Rightarrow x^4 = \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$$
$$\Rightarrow x = [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{4}}$$
$$\Rightarrow x = \left[\cos\frac{(2k\pi + \pi)}{4} + i\sin\frac{(2k\pi + \pi)}{4}\right]$$

Putting $k = 0, 1, 2, 3$, we get all the four roots of the equation $(x^4 + 1) = 0$ in the following manner:

$$k = 0, \quad x_1 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$$

$$k = 1, \quad x_2 = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$$

$k = 2,\quad x_3 = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)$

$k = 3,\quad x_4 = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)$

Now, $(x^5 - 1) = 0 \Rightarrow x^5 = 1 = \cos 0 + i\sin 0$

$$\Rightarrow x^5 = \cos 2k\pi + i\sin 2k\pi$$

$$\Rightarrow x = [\cos 2k\pi + i\sin 2k\pi]^{\frac{1}{5}}$$

$$\Rightarrow x = \left[\cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)\right]$$

Putting $k = 0, 1, 2, 3, 4$, we get all the five roots of the equation $(x^5 - 1) = 0$ in the following manner:

$k = 0,\quad x_5 = \cos 0 + i\sin 0$

$k = 1,\quad x_6 = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$

$k = 2,\quad x_7 = \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{4}\right)$

$k = 3,\quad x_8 = \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right)$

$k = 4,\quad x_9 = \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right).$

18. Solve the equation $x^{10} + 11x^5 + 10 = 0$.

Solution: Let $y = x^5$, then

$$y^2 + 11y + 10 = 0$$

$$\Rightarrow (y + 10)(y + 1) = 0$$

$$\Rightarrow y = -10, -1$$

Now, $x^5 = -10 = 10(\cos\pi + i\sin\pi)$

$$\Rightarrow x = 10^{\frac{1}{5}}(\cos\pi + i\sin\pi)^{\frac{1}{5}}$$

$$\Rightarrow x = 10^{\frac{1}{5}}[\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{5}}$$

$$\Rightarrow x = 10^{\frac{1}{5}}\left[\cos\frac{(2k+1)\pi}{5} + i\sin\frac{(2k+1)\pi}{5}\right]$$

Putting $k = 0, 1, 2, 3, 4$, we get all the five roots of the equation $x^5 + 10 = 0$.

Now, consider $x^5 = -1 = \cos\pi + i\sin\pi$

$$\Rightarrow x^5 = \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$$

$$\Rightarrow x = [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{5}}$$

$$\Rightarrow x = \left[\cos\frac{(2k+1)\pi}{5} + i\sin\frac{(2k+1)\pi}{5}\right]$$

Putting $k = 0, 1, 2, 3, 4$, we get all the five roots of the equation $x^5 + 1 = 0$.

19. Show that all the roots of $(x + 1)^7 = (x - 1)^7$ are given by $\pm i\cot\frac{k\pi}{7}$, $k = 1, 2, 3 \ldots$

Solution: We have $(x + 1)^7 = (x - 1)^7$

$$\Rightarrow \left(\frac{x+1}{x-1}\right)^7 = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$$

$$\Rightarrow \frac{x+1}{x-1} = (\cos 2k\pi + i\sin 2k\pi)^{1/7}$$

$$\Rightarrow \frac{x+1}{x-1} = \cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7} = e^{i\frac{2k\pi}{7}}, \qquad \textit{where } k = 0, 1, 2, 3, 4 \ldots$$

For $k = 0$, $\quad \frac{x+1}{x-1} = \cos 0 + i\sin 0 = 1$

$$\Rightarrow x + 1 = x - 1 \text{ or } 1 = -1, \qquad \text{which is an absurd result.}$$

Hence, $k \neq 0$

$\therefore$ k takes values $1, 2, 3, 4, 5, 6$

$$\frac{x+1}{x-1} = e^{i\frac{2k\pi}{7}} = \frac{e^{i\theta}}{1}, \qquad \text{where } \theta = \frac{2k\pi}{7}$$

$$\Rightarrow \frac{2x}{2} = \frac{e^{i\theta} + 1}{e^{i\theta} - 1}, \qquad \text{by applying componendo \& } \textit{dividendo method}$$

$$\Rightarrow x = \frac{(e^{i\theta} + 1)e^{-i\theta/2}}{(e^{i\theta} - 1)e^{-i\theta/2}}$$

$$\Rightarrow x = \frac{e^{i\theta/2} + e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} = \frac{(e^{i\theta/2} + e^{-i\theta/2})/2}{i(e^{i\theta/2} + e^{-i\theta/2})/2i}$$

$$\Rightarrow x = \frac{2\cos\frac{\theta}{2}}{2i\sin\frac{\theta}{2}}$$

$$\Rightarrow x = -i\cot\frac{\theta}{2} \quad \left[\frac{1}{i} = \frac{i}{i^2} = -i\right]$$

$$\Rightarrow x = -i\cot\frac{k\pi}{7}$$

Putting $k = 1, 2, 3, 4, 5, 6$, the roots of $(x + 1)^7 = (x - 1)^7$ are, therefore,

$k = 1,\quad x_1 = -i\cot\frac{\pi}{7}$

$k = 2,\quad x_2 = -i\cot\frac{2\pi}{7}$

$k = 3,\quad x_3 = -i\cot\frac{3\pi}{7}$

$k = 4,\quad x_4 = -i\cot\frac{4\pi}{7} = -i\cot\left(\pi - \frac{3\pi}{7}\right) = i\cot\frac{3\pi}{7} = \overline{x_3}$

$k = 5,\quad x_5 = -i\cot\frac{5\pi}{7} = -i\cot\left(\pi - \frac{2\pi}{7}\right) = i\cot\frac{2\pi}{7} = \overline{x_2}$

$k = 6,\quad x_6 = -i\cot\frac{6\pi}{7} = -i\cot\left(\pi - \frac{\pi}{7}\right) = i\cot\frac{\pi}{7} = \overline{x_1}$

Hence all the roots of $(x + 1)^7 = (x - 1)^7$ are given by $\pm i\cot\frac{k\pi}{7}$, *where* $k = 1, 2, 3.$

20. If $\alpha, \alpha^2, \alpha^3, \alpha^4$ are roots of $x^5 - 1 = 0$, then show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$.

Solution: Let us consider the equation $x^5 - 1 = 0$, giving

$$x^5 = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$$

$$\Rightarrow x = (\cos 2k\pi + i\sin 2k\pi)^{1/5}$$

$$\Rightarrow x = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$$

Putting $k = 0, 1, 2, 3, 4$, we get all the five roots in the following manner:

$$k = 0, \quad x_1 = 1$$

$$k = 1, \quad x_2 = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) = \alpha$$

$$k = 2, \quad x_3 = \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{4}\right) = \alpha^2$$

$$k = 3, \quad x_4 = \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right) = \alpha^3$$

$$k = 4, \quad x_5 = \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right) = \alpha^4$$

$$\therefore x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\Rightarrow (x-1)(x^4 + x^3 + x^2 + x + 1) = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\Rightarrow x^4 + x^3 + x^2 + x + 1 = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

Substitute $x = 1$ on both the side to obtain

$$1 + 1 + 1 + 1 + 1 = (1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4)$$

$$\Rightarrow (1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5.$$

21. If w is 7^{th} root of unity, prove that $S = 1 + w^n + w^{2n} + w^{3n} + w^{4n} + w^{5n} + w^{6n} = 7$, when n is a multiple of 7 and is equal to zero otherwise.

Solution: Let $x = (1)^{1/7} = (\cos 0 + i\sin 0)^{1/7}$, then

$$x = (\cos 2k\pi + i\sin 2k\pi)^{1/7}$$

$$\Rightarrow x = \cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}, \quad \text{where } k = 0, 1, 2, 3, 4, 5, 6$$

Putting $k = 0, 1, 2, 3, 4, 5, 6$, we find that the 7^{th} roots of unity are $1, w, w^2, w^3, w^4, w^5, w^6$,

$$\text{where } w = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} \ldots\ldots\ldots (1)$$

$$\text{Let } S = 1 + w^n + w^{2n} + w^{3n} + w^{4n} + w^{5n} + w^{6n} = \frac{1 - w^{7n}}{1 - w^n} \ldots\ldots\ldots (2)$$

[$\because$ above series is in G.P. with common ratio w]

(i) When n is not a multiple of 7

$$w^{7n} = (w^7)^n = \left[\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7\right]^n = 1$$

$$\Rightarrow 1 - w^{7n} = 0$$

But $1 - w^n \neq 0$, as n is not a multiple of 7

$\therefore$ The sum $S = 0$, if n is not a multiple of 7.

(ii) If n is a multiple of 7, say $n = 7k$, then

$$S = 1 + (w^7)^k + (w^7)^{2k} + (w^7)^{3k} + (w^7)^{4k} + (w^7)^{5k} + (w^7)^{6k} = 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7.$$

EXERCISE - 1.4

1. Solve the equation $x^6 + 1 = 0$.

   [Ans.: $x_0 = \frac{\sqrt{3}}{2} + \frac{i}{2}, x_1 = i, x_2 = -\frac{\sqrt{3}}{2} + \frac{i}{2}, x_3 = -\frac{\sqrt{3}}{2} - \frac{i}{2}, x_4 = -i, x_5 = \frac{\sqrt{3}}{2} - \frac{i}{2}$]

2. Solve the equation $x^6 - i = 0$.

   [Ans.: $x_0 = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}, x_1 = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}, x_2 = \cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}, x_3 = -\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}, x_4 = -\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12}, x_5 = -\cos\frac{9\pi}{12} - i\sin\frac{9\pi}{12}$]

3. Solve the equation $x^7 + x^4 + i(x^3 + 1) = 0$.

   [Ans.: $x_1 = \cos\frac{\pi}{8} - i\sin\frac{\pi}{8}, x_2 = \cos\frac{5\pi}{8} - i\sin\frac{5\pi}{8}, x_3 = \cos\frac{9\pi}{8} - i\sin\frac{9\pi}{8}, x_4 = \cos\frac{13\pi}{8} - i\sin\frac{13\pi}{8}$ are the four roots of the equation $x^4 + i = 0$, $x_5 = \frac{1}{2} + \frac{i\sqrt{3}}{2}, x_6 = -1, x_7 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ are the three roots of $x^3 + 1 = 0$]

4. Find all the values of $\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{3/4}$ and hence show that their continued product is 1.

   [Ans.: Four roots are $\alpha_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}, \alpha_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}, \alpha_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}, \alpha_4 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$]

5. If ω is the 7^{th} root of unity, prove that $S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n} = 7$ when n is a multiple of 7 and is equal to zero otherwise.

6. Find the cube roots of unity and hence show that they form equilateral triangle in the Argand diagram.

   [Ans.: Three roots are $1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$]

7. Use De-Moivre's theorem to solve the equation $x^4 - x^3 + x^2 - x + 1 = 0$.

   [Ans.: Four roots are $\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}, \cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}$]

8. Show that the roots of the equation $(x-1)^n = x^n$, n being a positive integer, are $\frac{1}{2}\left(1 + i\cot\frac{r\pi}{n}\right)$ where $r = 1, 2, 3, \ldots\ldots\ldots, n-1$.

1.18 Exponential function of a complex variable

When x is real, we have

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \ldots\ldots + \frac{x^n}{n!} + \cdots \ldots\ldots \infty \ldots\ldots\ldots (1)$$

On the same lines, we define the exponential function of the complex variable $z = x + iy$ as

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \ldots\ldots + \frac{z^n}{n!} + \cdots \ldots\ldots \infty \ldots\ldots\ldots (2)$$

From (1), we have

$$e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots \ldots\ldots\ldots$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)$$

$$= \cos y + i \sin y$$

Thus, $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$.

Also, $x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

Thus, $z = re^{i\theta}$

Also, $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

$$= e^x\{\cos(2n\pi + y) + i\sin(2n\pi + y)\} = e^x e^{i(2n\pi + y)}$$

$$= e^x e^{2n\pi i} e^{iy} = e^{x+iy+2n\pi i}$$

$$= e^{z+2n\pi i}$$

Thus, the value of e^z remains unchanged if z is increased by any integral multiple of $2\pi i$. Hence, the period of e^z is $2\pi i$.

1.19 Circular functions of Complex Numbers

We have proved that

$$e^{iy} = \cos y + i \sin y \text{ ----- } (1)$$

$$e^{-iy} = \cos y - i \sin y \text{ ------ } (2)$$

From (1) and (2), we have

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}, \sin y = \frac{e^{iy} - e^{-iy}}{2i},$$

which are known as Euler's exponential formulae for circular functions $\cos y$ and $\sin y$.

Similarly, if z is a complex number, we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

The other circular functions of z are defined by

$$\tan z = \frac{\sin z}{\cos z}, \cot z = \frac{\cos z}{\sin z}, \sec z = \frac{1}{\cos z}, cosec\, z = \frac{1}{\sin z}.$$

Using these definitions, it can be easily verified that these circular functions of the complex number obey all the formulae / identities satisfied by circular functions of real numbers; i.e.,

$$\sin^2 z + \cos^2 z = 1$$

$$\sin 2z = 2 \sin z \cos z$$

$$\cos 2z = 2\cos^2 z - 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2.$$

1.20 Periodicity of Circular Functions

Here $\sin(z + 2n\pi) = \sin z \cos 2n\pi + \cos z \sin 2n\pi = \sin z$

$\cos(z + 2n\pi) = \cos z \cos 2n\pi - \sin z \sin 2n\pi = \cos z$.

From these, we conclude that $\sin z$ and $\cos z$ are periodic functions with period 2π. Further, $\tan(z + \pi) = \tan z$. Hence, $\tan z$ is a periodic function with period .

1.21 Hyperbolic Functions

If z is a complex number, then sine hyperbolic of z is denoted by $\sinh z$ and is given by

$$\sinh z = \frac{e^z - e^{-z}}{2}.$$

Similarly, the cosine hyperbolic of z is denoted as $\cosh z$ and is defined as

$$\cosh z = \frac{e^z + e^{-z}}{2}.$$

The other hyperbolic functions are

$$\tanh z = \frac{\sinh z}{\cosh z}, \text{sech}\, z = \frac{1}{\cosh z}, \text{cosech}\, z = \frac{1}{\sinh z}, \coth z = \frac{1}{\tanh z}.$$

1.22 Relation between Circular and Hyperbolic Functions

We know that $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$.

Let $\theta \Rightarrow ix$, then $\sin ix = \frac{e^{-x} - e^{x}}{2i} = -\frac{e^{x} - e^{-x}}{2i}, \cos ix = \frac{e^{-x} + e^{x}}{2}$

$$\Rightarrow \sin ix = i^2\left[\frac{e^x - e^{-x}}{2i}\right], \cos ix = \cosh x$$

$$\Rightarrow \sin ix = i \sinh x, \cos ix = \cosh x$$

$$\therefore \tan ix = i \tanh x$$

$$\sinh ix = i \sin x$$

$$\cosh ix = \cos x$$

$$\tanh ix = i \tan x.$$

1.23 Formulae of Hyperbolic Functions

(a). Fundamental Formulae

(1) $\cosh^2 x - \sinh^2 x = 1$

(2) $\text{sech}^2 x + \tanh^2 x = 1$

(3) $\coth^2 x - cosech^2 x = 1$.

cosh = $\frac{e^\theta + e^{-\theta}}{2}$

sinh $\frac{e^\theta - e^{-\theta}}{2}$

(b). Addition Formulae

(4) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

(5) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(6) $\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$.

(c). Functions of 2x

(7) $\sinh 2x = 2 \sinh x \cosh x$

(8) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

(9) $\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$.

(d). Functions of 3x

(10) $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$

(11) $\cosh 3x = 4\cosh^3 x - 3 \cosh x$

(12) $\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$.

(e). Some useful Formulae

(13) $\sinh x + \sinh y = 2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2}$

(14) $\sinh x - \sinh y = 2\cosh\frac{x+y}{2}\sinh\frac{x-y}{2}$

(15) $\cosh x + \cosh y = 2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2}$

(16) $\cosh x - \cosh y = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}$.

Proofs: (1) We know that

$$\cos^2\theta + \sin^2\theta = 1 \ldots\ldots\ldots\ldots (1)$$

Let $\theta \to ix$, then we get from (1)

$$\cos^2 ix + \sin^2 ix = 1$$

or, $(\cos ix)^2 + (\sin ix)^2 = 1$

or, $(\cosh x)^2 + (i\sinh x)^2 = 1$

or, $\cosh^2 x - \sinh^2 x = 1$

Similarly, we can establish the formulae (2) and (3).

(4) $\sinh(x \pm y) = \frac{1}{i}\sin i(x \pm y) = -i\sin(ix \pm iy)$

$= -i\{\sin ix\cos iy \pm \cos ix\sin iy\}$

$= -i\{i\sinh x\cosh y \pm \cosh x\,(i\sinh y)\}$

$= \sinh x\cosh y \pm \cosh x\sinh y$

Similarly, we can establish the formulae (5) and (6).

(7) We know that

$$\sin 2\theta = 2\sin\theta\cos\theta \ldots\ldots\ldots\ldots (2)$$

Let $\theta \to ix$, then from (2)

$\sin 2ix = 2\sin ix\cos ix$

$\Rightarrow \sin i(2x) = 2\sin ix\cos ix$

$\Rightarrow i\sinh 2x = 2i\sinh x\cosh x$

$\Rightarrow \sinh 2x = 2\sinh x\cosh x$

Similarly, we can establish the formulae (8) and (9).

(10) We know that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \ldots\ldots\ldots\ldots (3)$$

Let $\theta \to ix$, then

$\sin 3ix = 3\sin ix - 4\sin^3 ix$

$\Rightarrow \sin i(3x) = 3\sin ix - 4\sin^3 ix$

$\Rightarrow i\sinh 3x = 3i\sinh x - 4(i\sinh x)^3$

$\Rightarrow i\sinh 3x = 3i\sinh x + 4i\sinh^3 x$

$\Rightarrow \sinh 3x = 3\sinh x + 4\sinh^3 x$

Similarly, we can establish the formulae (11) and (12) also.

(13) We know that

$$\sin\theta + \sin\phi = 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} \ldots\ldots\ldots (4)$$

Let $\theta \to ix$ and $\phi \to iy$ in (4) to obtain

$$\sin ix + \sin iy = 2\sin\frac{ix+iy}{2}\cos\frac{ix-iy}{2}$$

$$\Rightarrow i\sinh x + i\sinh y = 2\sin i\left(\frac{x+y}{2}\right)\cos i\left(\frac{x-y}{2}\right) = 2i\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$$

$$\Rightarrow \sinh x + \sinh y = 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$$

Similarly, we can establish the formulae (14) (15) and (16).

1.24 Solved Examples

(1) If $\log\tan x = y$, prove that $\sinh ny = \frac{1}{2}(\tan^n x - \cot^n x)$, and $\cosh(n+1)y + \cosh(n-1)y = 2\cosh ny\ cosec2x$.

Solution: Given that $y = \log\tan x$, we have $e^y = \tan x \Rightarrow e^{-y} = \cot x$

$$\therefore \sinh ny = \frac{e^{ny} - e^{-ny}}{2} = \frac{\tan^n x - \cot^n x}{2}$$

Also, $\cosh(n+1)y + \cosh(n-1)y = 2\cosh ny\cosh y$

$$= 2\cosh ny\left(\frac{e^y + e^{-y}}{2}\right)$$

$$= \cosh ny(\tan x + \cot x)$$

$$= \cosh ny\left(\frac{\sin^2 x + \cos^2 x}{\sin x\cos x}\right)$$

$$= \frac{2\cosh ny}{2\sin x\cos x} = 2\cosh ny\ cosec2x.$$

(2) Prove that $\left(\frac{1+\tanh x}{1-\tanh x}\right)^3 = \cosh 6x + \sinh 6x$.

Solution: $\left(\frac{1+\tanh x}{1-\tanh x}\right)^3 = \left[\frac{\cosh x + \sinh x}{\cosh x - \sinh x}\right]^3 = \left[\frac{\frac{e^x+e^{-x}}{2} + \frac{e^x-e^{-x}}{2}}{\frac{e^x+e^{-x}}{2} - \frac{e^x-e^{-x}}{2}}\right]^3$

$$= \left[\frac{e^x}{e^{-x}}\right]^3 = e^{6x}$$

$$= \frac{e^{6x} + e^{-6x}}{2} + \frac{e^{6x} - e^{-6x}}{2}$$

$$= \cosh 6x + \sinh 6x.$$

(3) Solve the equation $7\cosh x + 8\sinh x = 1$ for real values of .

Solution: Here $7\cosh x + 8\sinh x = 1$

or, $7\left(\frac{e^x + e^{-x}}{2}\right) + 8\left(\frac{e^x - e^{-x}}{2}\right) = 1$

or, $15e^x - e^{-x} = 2$

or, $15e^{2x} - 1 = 2e^x$

or, $15e^{2x} - 2e^{x} - 1 = 0$,

which is quadratic in e^x. Hence

$$e^x = \frac{2 \pm \sqrt{4+60}}{30} = \frac{2 \pm 8}{30} = \frac{10}{30}, \frac{-6}{30}$$

i.e. $e^x = \frac{1}{3}, -\frac{1}{5}$.

(4) If $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, prove that

(a). $\cosh u = \sec\theta$

(b). $\sinh u = \tan\theta$

(c). $\tanh u = \sin\theta$

(d). $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$.

Solution: Given that $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$,

we have $\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = e^u$

$$\Rightarrow e^u = \frac{\tan\frac{\pi}{4} + \tan\frac{\theta}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{\theta}{2}} = \frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}$$

$$= \frac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)} = \frac{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}$$

$$\Rightarrow e^u = \frac{1 + \sin\theta}{\cos\theta} = \sec\theta + \tan\theta$$

$$\Rightarrow e^{-u} = \frac{1}{\sec\theta + \tan\theta} = \frac{\sec\theta - \tan\theta}{\sec^2\theta - \tan^2\theta} = \sec\theta - \tan\theta$$

(a). $\cosh u = \dfrac{e^u + e^{-u}}{2} = \dfrac{(\sec\theta + \tan\theta) + (\sec\theta - \tan\theta)}{2} = \sec\theta$

(b). $\sinh u = \dfrac{e^u - e^{-u}}{2} = \dfrac{(\sec\theta + \tan\theta) - (\sec\theta - \tan\theta)}{2} = \tan\theta$

(c). $\tanh u = \dfrac{\sinh u}{\cosh u} = \dfrac{\tan\theta}{\sec\theta} = \sin\theta$

(d). $\tanh\dfrac{u}{2} = \dfrac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \dfrac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} \cdot \dfrac{e^{u/2}}{e^{u/2}} = \dfrac{e^u - 1}{e^u + 1}$

$$= \frac{\frac{1+\sin\theta}{\cos\theta} - 1}{\frac{1+\sin\theta}{\cos\theta} + 1} = \frac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta}$$

$$= \frac{1 + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \left(1 - 2\sin^2\frac{\theta}{2}\right)}{1 + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \left(2\cos^2\frac{\theta}{2} - 1\right)}$$

$$= \frac{2\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)} = \tan\frac{\theta}{2}.$$

(5) If $\tan\frac{x}{2} = \tanh\frac{u}{2}$, then show that $u = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$.

Solution: Given that $\tanh\frac{u}{2} = \tan\frac{x}{2}$,

we have $\dfrac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \dfrac{\tan\frac{x}{2}}{1}$

or, $\dfrac{\left(e^{u/2} - e^{-u/2}\right) + \left(e^{u/2} + e^{-u/2}\right)}{\left(e^{u/2} + e^{-u/2}\right) - \left(e^{u/2} - e^{-u/2}\right)} = \dfrac{\tan\frac{x}{2} + 1}{1 - \tan\frac{x}{2}}$

or, $\dfrac{2e^{u/2}}{2e^{-u/2}} = \dfrac{\sin\frac{x}{2} + \cos\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}} = \dfrac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}} = \dfrac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}}$

or, $e^u = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

or, $u = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$.

(6) If $\tanh x = \frac{2}{3}$, find the value of x and then $\cosh 2x$.

Solution: Given that $\tanh x = \frac{2}{3}$,

we have $\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{2}{3}$

or, $\dfrac{(e^x - e^{-x}) + (e^x + e^{-x})}{(e^x + e^{-x}) - (e^x - e^{-x})} = \dfrac{2+3}{3-2} = 5$

or, $\dfrac{2e^x}{2e^{-x}} = 5$

or, $e^{2x} = 5$ ……… (1)

or, $2x = \log_e 5$

or, $x = \frac{1}{2}\log_e 5$

From (1), we have $e^{-2x} = \frac{1}{5}$

$\therefore \cosh 2x = \dfrac{e^{2x} + e^{-2x}}{2} = \dfrac{5 + \frac{1}{5}}{2} = \dfrac{26}{10} = \dfrac{13}{5}$.

$\cosh^{-1} x$ $\log\left[x + \sqrt{x^2 - 1}\right]$

(7) If $\cosh x = \sec\theta$, prove that $x = \log(\sec\theta + \tan\theta)$.

Solution: We have $\cosh x = \sec\theta$

$\Rightarrow x = \cosh^{-1}(\sec\theta)$

$\Rightarrow x = \log\left[\sec\theta + \sqrt{\sec^2\theta - 1}\right]$

$\Rightarrow x = \log(\sec\theta + \tan\theta)$.

(8) If $\cos(x+iy)\cos(u+iv) = 1$ where x, y, u and v are real, show that $\tanh^2 y \cosh^2 v = \sin^2 u$.

Solution: We have $\cos(x+iy)\cos(u+iv) = 1$

$\Rightarrow \cos(x+iy) = \sec(u+iv)$

$\therefore \sin(x+iy) = \sqrt{1 - \cos^2(x+iy)} = \sqrt{1 - \sec^2(u+iv)} = \sqrt{-\tan^2(u+iv)}$

$= i\tan(u+iv)$

$$\therefore \tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{i\tan(u+iv)}{\sec(u+iv)} = i\sin(u+iv)$$

Similarly, $\tan(x-iy) = -i\sin(u-iv)$

$$\therefore \tan 2iy = \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$= \frac{i\sin(u+iv)-\{-i\sin(u-iv)\}}{1+\{i\sin(u+iv)\}\{-i\sin(u-iv)\}}$$

$$= \frac{i\sin(u+iv)+i\sin(u-iv)}{1+\sin(u+iv)\sin(u-iv)}$$

$$\Rightarrow i\tanh 2y = \frac{2i\sin u\cos iv}{1+\frac{1}{2}\{\cos 2iv - \cos 2u\}}$$

$$\Rightarrow \tanh 2y = \frac{2\sin u\cosh v}{1+\frac{1}{2}\{\cosh 2v - \cos 2u\}}$$

$$= \frac{2\sin u\cosh v}{1+\frac{1}{2}\{2\cosh^2 v - 1 - (1-2\sin^2 u)\}}$$

$$= \frac{2\sin u\cosh v}{1+\{\cosh^2 v - 1 + \sin^2 u\}}$$

$$= \frac{2\sin u\cosh v}{1+\cosh^2 v - \cos^2 u}$$

$$= \frac{2\sin u\cosh v}{\sin^2 u + \cosh^2 v}$$

$$\Rightarrow \frac{2\tanh y}{1+\tanh^2 y} = \frac{2\sin u\cosh v/\cosh^2 v}{(\sin^2 u + \cosh^2 v)/\cosh^2 v} = \frac{2\sin u/\cosh v}{1+\sin^2 u/\cosh^2 v}$$

$$\Rightarrow \frac{2\tanh y}{2\sin u/\cosh v} = \frac{1+\tanh^2 y}{1+\sin^2 u/\cosh^2 v} = k(say)$$

Hence $\dfrac{\tanh y}{\sin u/\cosh v} = k$, gives $\tanh^2 y = k^2\left(\dfrac{\sin^2 u}{\cosh^2 v}\right)$

$\Rightarrow \tanh^2 y\cosh^2 v = k^2\sin^2 u$

$\Rightarrow \tanh^2 y\cosh^2 v = \sin^2 u$ (Taking $k^2 = 1$).

EXERCISE - 1.5

1. If $\tanh x = \frac{1}{2}$, find the value of x and $\sinh 2x$.
2. Prove that $(\cosh x - \sinh x)^n = \cosh nx - \sinh nx$.
3. Find the value of $\tanh(\log\sqrt{5})$.
4. Prove that $16\cosh^5 x = \cosh 5x + 5\cosh 3x + 10\cosh x$.

1.25 Real and Imaginary parts of Circular and Hyperbolic Functions

(1) To separate the real and imaginary parts of

(a) $\sin(x+iy)$ **(b)** $\cos(x+iy)$ **(c)** $\tan(x+iy)$

(d) $\cot(x+iy)$ **(e)** $\sec(x+iy)$ **(f)** $\text{cosec}(x+iy)$.

Solution: (a) $\sin(x+iy) = \sin x\cos iy + \cos x\sin iy$

$= \sin x\cosh y + \cos x\,(i\sinh y)$

$= \sin x\cosh y + i\cos x\sinh y$

$\therefore$ Real part of $\sin(x+iy) = \sin x\cosh y$

Imaginary part of $\sin(x+iy) = \cos x\sinh y$

(b) $\cos(x+iy) = \cos x\cos iy - \sin x\sin iy$

$= \cos x\cosh y - \sin x\,(i\sinh x)$

$= \cos x\cosh y - i\sin x\sinh x$

$\therefore$ Real part of $\cos(x+iy) = \cos x\cosh y$

Imaginary part of $\cos(x+iy) = -\sin x\sinh x$

(c) Let $\alpha + i\beta = \tan(x+iy)$, then $\alpha - i\beta = \tan(x-iy)$

On adding, we have

$$2\alpha = \tan(x+iy)+\tan(x-iy) = \frac{\sin(x+iy)}{\cos(x+iy)} + \frac{\sin(x-iy)}{\cos(x-iy)}$$

$$= \frac{\sin(x+iy)\cos(x-iy)+\sin(x-iy)\cos(x+iy)}{\cos(x+iy)\cos(x-iy)}$$

$$= \frac{\sin 2x}{\cos(x+iy)\cos(x-iy)} = \frac{2\sin 2x}{2\cos(x+iy)\cos(x-iy)}$$

$$= \frac{2\sin 2x}{\cos 2x + \cos 2iy} = \frac{2\sin 2x}{\cos 2x + \cosh 2y}$$

$$\Rightarrow \alpha = \frac{\sin 2x}{\cos 2x + \cosh 2y} = \text{Real part of } \tan(x+iy)$$

On subtracting, we have

$$2i\beta = \tan(x+iy) - \tan(x-iy)$$

$$= \frac{\sin(x+iy)}{\cos(x+iy)} - \frac{\sin(x-iy)}{\cos(x-iy)}$$

$$= \frac{\sin(x+iy)\cos(x-iy)-\sin(x-iy)\cos(x+iy)}{\cos(x+iy)\cos(x-iy)}$$

$$= \frac{2\sin 2iy}{2\cos(x+iy)\cos(x-iy)} = \frac{2i\sinh 2y}{\cos 2x + \cos(i2y)}$$

$$= \frac{2i\sinh 2y}{\cos 2x + \cosh(2y)}$$

$$\Rightarrow \beta = \frac{\sinh 2y}{\cos 2x + \cosh(2y)} = \text{Imaginary part of } \tan(x+iy)$$

(d) Let $\alpha + i\beta = \cot(x+iy)$, then $\alpha - i\beta = \cot(x-iy)$

$$\Rightarrow 2\alpha = \cot(x+iy) + \cot(x-iy) = \frac{\cos(x+iy)}{\sin(x+iy)} + \frac{\cos(x-iy)}{\sin(x-iy)}$$

$$= \frac{\cos(x+iy)\sin(x-iy) + \cos(x-iy)\sin(x+iy)}{\sin(x+iy)\sin(x-iy)}$$

$$\Rightarrow \alpha = \frac{\sin 2x}{2\sin(x+iy)\sin(x-iy)}$$

$$\Rightarrow \alpha = \frac{\sin 2x}{\cos(2iy) - \cos(2x)} = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

Also, $$2i\beta = \cot(x+iy) - \cot(x-iy) = \frac{\cos(x+iy)}{\sin(x+iy)} - \frac{\cos(x-iy)}{\sin(x-iy)}$$

$$= \frac{\cos(x+iy)\sin(x-iy) - \cos(x-iy)\sin(x+iy)}{\sin(x+iy)\sin(x-iy)}$$

$$\Rightarrow i\beta = \frac{\sin(-2iy)}{2\sin(x+iy)\sin(x-iy)} = \frac{-i\sinh 2y}{\cos 2iy - \cos 2x}$$

$$= \frac{-i\sinh 2y}{\cosh 2y - \cos 2x}$$

$$\Rightarrow \beta = \frac{-\sinh 2y}{\cosh 2y - \cos 2x}$$

(e) Let $\alpha + i\beta = \sec(x+iy)$, then $\alpha - i\beta = \sec(x-iy)$

$$\Rightarrow 2\alpha = \sec(x+iy) + \sec(x-iy) = \frac{\cos(x-iy) + \cos(x+iy)}{\cos(x+iy)\cos(x-iy)}$$

$$\text{or,}\quad \alpha = \frac{2\cos x\cos iy}{\cos 2x + \cosh 2y}$$

Also, $$2i\beta = \sec(x+iy) - \sec(x-iy) = \frac{\cos(x-iy) - \cos(x+iy)}{\cos(x-iy)\cos(x+iy)}$$

$$i\beta = \frac{2\sin x\sin iy}{\cos 2x + \cosh 2y} = \frac{2\sin x\,(i\sinh y)}{\cos 2x + \cosh 2y}$$

$$\Rightarrow \beta = \frac{2\sin x\sinh y}{\cos 2x + \cosh 2y}$$

(f) Let $\alpha + i\beta = \operatorname{cosec}(x+iy)$, then $\alpha - i\beta = \operatorname{cosec}(x-iy)$

$$\Rightarrow 2\alpha = \operatorname{cosec}(x+iy) + \operatorname{cosec}(x-iy) = \frac{\sin(x-iy) + \sin(x+iy)}{\sin(x+iy)\sin(x-iy)}$$

$$\Rightarrow \alpha = \frac{2\sin x\cos hy}{\cosh 2y - \cos 2x}$$

Also, $$2i\beta = \operatorname{cosec}(x+iy) - \operatorname{cosec}(x-iy) = \frac{\sin(x-iy) - \sin(x+iy)}{\sin(x+iy)\sin(x-iy)}$$

$$\text{or,}\quad i\beta = \frac{-2\cos x\sin iy}{\cosh 2y - \cos 2x} = \frac{-2\cos x\,(i\sinh y)}{\cosh 2y - \cos 2x}$$

$$\Rightarrow \beta = \frac{-2\cos x\sinh y}{\cosh 2y - \cos 2x}$$

(2) To separate the real and imaginary parts of

(a). $\sinh(x+iy)$ **(b). $\cosh(x+iy)$** **(c). $\tanh(x+iy)$**

Solution: (a). $\sinh(x+iy) = \frac{1}{i}\sin i(x+iy) = -i\sin(ix-y)$

$$= -i\{\sin ix\cos y - \cos ix\sin y\}$$

$$= -i\{i\sinh x\cos y - \cosh x\sin y\}$$

$$= \sinh x\cos y + i\cosh x\sin y$$

(b). $\text{Cosh}(x+iy) = \cos i(x+iy) = \cos(ix-y)$

$$= \cos ix\cos y + \sin ix\sin y$$

$$= \cosh x\cos y + i\sinh x\sin y$$

(c). Let $\alpha + i\beta = \tanh(x+iy) = \frac{1}{i}\tan i(x+iy) = -i\tan(ix-y)$, then

$$\alpha - i\beta = -(-i)\tan(-ix-y) = -i\tan(ix+y)$$

$$\therefore 2\alpha = -i\tan(ix-y) - i\tan(ix+y) = -i\frac{\sin(ix-y)}{\cos(ix-y)} - \frac{i\sin(ix+y)}{\cos(ix+y)}$$

$$= -i\left[\frac{\sin(ix-y)}{\cos(ix-y)} + \frac{\sin(ix+y)}{\cos(ix+y)}\right]$$

$$\Rightarrow \alpha = -i\left[\frac{\sin(ix-y)\cos(ix+y) + \cos(ix-y)\sin(ix+y)}{2\cos(ix-y)\cos(ix+y)}\right]$$

$$= -i\left[\frac{\sin 2ix}{\cos 2ix + \cos 2y}\right] = \frac{-i(i\sinh 2x)}{\cosh 2x + \cos 2y}$$

$$\Rightarrow \alpha = \frac{\sinh 2x}{\cosh 2x + \cos 2y}$$

Also, $2i\beta = -i\tan(ix-y) + i\tan(ix+y)$

$$= i[\tan(ix+y) - \tan(ix-y)]$$

$$\Rightarrow \beta = \frac{1}{2}\left[\frac{\sin(ix+y)}{\cos(ix+y)} - \frac{\sin(ix-y)}{\cos(ix-y)}\right]$$

$$= \frac{1}{2}\left[\frac{\sin(ix+y)\cos(ix-y) - \sin(ix-y)\cos(ix+y)}{\cos(ix+y)\cos(ix-y)}\right]$$

$$\Rightarrow \beta = \frac{\sin 2y}{\cosh 2x + \cos 2y}$$

1.26 Solved Examples

1. If $\sin(A+iB) = x+iy$, then prove that (i) $\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$ (ii) $\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$.

Solution: Here $x + iy = \sin(A+iB) = \sin A\cos iB + \cos A\sin iB$

$$= \sin A\cosh B + \cos A(i\sinh B)$$

$$= \sin A\cosh B + i(\cos A\sinh B)$$

$$\Rightarrow \sin A\cosh B = x$$

$$\cos A\sinh B = y$$

$$\Rightarrow \frac{x}{\cosh B} = \sin A \ \ and \ \frac{y}{\sinh B} = \cos A$$

$$\Rightarrow \frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \sin^2 A + \cos^2 A = 1$$

Also, $$\frac{x}{\sin A} = \cosh B \ \ and \ \frac{y}{\cos A} = \sinh B$$

$$\Rightarrow \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B = 1$$

2. If $\cosh(u + iv) = x + iy$, prove that (i) $\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1$ (ii) $\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1$.

Solution: Given that $\cosh(u + iv) = x + iy$,

we have $\cos i(u + iv) = x + iy$

$$\Rightarrow \cos(iu - v) = x + iy$$

$$\Rightarrow \cos iu \cos v + \sin iu \sin v = x + iy$$

$$\Rightarrow \cosh u \cos v + i \sinh u \sin v = x + iy$$

$$\Rightarrow x = \cosh u \cos v \ , y = + \sinh u \sin v$$

$$\Rightarrow \left(\frac{x}{\cosh u}\right)^2 = \cos^2 v \ \ and \ \left(\frac{y}{\sinh u}\right)^2 = \sin^2 v$$

$$\therefore \frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = \cos^2 v + \sin^2 v = 1$$

Similarly, $$\left(\frac{x}{\cos v}\right)^2 - \left(\frac{y}{\sin v}\right)^2 = \cosh^2 u - \sinh^2 u = 1$$

$$\Rightarrow \frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1.$$

3. If $\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$, prove that $\cos^2\theta = \pm\sin\alpha$.

$$\Rightarrow \sin\theta \cos i\phi + \cos\theta \sin i\phi = \cos\alpha + i\sin\alpha$$

$$\Rightarrow \sin\theta \cosh\phi + \cos\theta(i\sinh\phi) = \cos\alpha + i\sin\alpha$$

$$\Rightarrow \sin\theta\cosh\phi = \cos\alpha \ \ and \ \cos\theta\sinh\phi = \sin\alpha$$

$$\therefore \cosh\phi = \frac{\cos\alpha}{\sin\theta} \ \ and \ \sinh\phi = \frac{\sin\alpha}{\cos\theta}$$

$$\Rightarrow \cosh^2\phi - \sinh^2\phi = \frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta}$$

$$\Rightarrow 1 = \frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta} = \frac{\cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta}{\sin^2\theta\cos^2\theta}$$

$$\Rightarrow \sin^2\theta\cos^2\theta = \cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta$$

$$\Rightarrow (1 - \cos^2\theta)\cos^2\theta = \cos^2\alpha\cos^2\theta - \sin^2\alpha(1 - \cos^2\theta)$$

$$\Rightarrow \cos^2\theta - \cos^4\theta = \cos^2\alpha\cos^2\theta - \sin^2\alpha + \sin^2\alpha\cos^2\theta$$

$$\Rightarrow \cos^4\theta + (\sin^2\alpha + \cos^2\alpha - 1)\cos^2\theta - \sin^2\alpha = 0$$

$$\Rightarrow \cos^4\theta = \sin^2\alpha$$

$$\therefore \cos^2\theta = \pm\sin\alpha.$$

4. If $\tan(A + iB) = x + iy$, prove that (i) $\tan 2A = \frac{2x}{1-x^2-y^2}$ (ii) $\tanh 2B = \frac{2y}{1+x^2+y^2}$.

Solution: Let $\tan(A + iB) = x + iy$,

Then $\tan(A - iB) = x - iy$

$$\therefore \tan 2A = \tan[(A + iB) + (A - iB)] = \frac{\tan(A + iB) + \tan(A - iB)}{1 - \tan(A + iB)\tan(A - iB)}$$

$$= \frac{(x + iy) + (x - iy)}{1 - (x + iy)(x - iy)} = \frac{2x}{1 - (x^2 + y^2)} = \frac{2x}{1 - x^2 - y^2}$$

Also, $$\tan 2iB = \tan[(A + iB) - (A - iB)] = \frac{\tan(A + iB) - \tan(A - iB)}{1 + \tan(A + iB)\tan(A - iB)}$$

or, $$i\tanh 2B = \frac{(x + iy) - (x - iy)}{1 + (x + iy)(x - iy)} = \frac{2iy}{1 + x^2 + y^2}$$

or, $$\tanh 2B = \frac{2y}{1 + x^2 + y^2}.$$

5. If $\cos(\theta + i\phi) = Re^{i\alpha}$, show that $\phi = \frac{1}{2}\log_e \frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$.

Solution: Here $\cos(\theta + i\phi) = Re^{i\alpha}$

gives $\cos\theta\cos i\phi - \sin\theta\sin i\phi = R(\cos\alpha + i\sin\alpha)$

$$\Rightarrow \cos\theta\cosh\phi - \sin\theta\,(i\sinh\phi) = R\cos\alpha + iR\sin\alpha$$

$$\Rightarrow R\cos\alpha = \cos\theta\cosh\phi \ \ and \ R\sin\alpha = -\sin\theta\sinh\phi$$

$$\therefore \frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{-\sin\theta\sinh\phi}{\cos\theta\cosh\phi} = -\tan\theta\tanh\phi$$

$$\text{or, } \tanh\phi = -\frac{\tan\alpha}{\tan\theta} \Rightarrow \frac{e^{\phi} - e^{-\phi}}{e^{\phi} + e^{-\phi}} = -\frac{\sin\alpha\cos\theta}{\cos\alpha\sin\theta}$$

$$\Rightarrow \frac{(e^{\phi} - e^{-\phi}) + (e^{\phi} + e^{-\phi})}{(e^{\phi} + e^{-\phi}) - (e^{\phi} - e^{-\phi})} = \frac{-\sin\alpha\cos\theta + \cos\alpha\sin\theta}{\cos\alpha\sin\theta + \sin\alpha\cos\theta}$$

$$\Rightarrow \frac{2e^{\phi}}{2e^{-\phi}} = \frac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)}$$

$$\Rightarrow e^{2\phi} = \frac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)}$$

$$\Rightarrow \phi = \frac{1}{2}\log_e \frac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)}.$$

6. If $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$, prove that (i) $\theta = \frac{n\pi}{2} + \frac{\pi}{4}$ (ii) $\phi = \frac{1}{2}\log_e \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$.

Solution: Given that $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$,

we have $\tan(\theta - i\phi) = \cos\alpha - i\sin\alpha$

$$\therefore \tan 2\theta = \tan\{\overline{\theta + i\phi} + \overline{\theta - i\phi}\} = \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

$$= \frac{(\cos\alpha + i\sin\alpha) + (\cos\alpha - i\sin\alpha)}{1 - (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

$$= \frac{2\cos\alpha}{1 - (\cos^2\alpha + \sin^2\alpha)} = \tan\frac{\pi}{2}$$

$\therefore \quad 2\theta = \dfrac{\pi}{2}$

or, $\quad 2\theta = n\pi + \dfrac{\pi}{2}$ *(general value)*

or, $\quad \theta = \dfrac{n\pi}{2} + \dfrac{\pi}{4}$

Similarly,

$$\tan(2i\phi) = \tan\{\theta + i\phi - \overline{\theta - i\phi}\}$$

or, $\quad i\tanh(2\phi) = \dfrac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)} = \dfrac{2i\sin\alpha}{1 + \cos^2\alpha + \sin^2\alpha} = i\sin\alpha$

or, $\quad \tanh(2\phi) = \sin\alpha$

or, $\quad \dfrac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \dfrac{\sin\alpha}{1}$

or, $\quad \dfrac{(e^{2\phi} - e^{-2\phi}) + (e^{2\phi} + e^{-2\phi})}{(e^{2\phi} + e^{-2\phi}) - (e^{2\phi} - e^{-2\phi})} = \dfrac{1 + \sin\alpha}{1 - \sin\alpha}$

or, $\quad \dfrac{2e^{2\phi}}{2e^{-2\phi}} = \dfrac{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$

or, $\quad e^{4\phi} = \left(\dfrac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}\right)^2$

or, $\quad e^{2\phi} = \dfrac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}} = \dfrac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}} = \dfrac{\tan\frac{\pi}{4} + \tan\frac{\alpha}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{\alpha}{2}}$

or, $\quad e^{2\phi} = \tan\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right)$

or, $\quad \phi = \dfrac{1}{2}\log_e \tan\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right).$

7. Separate $\tan^{-1}(x + iy)$ into real and imaginary parts.

Solution: Let $\alpha + i\beta = \tan^{-1}(x + iy)$, then $\alpha - i\beta = \tan^{-1}(x - iy)$

$\therefore 2\alpha = \tan^{-1}(x + iy) + \tan^{-1}(x - iy)$

$= \tan^{-1}\left\{\dfrac{(x + iy) + (x - iy)}{1 - (x + iy)(x - iy)}\right\} = \tan^{-1}\left\{\dfrac{2x}{1 - (x^2 + y^2)}\right\}$

or, $\quad \alpha = \dfrac{1}{2}\tan^{-1}\left(\dfrac{2x}{1 - x^2 - y^2}\right)$

Also, $\quad 2i\beta = \tan^{-1}(x + iy) - \tan^{-1}(x - iy)$

$= \tan^{-1}\left\{\dfrac{(x + iy) - (x - iy)}{1 + (x + iy)(x - iy)}\right\}$

$= \tan^{-1}\left\{\dfrac{2iy}{1 + x^2 + y^2}\right\} = i\tanh^{-1}\left(\dfrac{2y}{1 + x^2 + y^2}\right)$

$\Rightarrow \beta = \dfrac{1}{2}\tanh^{-1}\left(\dfrac{2y}{1 + x^2 + y^2}\right).$

8. Separate $\sin^{-1}(\cos\theta + i\sin\theta)$ into real and imaginary parts, where θ is positive acute angle.

Solution: Let $\sin^{-1}(\cos\theta + i\sin\theta) = x + iy$, then

$\cos\theta + i\sin\theta = \sin(x + iy) = \sin x\cos iy + \cos x\sin iy$

or, $\quad \cos\theta + i\sin\theta = \sin x\cosh y + \cos x\,(i\sinh y)$

$\Rightarrow \cos\theta = \sin x\cosh y\ ,\ \sin\theta = \cos x\sinh y \ldots\ldots\ldots (1)$

Squaring and adding, we have

$1 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y = \sin^2 x\,(1 + \sinh^2 y) + \cos^2 x\sinh^2 y$

or, $\quad 1 = \sin^2 x + \sinh^2 y$

or, $\quad 1 - \sin^2 x = \sinh^2 y$

or, $\quad \cos^2 x = \sinh^2 y$

sin²x + sin²x sinh²y + cos²x sinh²y

sin²x + sinh²y [sin²x + cos²x]

From (1), we have

$\sin^2\theta = \cos^2 x\sinh^2 y = \cos^2 x\,(\cos^2 x)$

$\therefore \quad \cos^4 x = \sin^2\theta$

or, $\quad \cos^2 x = \sin\theta, \quad [\because \theta$ is a positive acute angle$]$

As x is to be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, therefore, we have

$\cos x = \sqrt{\sin\theta}$

or, $\quad x = \cos^{-1}\sqrt{\sin\theta}$

sin²x(1+sinh²y)

Also, $\sin\theta = \cos x\sinh y = \sqrt{\sin\theta}\sinh y$

$\Rightarrow \sinh y = \sqrt{\sin\theta}$

$\Rightarrow y = \sinh^{-1}\sqrt{\sin\theta}.$

1.27 Inverse hyperbolic function

If $x = \sinh u$, then $u = \sinh^{-1} x$ is called sine hyperbolic inverse of x where x is real. Similarly, we can define $\cosh^{-1} x$, $\tanh^{-1} x$, $\coth^{-1} x$, $\operatorname{sech}^{-1} x$ and $\operatorname{cosech}^{-1} x$.

If x is real, we have

(i) $\sinh^{-1} x = \log(x + \sqrt{x^2 + 1})$

(ii) $\cosh^{-1} x = \log(x + \sqrt{x^2 - 1})$

(iii) $\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Proof (i): Let $x = \sinh y$, then $x = \dfrac{e^y - e^{-y}}{2}$

$\Rightarrow 2x = e^y - \dfrac{1}{e^y}$

$\Rightarrow e^{2y} - 2xe^y - 1 = 0$

This equation is quadratic in e^y. So, we have

$e^y = \dfrac{2x \pm \sqrt{4x^2 + 4}}{2}$

$\Rightarrow e^y = x \pm \sqrt{x^2 + 1}$

$\Rightarrow \quad y = \log\left(x \pm \sqrt{x^2 + 1}\right)$

Since $\log\left(x-\sqrt{x^2+1}\right)$ is not defined as $x-\sqrt{x^2+1}<0$, we have

$$y = \log\left(x+\sqrt{x^2+1}\right)$$

$$\Rightarrow \sinh^{-1} x = \log\left(x+\sqrt{x^2+1}\right)$$

(ii) The proof of (ii) is similar to the proof of (i).

(iii) Let $x = \tanh y$, then $\quad \tanh^{-1} x = y$

Also, $\quad x = \tanh y \Rightarrow \frac{x}{1} = \frac{\frac{e^y-e^{-y}}{2}}{\frac{e^y+e^{-y}}{2}}$

$$\Rightarrow \frac{x}{1} = \frac{e^y-e^{-y}}{e^y+e^{-y}}$$

Applying componendo & dividendo, we obtain

$$\frac{1+x}{1-x} = \frac{2e^y}{2e^{-y}}$$

$$\Rightarrow \frac{1+x}{1-x} = e^{2y}$$

$$\Rightarrow 2y = \log\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow \tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

1.28 Solved Problems

1. Prove that $\sinh^{-1}(\tan\theta) = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$.

Solution: We have $\sinh^{-1}(\tan\theta) = \log\left(\tan\theta+\sqrt{\tan^2\theta+1}\right)$

$$= \log(\tan\theta+\sec\theta) = \log\left(\frac{\sin\theta+1}{\cos\theta}\right)$$

$$= \log\left[\frac{\cos\left(\frac{\pi}{2}-\theta\right)+1}{\sin\left(\frac{\pi}{2}-\theta\right)}\right] = \log\left[\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right]$$

$$= \log\left[\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right] = \log\tan\left[\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$$= \log\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right].$$

2. Prove that (i) $\tanh^{-1} x = \sinh^{-1}\frac{x}{\sqrt{1-x^2}}$, and (ii) $\sinh^{-1} x = \cosh^{-1}\left(\sqrt{1+x^2}\right)$.

Solution: (i) $\sinh^{-1}\frac{x}{\sqrt{1-x^2}} = \log\left(\frac{x}{\sqrt{1-x^2}}+\sqrt{\frac{x^2}{1-x^2}+1}\right)$

$$= \log\left(\frac{x}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-x^2}}\right) = \log\left(\frac{x+1}{\sqrt{1-x^2}}\right)$$

$$= \log\left(\frac{\sqrt{x+1}\sqrt{x+1}}{\sqrt{1+x}\sqrt{1-x}}\right) = \log\left(\frac{\sqrt{x+1}}{\sqrt{1-x}}\right)$$

$$= \frac{1}{2}\log\left(\frac{x+1}{1-x}\right) = \tanh^{-1} x$$

(ii) $\cosh^{-1}\left(\sqrt{(1+x^2)}\right) = \log\left(\sqrt{1+x^2}+\sqrt{x^2+1-1}\right)$

$$= \log\left(\sqrt{1+x^2}+x\right)$$

$$= \sinh^{-1} x.$$

3. Separate into real and imaginary parts of $\sin^{-1}(e^{i\theta})$.

Solution: Let $\sin^{-1}(e^{i\theta}) = x+iy$, then

$$e^{i\theta} = \sin(x+iy)$$

$$\Rightarrow \cos\theta + i\sin\theta = \sin x\cos iy + \cos x\sin iy$$

$$= \sin x\cosh y + i\cos x\sinh y$$

Comparing real and imaginary parts on both the sides, we have

$$\cos\theta = \sin x\cosh y \ldots\ldots\ldots\ldots\ldots (1)$$

$$\sin\theta = \cos x\sinh y \ldots\ldots\ldots\ldots\ldots (2)$$

Eliminating y from equations (1) & (2), we have

$$\cosh^2 y - \sinh^2 y = \frac{\cos^2\theta}{\sin^2 x} - \frac{\sin^2\theta}{\cos^2 x}$$

$$\Rightarrow 1 = \frac{\cos^2\theta\cos^2 x - \sin^2\theta\sin^2 x}{\sin^2 x\cos^2 x}$$

$$\Rightarrow \sin^2 x\cos^2 x = \cos^2\theta\cos^2 x - \sin^2\theta\,(1-\cos^2 x)$$

$$\Rightarrow (1-\cos^2 x)\cos^2 x = \cos^2\theta\cos^2 x - \sin^2\theta + \sin^2\theta\cos^2 x$$

$$\Rightarrow \cos^2 x - \cos^4 x = \cos^2 x\,(\cos^2\theta+\sin^2\theta) - \sin^2\theta$$

$$\Rightarrow \cos^2 x - \cos^4 x = \cos^2 x - \sin^2\theta$$

$$\Rightarrow \cos^4 x = \sin^2\theta$$

$$\Rightarrow \cos^2 x = \sin\theta$$

$$\Rightarrow \cos x = \pm\sqrt{\sin\theta}$$

$$\Rightarrow x = \cos^{-1}\{\pm\sqrt{\sin\theta}\}$$

From equation (2), $\sin^2\theta = \cos^2 x\sinh^2 y$

Putting $\cos^2 x = \sin\theta$, we have

$$\sin^2\theta = \sin\theta\sinh^2 y$$

$$\Rightarrow \sinh^2 y = \sin\theta$$

$$\Rightarrow \sinh y = \pm\sqrt{\sin\theta}$$

$$\Rightarrow y = \sinh^{-1}\left(\pm\sqrt{\sin\theta}\right) = \log\left(\pm\sqrt{\sin\theta}+\sqrt{\sin\theta+1}\right)$$

Hence $\sin^{-1}(e^{i\theta}) = \cos^{-1}\left(\pm\sqrt{\sin\theta}\right) + i\log\left(\pm\sqrt{\sin\theta}+\sqrt{\sin\theta+1}\right)$.

4. Separate into real and imaginary parts $\cos^{-1}\left(\frac{3i}{4}\right)$.

Solution: Let $\cos^{-1}\left(\frac{3i}{4}\right) = x + iy$, then

$$\frac{3i}{4} = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$$

Comparing real and imaginary parts, we have

$$\cos x \cosh y = 0 \ldots\ldots\ldots\ldots (1)$$

$$\sin x \sinh y = -\frac{3}{4} \ldots\ldots\ldots\ldots (2)$$

From equation (1), $\cos x = 0$ $[\because \cosh y \neq 0]$

$$\Rightarrow x = \frac{\pi}{2}$$

Putting $x = \frac{\pi}{2}$ in equation (2), we have

$$\sinh y = -\frac{3}{4}$$

$$\Rightarrow y = \sinh^{-1}\left(-\frac{3}{4}\right)$$

$$\Rightarrow y = \log\left(-\frac{3}{4} + \sqrt{\frac{9}{16}+1}\right)$$

$$\Rightarrow y = \log\left(-\frac{3}{4} + \frac{5}{4}\right)$$

$$\Rightarrow y = \log\frac{1}{2}$$

$$\Rightarrow \cos^{-1}\left(\frac{3i}{4}\right) = \frac{\pi}{2} + i\log\frac{1}{2}.$$

5. Separate into real and imaginary parts of $\tan^{-1}(e^{i\theta})$.

Solution: Let $\tan^{-1}(e^{i\theta}) = x + iy$, then

$$\tan(x+iy) = e^{i\theta}$$

$$\Rightarrow \tan(x-iy) = e^{-i\theta}$$

Now, $\tan 2x = \tan(\overline{x+iy} + \overline{x-iy})$

$$= \frac{\tan(x+iy) + \tan(x-iy)}{1 - \tan(x+iy)\tan(x-iy)}$$

$$= \frac{e^{i\theta} + e^{-i\theta}}{1 - e^{i\theta}.e^{-i\theta}}$$

$$= \frac{2\cos\theta}{0} = \infty$$

$\therefore \tan 2x = \infty$

$$\Rightarrow 2x = n\pi + \frac{\pi}{2}$$

$$\Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{4}$$

Similarly, $\tan 2iy = \tan(\overline{x+iy} - \overline{x-iy})$

$$= \frac{\tan(x+iy) - \tan(x-iy)}{1 + \tan(x+iy)\tan(x-iy)}$$

$$= \frac{e^{i\theta} - e^{-i\theta}}{1 + e^{i\theta}.e^{-i\theta}}$$

$$= \frac{\cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)}{1+1}$$

$$= \frac{2i\sin\theta}{2} = i\sin\theta$$

$$\Rightarrow i\tanh 2y = i\sin\theta$$

$$\Rightarrow \tanh 2y = \sin\theta$$

$$\Rightarrow 2y = \tanh^{-1}\sin\theta$$

$$\Rightarrow 2y = \frac{1}{2}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$$

$$\Rightarrow 2y = \frac{1}{2}\log\left(\frac{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$\Rightarrow 2y = \frac{1}{2}\log\frac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2}$$

$$\Rightarrow 2y = \log\frac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)}$$

$$\Rightarrow 2y = \log\frac{\left(1 + \tan\frac{\theta}{2}\right)}{\left(1 - \tan\frac{\theta}{2}\right)}$$

$$\Rightarrow 2y = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$\Rightarrow y = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$\Rightarrow \tan^{-1} e^{i\theta} = \frac{n\pi}{2} + \frac{\pi}{4} + \frac{i}{2}\log\left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$

6. Prove that $\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = -\frac{i}{2}\log\left(\frac{a}{x}\right)$.

Solution: Let $\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = u + iv$, then

$$\tan(u+iv) = i\left(\frac{x-a}{x+a}\right)$$

$$\therefore \tan(u-iv) = -i\left(\frac{x-a}{x+a}\right)$$

$$\tan 2u = \tan(\overline{u+iv} + \overline{u-iv}) = \frac{\tan(u+iv) + \tan(u-iv)}{1 - \tan(u+iv)\tan(u-iv)}$$

$$= \frac{i\left(\frac{x-a}{x+a}\right) - i\left(\frac{x-a}{x+a}\right)}{1 + i^2\left(\frac{x-a}{x+a}\right)\left(\frac{x-a}{x+a}\right)} = 0$$

$$\Rightarrow 2u = \tan^{-1} 0 = 0$$

$$\Rightarrow u = 0$$

Similarly, we have

$$\tan 2iv = \tan\left(\overline{u+iv} - \overline{u-iv}\right)$$

$$= \frac{\tan(u+iv) - \tan(u-iv)}{1 + \tan(u+iv)\tan(u-iv)}$$

$$= \frac{i\left(\frac{x-a}{x+a}\right) + i\left(\frac{x-a}{x+a}\right)}{1 - i^2\left(\frac{x-a}{x+a}\right)\left(\frac{x-a}{x+a}\right)}$$

$$= \frac{2i\left(\frac{x-a}{x+a}\right)}{1 + \left(\frac{x-a}{x+a}\right)^2} = \frac{2i\left(\frac{x-a}{x+a}\right)}{\frac{(x+a)^2 + (x-a)^2}{(x+a)^2}}$$

$$= \frac{2i\,(x-a)(x+a)}{x^2 + a^2 + 2ax + x^2 + a^2 - 2ax}$$

$$= \frac{2i(x^2 - a^2)}{2(x^2 + a^2)}$$

$$\Rightarrow i \tanh 2v = i\left(\frac{x^2 - a^2}{x^2 + a^2}\right)$$

$$\Rightarrow 2v = \tanh^{-1}\left(\frac{x^2 - a^2}{x^2 + a^2}\right)$$

$$\Rightarrow 2v = \frac{1}{2}\log\left(\frac{1 + \frac{x^2 - a^2}{x^2 + a^2}}{1 - \frac{x^2 - a^2}{x^2 + a^2}}\right)$$

$$\Rightarrow 2v = \frac{1}{2}\log\frac{2x^2}{2a^2}$$

$$\Rightarrow v = -\frac{1}{2}\log\frac{a}{x}$$

$$\therefore \tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = -\frac{i}{2}\log\left(\frac{a}{x}\right).$$

7. Prove that $\operatorname{sech}^{-1}(\sin\theta) = \log\left(\cot\frac{\theta}{2}\right)$.

Solution: Let $\operatorname{sech}^{-1}(\sin\theta) = x$, then $\sin\theta = \operatorname{sech} x$, $\therefore \cosh x = \operatorname{cosec}\theta$

$$x = \cosh^{-1}(\operatorname{coses}\theta) = \log\left[\operatorname{cosec}\theta + \sqrt{\operatorname{cosec}^2\theta - 1}\right]$$

$$= \log\left[\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right] = \log\left[\frac{1+\cos\theta}{\sin\theta}\right]$$

$$= \log\left[\frac{2\cos^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\right]$$

$$= \log\left(\cot\frac{\theta}{2}\right).$$

8. Separate into real and imaginary parts of $\tanh^{-1}(x+iy)$.

Solution: Let $\tanh^{-1}(x+iy) = u + iv$, then

$$x + iy = \tanh(u+iv) = \frac{1}{i}\tan(iu - v)$$

$$\therefore x - iy = \tanh(u - iv) = \frac{1}{i}\tan(iu + v)$$

Adding, we get $\tan(2iu) = \tan\left(\overline{iu+v} + \overline{iu-v}\right)$

$$= \frac{\tan(iu+v) + \tan(iu-v)}{1 - \tan(iu+v).\tan(iu-v)}$$

$$= \frac{ix + y + ix - y}{1 - (ix+y)(ix-y)}$$

$$\Rightarrow i\tanh 2u = \frac{2ix}{1 + x^2 + y^2}$$

$$\Rightarrow u = \frac{1}{2}\tanh^{-1}\left(\frac{2x}{1 + x^2 + y^2}\right)$$

Now by subtracting, we have

$$\tan(2v) = \tan\left(\overline{iu+v} - \overline{iu-v}\right)$$

$$= \frac{\tan(iu+v) - \tan(iu-v)}{1 + \tan(iu+v).\tan(iu-v)}$$

$$= \frac{ix + y - ix + y}{1 + (ix+y)(ix-y)}$$

$$\Rightarrow \tan 2v = \frac{2y}{1 - x^2 - y^2}$$

$$\Rightarrow v = \frac{1}{2}\tan^{-1}\left(\frac{2y}{1 - x^2 - y^2}\right).$$

EXERCISE - 1.6

1. If $\tan(A + iB) = x + iy$, prove that (i) $x^2 + y^2 + 2x\cot 2A = 1$ (ii) $x^2 + y^2 - 2y\coth 2B + 1 = 0$.
2. If $\tan(x + iy) = \sin(u + iv)$, prove that $\frac{\sin 2x}{\sinh 2y} = \frac{\tan u}{\tan v}$.
3. If $cosec\left(\frac{\pi}{4} + ix\right) = u + iv$, prove that $u^2 + v^2 = 2(u^2 - v^2)$.
4. If $\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha$, prove that $e^{2\phi} = \pm\cot\frac{\alpha}{2}$ and $2\theta = \left(n + \frac{1}{2}\right)\pi + \alpha$.
5. If $x = 2\cos\alpha\cosh\beta$, $y = 2\sin\alpha\sinh\beta$, prove that $\sec(\alpha + i\beta) + \sec(\alpha - i\beta) = \frac{4x}{x^2+y^2}$.
6. If $a + ib = \tanh\left(v + \frac{i\pi}{4}\right)$, prove that $a^2 + b^2 = 1$.
7. Reduce $\tan^{-1}(\cos\theta + i\sin\theta)$ to the form $a + ib$.

   Hence show that $\tan^{-1}(e^{i\theta}) = \frac{n\pi}{2} + \frac{\pi}{4} - \frac{i}{2}\log\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$.
8. If $\cos^{-1}(x + iy) = \alpha + i\beta$, show that (i) $x^2\sec^2\alpha - y^2 cosec^2\alpha = 1$ (ii) $x^2\operatorname{sech}^2\beta + y^2 cosech^2\beta = 1$.
9. If $\sin(\theta + i\phi) = \tan(x + iy)$, prove that $\frac{\sin 2x}{\sinh 2y} = \frac{\tan\theta}{\tanh\phi}$.

10. If $\tan z = \frac{1}{2}(1-i)$, prove that $z = \frac{n\pi}{2} + \frac{1}{2}\tan^{-1}(2) - \frac{i}{4}\log 5$.

11. If $\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha$, prove that $2\theta = n\pi + \frac{\pi}{2} + \alpha, e^{2\phi} = \pm\cot\frac{\alpha}{2}$.

12. If $\log\cos(x - iy) = \alpha + i\beta$, prove that $\alpha = \frac{1}{2}\log\left(\frac{\cos 2x \cosh 2y}{2}\right)$.

13. Prove that $\sinh^{-1}(\tan\theta) = \log\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$.

14. Prove that $\cosh^{-1}\left(\frac{3i}{4}\right) = \log 2 + \tan\frac{i\pi}{2}$.

15. Prove that $\cos^{-1}(ix) = \frac{\pi}{2} - i\log(x - \sqrt{x^2+1})$.

16. Prove that $\sin^{-1}(\operatorname{cosec}\theta) = \frac{\pi}{2} + i\log\cos\frac{\theta}{2}$.

17. If $\cos\left(\frac{\pi}{4} + ia\right).\cosh\left(b + \frac{i\pi}{4}\right) = 1$, where a and b real, prove that $2b = \log(2+\sqrt{3})$.

18. Prove that $\tan^{-1}(\sin\theta) = \cosh^{-1}(\sec\theta)$.

19. If $\tan\left(\frac{\pi}{2} + iv\right) = re^{i\theta}$, show that

    (a) $r = 1$ (b) $\tan\theta = \sinh 2v$ (c) $\tanh v = \tan\frac{\theta}{2}$.

20. If $\cosh x = \sec\theta$, prove that

    (a) $x\log(\sec\theta + \tan\theta)$ (b) $\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$ (c) $\tanh\left(\frac{x}{2}\right) = \tan\left(\frac{\theta}{2}\right)$.

21. If $\tan(\alpha + i\beta) = \cos\theta + i\sin\theta$, prove that $\alpha = \frac{n\pi}{2} + \frac{\pi}{4}, \beta = \frac{1}{2}\log\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

1.29 Logarithms of Complex Quantities

Let $z = \log(x+iy)$ and $x = r\cos\theta$ and $= r\sin\theta$. Then $r = \sqrt{(x^2+y^2)}$ and $= \tan^{-1}\frac{y}{x}$.

$$\therefore z = \log(r\cos\theta + ri\sin\theta) = \log r(\cos\theta + i\sin\theta)$$
$$= \log re^{i\theta} = \log r + \log e^{i\theta}$$
$$= \log r + i\theta$$

$$\text{or,}\quad \log(x+iy) = \log\sqrt{x^2+y^2} + i\,\tan^{-1}\left(\frac{y}{x}\right)$$

This is known as the principal value of $\log(x+iy)$.

The general value of $\log(x+iy)$ is denoted as $\mathrm{Log}(x+iy)$ and is given by

$$\mathrm{Log}(x+iy) = \log r + i(\theta + 2n\pi),$$ where r and θ are as stated above.

The principal value of logarithm of $(x+iy)$ is obtained by taking $= 0$.

1.30 Solved Examples

1. **Find the general value of (a) $\log_e(-i)$ (b) $\log_e(-3)$ (c) $\log_e(4+3i)$.**

Solution: (a) Let $-i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = e^{-\frac{i\pi}{2}}$

$$\therefore \log_e(-i) = \log_e e^{-\frac{i\pi}{2}} = -\frac{i\pi}{2}$$

Hence, by definition, we have

$$\mathrm{Log}_e(-i) = 2n\pi i + \log_e(-i) = 2n\pi i - \frac{i\pi}{2} = (4n-1)\frac{\pi}{2}i$$

(b) $-3 = 3(\cos\pi + i\sin\pi) = 3e^{\pi i}$

$$\Rightarrow \log_e(-3) = \log_e(3e^{\pi i}) = \log_e 3 + \log_e e^{\pi i} = \log_e 3 + \pi i$$

$$\therefore \mathrm{Log}_e(-3) = \log_e 3 + 2n\pi i + \pi i = \log_e 3 + (2n+1)\pi i$$

(c) $4 + 3i = r(\cos\theta + i\sin\theta)$, so that $r = 5$ *and* $\theta = \tan^{-1}\left(\frac{3}{4}\right)$

$$\log_e(4+3i) = \log_e r(\cos\theta + i\sin\theta) = \log_e re^{i\theta} = \log_e r + \log_e e^{i\theta}$$
$$= \log_e r + i\theta = \log_e 5 + i\tan^{-1}\left(\frac{3}{4}\right)$$

$$\therefore \mathrm{Log}_e(4+3i) = 2n\pi i + \log_e(4+3i) = 2n\pi i + \log_e 5 + i\tan^{-1}\left(\frac{3}{4}\right)$$

2. **Prove that $\log(1+e^{2i\theta}) = \log(2\cos\theta) + i\theta$.**

Solution: $\log(1+e^{2i\theta}) = \log(1+\cos 2\theta + i\sin 2\theta)$ by Euler's Formula

$$= \log(1 + 2\cos^2\theta - 1 + 2i\sin\theta\cos\theta)$$
$$= \log(2\cos^2\theta + 2i\sin\theta\cos\theta)$$
$$= \log\{2\cos\theta\,(\cos\theta + i\sin\theta)\}$$
$$= \log(2\cos\theta\, e^{i\theta})$$
$$= \log 2\cos\theta + \log e^{i\theta} = \log(2\cos\theta) + i\theta .$$

3. **Prove that $\log(e^{i\alpha} + e^{i\beta}) = \log\left(2\cos\frac{\alpha-\beta}{2}\right) + i\left(\frac{\alpha+\beta}{2}\right)$.**

Solution: $\log(e^{i\alpha} + e^{i\beta}) = \log\{(\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta)\}$

$$= \log\{(\cos\alpha + \cos\beta) + i(\sin\alpha + \sin\beta)\}$$
$$= \log\left\{2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2i\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right\}$$
$$= \log\left\{2\cos\frac{\alpha-\beta}{2}\left(\cos\frac{\alpha+\beta}{2} + i\sin\frac{\alpha+\beta}{2}\right)\right\}$$
$$= \log\left\{2\cos\frac{\alpha-\beta}{2}e^{\frac{\alpha+\beta}{2}i}\right\}$$
$$= \log 2\cos\frac{\alpha-\beta}{2} + \log e^{\frac{\alpha+\beta}{2}i}$$
$$= \log 2\cos\frac{\alpha-\beta}{2} + i\left(\frac{\alpha+\beta}{2}\right).$$

4. **Prove that $\log\frac{1}{1-e^{i\theta}} = \log\left(\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right) + i\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$.**

Solution: $\log\frac{1}{1-e^{i\theta}} = \log\frac{1}{1-(\cos\theta + i\sin\theta)} = \log\frac{1}{1-\cos\theta - i\sin\theta}$

$$= \log\frac{1}{1 - 1 + 2\sin^2\frac{\theta}{2} - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$
$$= \log\frac{1}{2\sin\frac{\theta}{2}\left\{\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right\}}$$

$$= \log 1 - \log\left(2\sin\frac{\theta}{2}\right) - \log\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)$$

$$= 0 + \log\frac{1}{2\sin\frac{\theta}{2}} - \log\left\{\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}$$

$$= \log\left(\frac{1}{2}cosec\frac{\theta}{2}\right) - \log_e e^{-i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}$$

$$= \log\left(\frac{1}{2}cosec\frac{\theta}{2}\right) - \left\{-i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}$$

$$= \log\left(\frac{1}{2}cosec\frac{\theta}{2}\right) + i\left(\frac{\pi}{2}-\frac{\theta}{2}\right).$$

5. Prove that $\tan\left[i\log\frac{a-ib}{a+ib}\right] = \frac{2ab}{a^2-b^2}$.

Solution: $\log(a+ib) = \log\sqrt{a^2+b^2} + i\tan^{-1}\left(\frac{b}{a}\right)$

$$\log(a-ib) = \log\sqrt{a^2+b^2} - i\tan^{-1}\left(\frac{b}{a}\right)$$

$$\therefore \log(a-ib) - \log(a+ib) = -2i\tan^{-1}\left(\frac{b}{a}\right)$$

$$\Rightarrow \log\left(\frac{a-ib}{a+ib}\right) = -2i\tan^{-1}\left(\frac{b}{a}\right)$$

$$\Rightarrow i\log\left(\frac{a-ib}{a+ib}\right) = 2\tan^{-1}\left(\frac{b}{a}\right)$$

$$\Rightarrow \tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \tan\left[2\tan^{-1}\left(\frac{b}{a}\right)\right]$$

$$= \tan 2\theta, \quad \text{where } \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$= \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{b}{a}\right)}{1-\frac{b^2}{a^2}} = \frac{2ab}{a^2-b^2}.$$

6. Prove that $\log(-\log i) = \log\frac{\pi}{2} - \frac{i\pi}{2}$.

Solution: Since $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

$$\therefore \log i = \frac{i\pi}{2} \Rightarrow -\log i = -\frac{i\pi}{2}$$

or, $\log(-\log i) = \log\left(-\frac{i\pi}{2}\right)$

Now $\log(x+iy) = \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)$

gives $\log\left(-\frac{i\pi}{2}\right) = \log\sqrt{(0)^2 + \left(-\frac{\pi}{2}\right)^2} + i\tan^{-1}\left(\frac{-\pi/2}{0}\right)$

$$= \log\frac{\pi}{2} - \frac{i\pi}{2}.$$

7. If $p\log(a+ib) = (x+iy)\log m$, prove that $\frac{y}{x} = \frac{2\tan^{-1}\frac{b}{a}}{\log(a^2+b^2)}$.

Solution: $p\log(a+ib) = (x+iy)\log m$

$$\Rightarrow p\left[\log\sqrt{a^2+b^2} + i\tan^{-1}\left(\frac{b}{a}\right)\right] = (x+iy)\log m$$

$$= p\log\sqrt{a^2+b^2} = x\log m \ \ldots\ldots\ldots (1)$$

and $p\tan^{-1}\left(\frac{b}{a}\right) = y\log m \ \ldots\ldots\ldots (2)$

From (1) and (2), we have

$$\frac{\tan^{-1}\left(\frac{b}{a}\right)}{\log\sqrt{a^2+b^2}} = \frac{y}{x} \Rightarrow \frac{\tan^{-1}\left(\frac{b}{a}\right)}{\frac{1}{2}\log(a^2+b^2)} = \frac{y}{x}$$

$$\Rightarrow \frac{2\tan^{-1}\left(\frac{b}{a}\right)}{\log(a^2+b^2)} = \frac{y}{x}.$$

8. If $\log\log(x+iy) = p+iq$, then prove that $y = x\tan\left[\tan q\log\sqrt{x^2+y^2}\right]$.

Solution: $\log\log(x+iy) = p+iq$ gives

$$\log(x+iy) = e^{p+iq} = e^p e^{iq}$$

$$\Rightarrow \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right) = e^p e^{iq} = e^p(\cos q + i\sin q)$$

$$\Rightarrow \log\sqrt{x^2+y^2} = e^p\cos q \ \textit{and}\ \tan^{-1}\left(\frac{y}{x}\right) = e^p\sin q$$

$$\therefore \frac{e^p\sin q}{e^p\cos q} = \frac{\tan^{-1}\left(\frac{y}{x}\right)}{\log\sqrt{x^2+y^2}}$$

or, $\tan q = \frac{\tan^{-1}\left(\frac{y}{x}\right)}{\log\sqrt{x^2+y^2}}$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \tan q\log\sqrt{x^2+y^2}$$

$$\Rightarrow \frac{y}{x} = \tan\left[\tan q\log\sqrt{x^2+y^2}\right]$$

$$\Rightarrow y = x\tan\left[\tan q\log\sqrt{x^2+y^2}\right].$$

9. Prove that (a) $i^i = e^{-(4n+1)\frac{\pi}{2}}$ and $\log i^i = -\left(2n+\frac{1}{2}\right)\pi$. (b) $\left(\sqrt{i}\right)^{\sqrt{i}} = e^{-\alpha}\operatorname{cis}\alpha$ where $\alpha = \frac{\pi}{4\sqrt{2}}$.

Solution: (a) We have by definition

$$i^i = e^{i\log i} = e^{i(2in\pi + \log i)} = e^{-2n\pi + i\log\left[\exp\left(\frac{i\pi}{2}\right)\right]}$$

$$= e^{-2n\pi + i\left(\frac{i\pi}{2}\right)} = e^{-\left(2n+\frac{1}{2}\right)\pi}$$

$$\Rightarrow \log i^i = \log e^{-\left(2n+\frac{1}{2}\right)\pi} = -\left(2n+\frac{1}{2}\right)\pi$$

(b) $(\sqrt{i})^{\sqrt{i}} = e^{\sqrt{i}\log\sqrt{i}}$

Now $\sqrt{i}\log\sqrt{i} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2}\log\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$$= \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\log\left(e^{\frac{i\pi}{2}}\right)$$

$$= \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\frac{i\pi}{2}$$

$$= \frac{i\pi}{4}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = -\frac{\pi}{4\sqrt{2}} + i\left(\frac{\pi}{4\sqrt{2}}\right)$$

Hence $(\sqrt{i})^{\sqrt{i}} = e^{-\alpha+i\alpha}$ where $\alpha = \frac{\pi}{4\sqrt{2}}$

$$= e^{-\alpha}\, e^{i\alpha} = e^{-\alpha}(\cos\alpha + i\sin\alpha) = e^{-\alpha} cis\, \alpha .$$

10. If $(a + ib)^p = m^{x+iy}$, prove that one of the values of $\frac{y}{x}$ is $\frac{2\tan^{-1}\frac{b}{a}}{\log(a^2+b^2)}$.

Solution: Taking the logarithm of $(a + ib)^p = m^{x+iy}$ gives

$$p\log(a + ib) = (x + iy)\log m = x\log m + iy\log m$$

or, $\quad p\left[\frac{1}{2}\log(a^2 + b^2) + i\tan^{-1}\frac{b}{a}\right] = x\log m + iy\log m$

$\Rightarrow \quad \frac{p}{2}\log(a^2 + b^2) = x\log m \ldots\ldots\ldots (1)$

$p\tan^{-1}\frac{b}{a} = y\log m \ldots\ldots\ldots\ldots\ldots (2)$

Division of (2) by (1) gives

$$\frac{y}{x} = \frac{2\tan^{-1}\left(\frac{b}{a}\right)}{\log(a^2 + b^2)}.$$

11. If $i^{i^{\cdots\infty}} = A + iB$, prove that $\tan\frac{\pi A}{2} = \frac{B}{A}$ and $A^2 + B^2 = e^{-\pi B}$.

Solution: Here $i^{i^{\cdots\infty}} = A + iB$ gives $i^{A+iB} = A + iB$

or, $\quad A + iB = e^{(A+iB)\log i} = e^{(A+iB)\log\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)}$

$$= \exp[(A + iB)\log(e^{i\pi/2})]$$

$$= e^{(A+iB)\frac{i\pi}{2}} = e^{-\frac{B\pi}{2}}\, e^{\frac{i\pi A}{2}}$$

$$= e^{-\frac{B\pi}{2}}\left(\cos\frac{\pi A}{2} + i\sin\frac{\pi A}{2}\right)$$

Equating real and imaginary parts, we get

$A = e^{-\frac{B\pi}{2}}\cos\frac{\pi A}{2} \ldots\ldots\ldots\ldots\ldots (1)$

and $B = e^{-\frac{B\pi}{2}}\sin\frac{\pi A}{2} \ldots\ldots\ldots\ldots (2)$

Division of (2) by (1) gives

$$\frac{B}{A} = \tan\frac{\pi A}{2}$$

Squaring and adding (1) and (2), we have

$$A^2 + B^2 = e^{-B\pi}$$

12. Separate into real and imaginary parts $\log\sin(x + iy)$.

Solution: $\log\sin(x + iy) = \log(\sin x\cos iy + \cos x\sin iy)$

$$= \log(\sin x\cosh y + i\cos x\sinh y)$$

$$= \log r(\cos\theta + i\sin\theta)$$

where $r\cos\theta = \sin x\cosh y$ *and* $r\sin\theta = \cos x\sinh y$

where $r = \sqrt{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y}$

$$= \sqrt{\frac{1-\cos 2x}{2}\cdot\frac{1+\cosh 2y}{2} + \frac{1+\cos 2x}{2}\cdot\frac{\cosh 2y - 1}{2}}$$

$$= \frac{1}{2}\sqrt{1 + \cosh 2y - \cos 2x - \cos 2x\cosh 2y + \cosh 2y - 1 + \cos 2x\cosh 2y - \cos 2x}$$

$$= \frac{1}{2}\sqrt{2\cosh 2y - 2\cos 2x}$$

$$= \sqrt{\frac{\cosh 2y - \cos 2x}{2}}$$

and $\theta = \tan^{-1}(\cot x\tanh y)$

Thus, $\log\sin(x + iy) = \log(re^{i\theta}) = \log r + i\theta$

$$= \log\sqrt{\frac{1}{2}(\cosh 2y - \cos 2x)} + i\,\tan^{-1}(\cot x\tanh y)$$

$$= \frac{1}{2}\log\left[\frac{1}{2}(\cosh 2y - \cos 2x)\right] + i\,\tan^{-1}(\cot x\tanh y).$$

EXERCISE - 1.7

1. Separate $\sin^{-1}(\cos\theta + i\sin\theta)$ into real and imaginary parts.

   Ans. R.P. $\cos^{-1}\sqrt{\sin\theta}$, I.P. $= \log\left[\sqrt{\sin\theta} + \sqrt{\sin\theta + 1}\right]$

2. If $\sin^{-1}(x + iy) = \log(A + iB)$, then show that $\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1$, where $A^2 + B^2 = e^{2u}$.

3. Prove that $\tan^{-1}\left(\frac{x-a}{x+a}\right) = \frac{1}{2} i \log\left(\frac{x}{a}\right)$.

4. Prove that

   (a) $\sinh^{-1} x = \log_e\left(x + \sqrt{x^2 + 1}\right)$

   (b) $\cosh^{-1} x = \log_e\left(x + \sqrt{x^2 - 1}\right)$

   (c) $\sinh^{-1} x = \cosh^{-1}\sqrt{1 + x^2}$

   (d) $\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$.

5. Prove that $\sinh^{-1} x = \log\left(x + \sqrt{x^2 + 1}\right)$.

6. Prove that

   (a) $\tanh^{-1}(\cos\theta) = \cosh^{-1}\operatorname{cosec}\theta$

   (b) $\sinh^{-1}(\tan\theta) = \log\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$

   (c) $\sinh^{-1}(\tan\theta) = \log(\sec\theta + \tan\theta)$

   (d) $\tanh^{-1}(\cos\theta) = \cosh^{-1}\operatorname{cosec}\theta$

   (e) $\operatorname{sech}^{-1}(\sin\phi) = \log\cot\frac{\phi}{2}$.

7. If $\tan\log(x + iy) = a + ib$, where $a^2 + b^2 \neq 1$, show that $\tan\log(x^2 + y^2) = \frac{2a}{1-a^2-b^2}$.

8. Find all the roots of the equation

   (a) $\sin z = \cosh 4$

   (b) $\sinh z = i$.

   Ans. (a) $z = n\pi + (-1)^n\left(\frac{\pi}{2} - 4i\right)$ (b) $z = i\left(2n + \frac{1}{2}\right)\pi$

OBJECTIVE TYPE QUESTIONS

Pick up the correct answers of the choices given in each questions

1. If $x + iy = \sqrt{2} + \sqrt{3}\,i$ then $x^2 + y^2$ is equal to

   (a) 5 (b) 7

   (c) 0 (d) none. [Ans.: (a)]

2. The real part of $(\sin x + i\cos x)^5$ is equal to

   (a) $-\cos 5x$ (b) $-\sin 5x$

   (c) $\sin 5x$ (d) none. [Ans.: (c)]

3. The real part of $(\cos x + i\sin x)^5$ is equal to

   (a) $\cos 5x$

   (c) $\cos^5 x$ (d) none. [Ans.: (a)]

4. The number $(i)^i$ is

   (a) Imaginary Number (b) Irrational Number

   (c) An integer (d) none. [Ans.: (b)]

5. The relation $|1 - z| + |2 + z| = 5$ represents

   (a) Circle (b) Parabolic

   (c) ellipse (d) none. [Ans.: (c)]

6. The value of the complex number z with $|z| = 1$ and $\arg(z) = \frac{3\pi}{4}$ is

   (a) $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ (b) $-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

   (c) $\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$ (d) none. [Ans.: (b)]

7. If $f(z) = e^{2z}$, then the real part of $f(z)$ is

   (a) $e^y \sin x$ (b) $e^x \cos 2y$

   (c) $e^{2x}\cos 2y$ (d) none. [Ans.: (c)]

8. If $f(z) = e^{2z}$, then the imaginary part of $f(z)$ is

   (a) $e^y \sin x$ (b) $e^{2x}\sin 2y$

   (c) $e^{2x}\cos 2y$ (d) none. [Ans.: (b)]

9. If $f(z) = 3\bar{z}$, then the value of $f(z)$ at $z = 2 + 4i$ is

   (a) $6 - 12i$ (b) $6 + 12i$

   (c) $-6 + 12i$ (d) none. [Ans.: (a)]

10. The real part of $\frac{2+3i}{3-4i}$ is

(a) $-\frac{6}{25}$ (b) $\frac{6}{25}$ [Ans.: (a)]

(c) $\frac{17}{28}$ (d) none.

11. The imaginary part of $\frac{2+3i}{3-4i}$ is

(a) $-\frac{17}{25}$ (b) $\frac{17}{25}$ [Ans.: (b)]

(c) $-\frac{6}{25}$ (d) none.

12. If $z = \cos\theta + i\sin\theta$, then the value of $z^n + \frac{1}{z^n}$ is

(a) $2\cos\theta$ (b) $2\sin\theta$ [Ans.: (a)]

(c) $\cos n\theta$ (d) none.

13. If $z = \cos\theta + i\sin\theta$, then the value of $z^n - \frac{1}{z^n}$ is

(a) $2i\sin\theta$ (b) $2i\cos n\theta$ [Ans.: (a)]

(c) $\sin n\theta$ (d) none.

14. Real part of $\cosh(x+iy)$ is

(a) $\cosh x\cos y$ (b) $\sinh x\sin y$ [Ans.: (a)]

(c) $\cosh x\sin y$ (d) none.

15. If $\tan\frac{x}{2} = \tanh\frac{y}{2}$, then the value of $\cos x\cosh y$ is

(a) 1 (b) -1 [Ans.: (a)]

(c) 0 (d) none.

16. The imaginary part of $\sin\bar{z}$ is

(a) $\cos x\sinh y$ (b) $-\cos x\sinh y$ [Ans.: (b)]

(c) $\cosh x\sin y$ (d) none.

17. The real part of $\sin z$ is

(a) $\sin x\cosh y$ (b) $\sinh x\cos y$ [Ans.: (a)]

(c) $\sinh x\cosh y$ (d) none.

18. The module of $(\sqrt{i})^{\sqrt{i}}$ is equal to

(a) $e^{\frac{\pi}{4\sqrt{2}}}$ (b) $e^{\frac{\pi}{2\sqrt{2}}}$ [Ans.: (c)]

(c) $e^{-\frac{\pi}{4\sqrt{2}}}$ (d) none.

19. If $\sin\alpha + \sin\beta + \sin\gamma = 0 = \cos\alpha + \cos\beta + \cos\gamma$, then $\sin 3\alpha + \sin 3\beta + \sin 3\gamma$ is

(a) $3\cos(\alpha+\beta+\gamma)$ (b) $3\sin(\alpha+\beta+\gamma)$ [Ans.: (b)]

(c) $\sin(\alpha+\beta+\gamma)$ (d) none.

20. If $\sin\alpha + \sin\beta + \sin\gamma = 0 = \cos\alpha + \cos\beta + \cos\gamma$, then $\sin 2\alpha + \sin 2\beta + \sin 2\gamma$ is

(a) 1 (b) 2

(c) 0 (d) none. [Ans.: (c)]

21. $(-i)^{-i}$ is

(a) purely real (b) purely imaginary

(c) complex (d) none. [Ans.: (c)]

22. Hyperbolic functions are

(a) periodic (b) non- periodic

(c) real (d) none. [Ans.: (a)]

23. The n^{th} roots of unity form

(a) G.P. (b) A.P.

(c) H.P. (d) none. [Ans.: (a)]

24. If $\left|\frac{z-a}{z-b}\right| = k(\neq 1)$, then the locus of z is

(a) a circle (b) a parabola

(c) an ellipse (d) none. [Ans.: (a)]

25. If $|z_1 + z_2| = |z_1 - z_2|$, then $amp(z_1) - amp(z_2)$ is equal to

(a) $-\frac{\pi}{2}$ (b) $\frac{\pi}{2}$

(c) π (d) none. [Ans.: (b)]

26. Cube roots of unity form

(a) right angled triangle (b) scalene triangle

(c) equilateral triangle (d) none. [Ans.: (c)]

27. If $\sin\theta = \tanh\phi$, then $\tan\theta$ is equal to

(a) $\sinh\phi$ (b) $\cosh\phi$

(c) $\cos\theta$ (d) none. [Ans.: (a)]

28. Imaginary part of $\tan(+i\phi)$ is given by

(a) $\sinh 2\phi/(\cosh 2\theta + \cosh 2\phi)$ (b) $\sinh 2\phi/(\cosh 2\phi + \sinh 2\theta)$

(c) $\cosh 2\phi/(\cosh 2\theta + \cosh 2\phi)$ (d) none. [Ans.: (a)]

29. $\log(-1)$ is equal to

(a) $2n\pi i$ (b) $-2n\pi i$

(c) $2n\pi$ (d) none. [Ans.: (a)]

30. The smallest positive integer n for which $(1+i)^{2n} = (1-i)^{2n}$ is

(a) 4 (b) 8

(c) 2 (d) none. [Ans.: (c)]

31. If the ω is the cube root of unity, then the value of $\begin{bmatrix} 1 & \omega & 2\omega^2 \\ 2 & 2\omega^2 & 4\omega^2 \\ 3 & 3\omega^3 & 6\omega^2 \end{bmatrix}$ is

(a) 1 (b) -1

(c) 0 (d) none. [Ans.: (c)]

32. The cube roots of unity lie on a circle

(a) $|z| = 1$ (b) $|z - 1| = 1$

(c) $|z + 1| = 1$ (d) none. [Ans.: (a)]

33. If z lies on $|z| = 1$, then $\frac{2}{z}$ will lie on

(a) a circle (b) an ellipse

(c) a parabola (d) none. [Ans.: (a)]

34. If real part of $\frac{z-8i}{z+6} = 0$, then z lies on the curve

(a) $x^2 + y^2 + 6x - 8y = 0$ (b) $x^2 + y^2 + 6x - 4y = 0$

(c) $x^2 + y^2 + 4x - 8y = 0$ (d) none. [Ans.: (a)]

35. If $2 + i\sqrt{3}$ is a root of the quadratic equation $x^2 + ax + b = 0$ where a and $b \in R$, then the values of a and b are respectively

(a) $4, 7$ (b) $-4, -7$

(c) $-4, 7$ (d) none. [Ans.: (c)]

36. The triangle formed by the points 1, $\frac{1+i}{\sqrt{2}}$ and i as vertices in the Argand diagram is

(a) Scalene (b) Equilateral

(c) isosceles (d) none. [Ans.: (c)]

37. If $c^2 + s^2 = 1$, then $\frac{1+c+is}{1+c-is}$ is equal to

(a) $c + is$ (b) $c - is$

(c) $s + ic$ (d) none. [Ans.: (a)]

38. The value of $\sin(\log i^i)$ is

(a) -1 (b) 1

(c) 0 (d) none. [Ans.: (a)]

39. If $\log[\log(x + iy)] = p + iq$, then the value of $\tan^{-1}(y/x)$ is

(a) $e^p \cos q$ (b) $e^q \sin p$

(c) $e^p \sin q$ (d) none. [Ans.: (c)]

40. If the root of $x^3 - 8x^2 + px + q = 0$, where p & q are real numbers, is $3 - i\sqrt{3}$, then its real root is

(a) 2 (b) 6

(c) 9 (d) none. [Ans.: (a)]

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